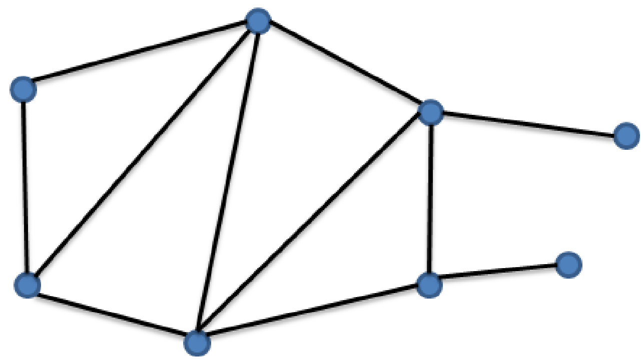
Lecture 27: Optimal control of undirected graphs



- Single-integrator dynamics

$$\dot{x}_i = u_i + d_i$$

- Relative information exchange with neighbors

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} k_{ij} \left(x_i(t) - x_j(t) \right)$$

- Closed-loop dynamics

$$\dot{x}(t) = -L(k)\, x(t) + d(t)$$

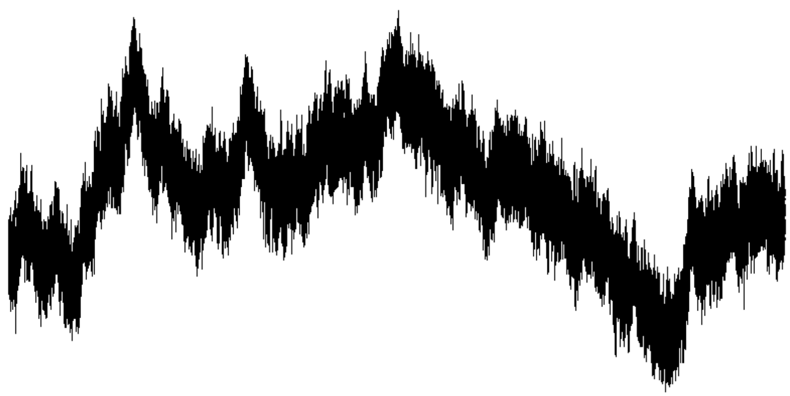
- Structured matrix L depends on $\left\{ \begin{array}{l} \text{graph topology} \\ \text{vector of feedback gains } k \end{array} \right.$

- Independent of graph topology and feedback gains

$$L(k)\,\mathbb{1} \;=\; 0 \cdot \mathbb{1}$$

Average mode

$$\bar{x}(t) \;=\; \frac{1}{N}\sum_{i\,=\,1}^{N} x_i(t): \quad \text{undergoes random walk}$$



If other modes are stable, $x_i(t)$ fluctuates around $\bar{x}(t)$

deviation from average: $\tilde{x}_i(t) \;=\; x_i(t) \;-\; \bar{x}(t)$

steady-state variance: $\lim\limits_{t\,\to\,\infty} \mathcal{E}\left(\tilde{x}^T(t)\,\tilde{x}(t)\right)$

Optimal control problem

What graph topologies lead to small variance?

How to design feedback gains to minimize variance?

$$
\begin{aligned}
\dot{x}(t) &= -L(k)\,x(t) \;+\; d(t) \\
z(t) &= \begin{bmatrix} \tilde{x}(t) \\ u(t) \end{bmatrix} \;=\; \begin{bmatrix} I \;-\; \frac{1}{N}\mathbb{1}\mathbb{1}^T \\ -\,L(k) \end{bmatrix} x(t)
\end{aligned}
$$

- Setup:
  - ⋆ Undirected graphs: bi-directional interaction between nodes
  - ⋆ Symmetric feedback gains

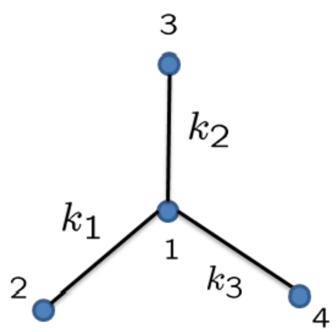
$$
k_{ij} \;=\; k_{ji} \;\;\Rightarrow\;\; L(k) \;=\; L^T(k)
$$

