$$\begin{cases} \dot{\psi}(t) = U^* A U \psi(t) + U^* d \\ \dot{\bar{x}}(t) = 0 \cdot \bar{x}(t) + \frac{1}{N} \mathbb{1}^T \cdot d \end{cases}$$

$$\dot{\psi}(t) = \bar{A} \psi(t) + \bar{B} \cdot d$$

$$\mathrm{Re}(\lambda_i(\bar{A})) < 0, \quad i = 1, \ldots, N-1$$

$$z = (I - \tfrac{1}{N} \mathbb{1}\mathbb{1}^T)\, x(t) =$$

$$(I - \tfrac{1}{N} \mathbb{1}\mathbb{1}^T)\begin{bmatrix} U & \mathbb{1} \end{bmatrix}\begin{bmatrix} \psi \\ \bar{x} \end{bmatrix}$$

$$= \left[(U - \tfrac{1}{N} \mathbb{1}\underbrace{\mathbb{1}^T U}_{0}) \;\vdots\; (\mathbb{1} - \cancel{\tfrac{1}{N}} \mathbb{1} \underbrace{\mathbb{1}^T \mathbb{1}}_{\cancel{N}}) \right] \begin{bmatrix} \psi \\ \bar{x} \end{bmatrix}$$

$$= U \psi$$

Thus, if we use DFT on systems with Circulant matrices, we obtain

$$\begin{cases} \dot{\hat{\psi}}_k(t) = \hat{a}_k \hat{\psi}_k(t) + \hat{d}_k \\ \hat{z}_k(t) = \hat{\psi}_k(t) \end{cases}$$

Lyapunov equation:

$$\bar{A} P + P \bar{A}^* = -\bar{B}\bar{B}^*$$

$$\|H\|_2^2 = \mathrm{trace}(\bar{C}^* P \bar{C})$$

Note: For spatially-invariant systems:

$$\hat{a}_k \hat{P}_k + \hat{P}_k \hat{a}_k^* = -1 \;; \qquad \|H\|_2^2 = \sum_{k=1}^{n} \frac{-1}{\hat{a}_k + \hat{a}_k^*}$$

An example:

Nearest neighbour information exchange.

Q. What would happen if instead we had paid attention to

$$y_n(t) = x_n(t) - x_{n-1}(t)$$

$$\hat{y}_k(t) = \underbrace{\left(1 - e^{-j\frac{2\pi}{N}k}\right)}_{\hat{C}_k} \hat{x}_k(t)$$

$$\hat{C}_k^* \hat{C}_k = 2\left(1 - \cos\frac{2\pi}{N}k\right)$$

Then, $\|H\|_2^2 = \sum_{k=1}^{N-1} \hat{C}_k^* \hat{C}_k \hat{P}_k = \sum_{k=1}^{N-1} \frac{1}{2} = \frac{N-1}{2}$

In this case

So, the total variance amplification of the system, is increasing linearly with N.

But, if we pay attention to the deviation from average or the slot length, it scales badly with N. (refer to lecture slides)

Role of dimensionality

Features:

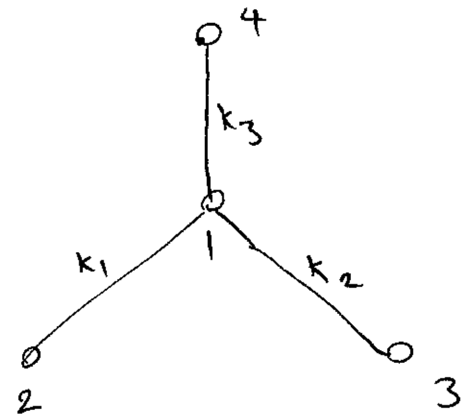
- spatial invariance
- locality $\Rightarrow$ fix the number of neighbours you are communicating with, then increase the total number of nodes.
- miror symmetry $\Rightarrow$ pay same attention to neighbours in all directions.

Incidence matrix:

$E \in \mathbb{R}^{N \times M}$

N ... # of nodes

M ... # of edges



$$E = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$E^T x = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \\ x_1 - x_4 \end{bmatrix} ; \quad E^T \mathbb{1} = 0$$

Laplacian: $\quad L(K) = E K E^T$

where K is structured feedback gain:

$$K = \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_M \end{bmatrix}$$