Incidence matrix

- Edge $l \sim (i, j)$: connects nodes i and j
  - ⋆ Define $e_l \in \mathbb{R}^N$ with only two nonzero entries

$$(e_l)_i = 1 \qquad (e_l)_j = -1$$

Incidence matrix: $E = [\, e_1 \cdots e_m \,]$



$$E = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad E^T x = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \\ x_1 - x_4 \end{bmatrix}, \quad E^T \mathbb{1} = 0$$

Edge $l \sim (i, j)$: $k_l := k_{ij} = k_{ji}$

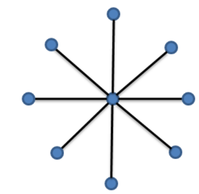
Laplacian: $L(K) = E\, K\, E^T = \sum_{l=1}^{m} k_l\, e_l\, e_l^T$

Structured feedback gain: $K = \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_m \end{bmatrix}$

Tree graphs

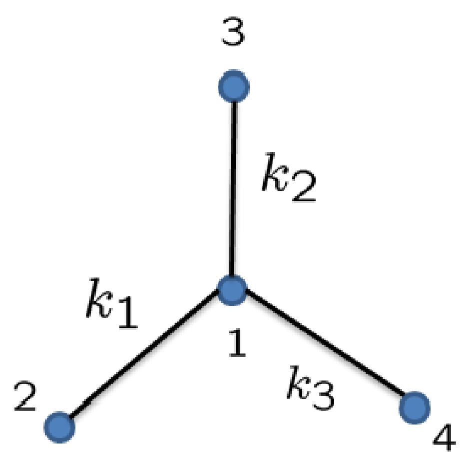
- Trees: connected graphs with no cycles

path 

star 

Incidence matrix of a tree $E_t \in \mathbb{R}^{N \times (N-1)}$

$$E_t = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$



$$E_t = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- Coordinate transformation

$$\begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} E_t^T \\ \frac{1}{N}\mathbb{1}^T \end{bmatrix}}_{T} x(t) \;\Leftrightarrow\; x(t) = \underbrace{\begin{bmatrix} E_t\,(E_t^T E_t)^{-1} & \mathbb{1} \end{bmatrix}}_{T^{-1}} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix}$$

In new coordinates

$$\begin{aligned}
\begin{bmatrix} \dot{\psi}(t) \\ \dot{\bar{x}}(t) \end{bmatrix} &= -\begin{bmatrix} E_t^T \\ \frac{1}{N}\mathbb{1}^T \end{bmatrix} E_t\,K\,E_t^T \begin{bmatrix} E_t\,(E_t^T E_t)^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix} + \begin{bmatrix} E_t^T \\ \frac{1}{N}\mathbb{1}^T \end{bmatrix} d(t) \\
&= \begin{bmatrix} -E_t^T E_t\,K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix} + \begin{bmatrix} E_t^T \\ \frac{1}{N}\mathbb{1}^T \end{bmatrix} d(t) \\
z(t) &= \begin{bmatrix} I - \frac{1}{N}\mathbb{1}\mathbb{1}^T \\ -E_t\,K\,E_t^T \end{bmatrix} \begin{bmatrix} E_t\,(E_t^T E_t)^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix}
\end{aligned}$$

Tree graphs: structured optimal H_2 design

$$
\begin{aligned}
\dot{\psi}(t) &= -\, E_t^T E_t\, K\, \psi(t) \;+\; E_t^T\, d(t) \\
z(t) &= \begin{bmatrix} E_t\,(E_t^T E_t)^{-1} \\ -\, E_t\, K \end{bmatrix} \psi(t)
\end{aligned}
$$

H_2 norm (from d to z)

$$
J(K) \;=\; \frac{1}{2}\,\mathrm{trace}\Big((E_t^T E_t)^{-1} K^{-1} \;+\; K\, E_t^T E_t\Big)
$$

$$
\text{Diagonal matrix:} \quad K \;=\; \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_{N-1} \end{bmatrix}
$$

- Structured optimal feedback gains

$$
k_i \;=\; \sqrt{\frac{\left(E_t^T E_t\right)_{ii}^{-1}}{2}}, \quad i \;=\; 1, \ldots, N-1
$$

- In Lecture 28, I made a blunder on board while deriving the optimal values of k_i

Here is correct derivation:

$\star$ $G := \left(E_t^T E_t\right)^{-1}$ $\Rightarrow$ diagonal elements of G determined by $G_{ii} = \left(E_t^T E_t\right)_{ii}^{-1}$

$\star$ All diagonal elements of $E_t^T E_t$ are equal to 2

$$E_t^T E_t \;=\; \begin{bmatrix} e_1 & \cdots & e_{N-1} \end{bmatrix}^T \begin{bmatrix} e_1 & \cdots & e_{N-1} \end{bmatrix} \;=\; \begin{bmatrix} e_1^T \\ \vdots \\ e_{N-1}^T \end{bmatrix} \begin{bmatrix} e_1 & \cdots & e_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} e_1^T e_1 & \cdots & e_1^T e_{N-1} \\ \vdots & \ddots & \vdots \\ e_{N-1}^T e_1 & \cdots & e_{N-1}^T e_{N-1} \end{bmatrix} = \begin{bmatrix} 2 & \cdots & e_1^T e_{N-1} \\ \vdots & \ddots & \vdots \\ e_{N-1}^T e_1 & \cdots & 2 \end{bmatrix}$$

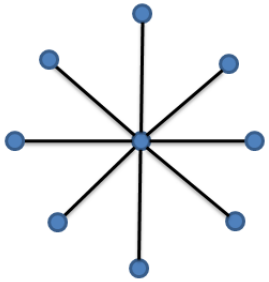
$\star$ K – diagonal matrix $\Rightarrow$ $J(K)$ can be written as

$$J(K) \;=\; \sum_{i\,=\,1}^{N-1} \left(\frac{G_{ii}}{2k_i} \;+\; k_i \right)$$

$\star$ $J(K)$ in a separable form $\Rightarrow$ element-wise minimization will do

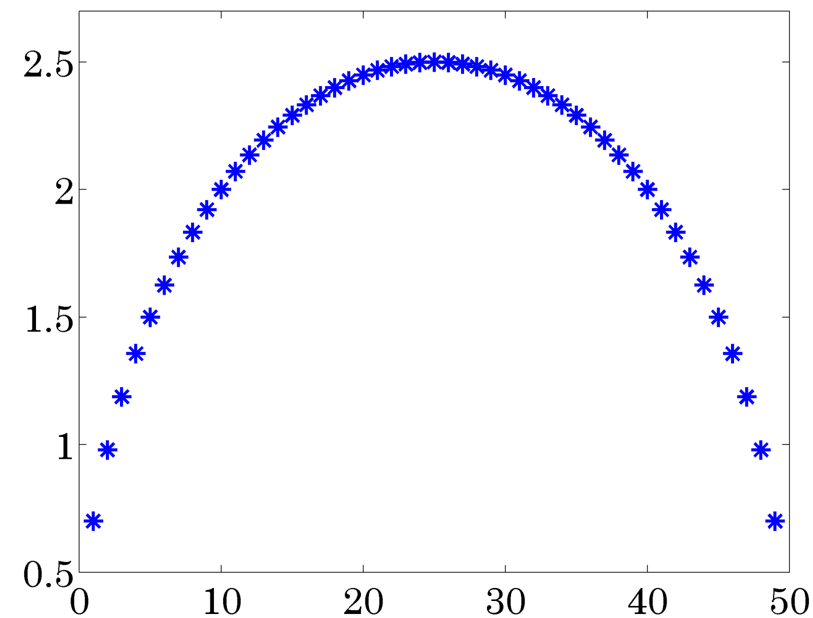
$$\frac{\mathrm{d}}{\mathrm{d}k_i} \left(\frac{G_{ii}}{2k_i} \;+\; k_i \right) \;=\; -\frac{G_{ii}}{2k_i^2} \;+\; 1 \;=\; 0 \;\Rightarrow\; k_i \;=\; \sqrt{\frac{G_{ii}}{2}}, \;\; i \;=\; 1, \ldots, N-1$$

Optimal gains for star and path

- Star:  uniform gain $k = \sqrt{\dfrac{N-1}{2N}} \approx \dfrac{1}{\sqrt{2}}$ for large N

- Path: 

$$k_i = \sqrt{\frac{i\,(N-i)}{2N}}$$



largest gains in the center

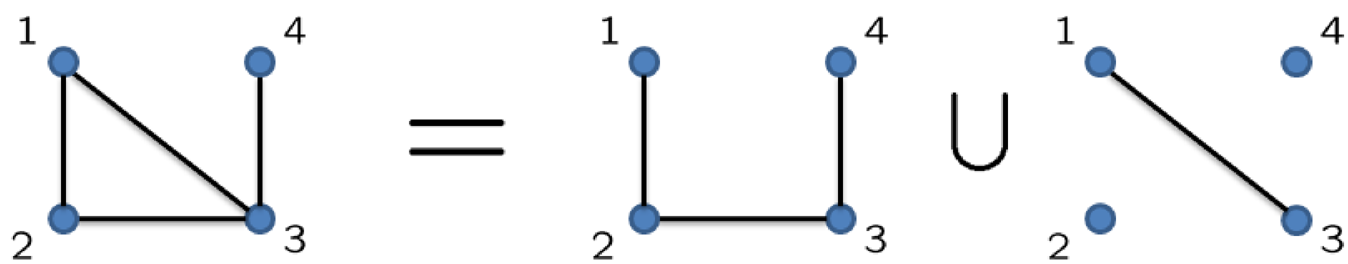
General undirected graphs

- Decompose graph into a tree subgraph and remaining edges

$$\text{Incidence matrix:} \quad E \;=\; \left[\; E_t \quad E_c \;\right]$$

$$\text{Projection matrix:} \quad \Pi \;=\; E_t\, E_t^{+} \;=\; E_t \left(E_t^T E_t\right)^{-1} E_t^T$$

$$E_c \in \text{range}\,(\Pi): \quad E_c \;=\; \Pi\, E_c$$



$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \cup \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} E \;&=\; \left[\; E_t \quad E_c \;\right] \;=\; \left[\; E_t \quad \Pi\, E_c \;\right] \\ &=\; E_t \left[\; I \quad (E_t^T E_t)^{-1} E_t^T\, E_c \;\right] \;=\; E_t\, M \end{aligned}$$

General graphs: structured optimal H_2 design

$$
\begin{aligned}
\dot{\psi}(t) &= -E_t^T E_t \, M \, K \, M^T \psi(t) + E_t^T d(t) \\
z(t) &= \begin{bmatrix} E_t \left(E_t^T E_t\right)^{-1} \\ -E_t \, M \, K \, M^T \end{bmatrix} \psi(t)
\end{aligned}
$$

tree graphs: $M = I$

H_2 norm (from d to z)

$$
J(K) = \frac{1}{2} \operatorname{trace} \Big(\left(E_t^T E_t\right)^{-1} \left(M \, K \, M^T\right)^{-1} + M \, K \, M^T E_t^T E_t \Big)
$$

- Main result:

  ⋆ Closed-loop stability $\Leftrightarrow$ $M \, K \, M^T > 0$

  $$\{W_1 > 0, W_2 = W_2^T\} \text{ then } -W_1 W_2 \text{ Hurwitz } \Leftrightarrow W_2 > 0$$

  ⋆ $M \, K \, M^T > 0 \Rightarrow$ convexity of $J(K)$

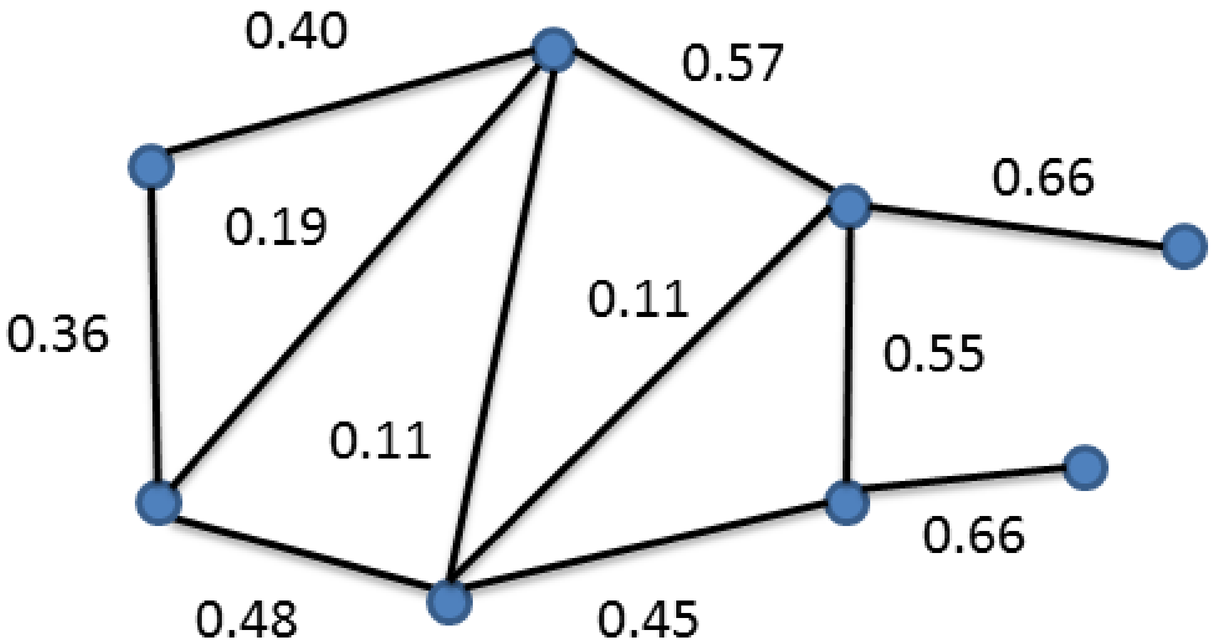
- Semi-definite program

$$\begin{array}{ll} \text{minimize} & \dfrac{1}{2}\,\text{trace}\left(X \,+\, M\,K\,M^T E_t^T\,E_t\right) \\ \text{subject to} & \left[\begin{array}{cc} X & (E_t^T E_t)^{-1/2} \\ (E_t^T E_t)^{-1/2} & M\,K\,M^T \end{array}\right] \;>\; 0 \\ & K \text{ diagonal} \end{array}$$

- Use CVX to solve it

```
cvx_begin sdp

    variable k(Ne) % vector of unknown feedback gains

    variable X(Nv-1,Nv-1) symmetric;
    X == semidefinite(Nv-1); % Schur complement variable

    Mk = M*diag(k)*M'; % Matrix Mk

    minimize(0.5*trace( q*X + r*Mk*W ))
    subject to [X, invWh; invWh, Mk] > 0;

cvx_end
```

Examples

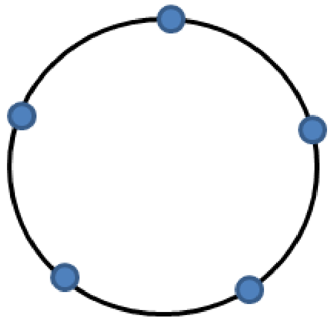


- Compare with performance of uniform gain design

J^*	$J(k=1)$	$(J - J^*)/J^*$
9.1050	13.1929	45%

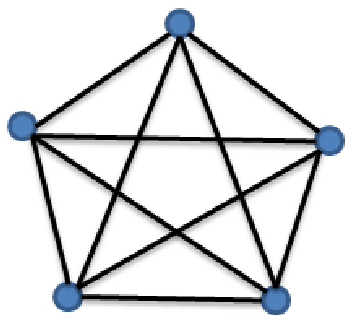
- Analytical results for circle and complete graphs

⋆ Circle



uniform gain $k = \sqrt{\dfrac{N^2 - 1}{24N}}$

⋆ Complete graph



uniform gain $k = \dfrac{2}{N}$

Additional material

- Papers to read
  - ⋆ *Xiao, Boyd, Kim, J. Parallel Distrib. Comput. '07*
  - ⋆ *Zelazo & Mesbahi, IEEE TAC '11*
  - ⋆ *Lin, Fardad, Jovanovic, CDC '10*