Finite platoons :

$$Q_p = V \Lambda V^*$$

$$Q_p \sim \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{M \times M}$$

$$\boxed{\lambda_n(Q_p) = 2\left(1 - \cos n\frac{\pi}{M+1}\right)}$$

Can show : solution to ARE

$$P = \begin{bmatrix} P_1 & P_0^* \\ P_0 & P_2 \end{bmatrix} ; \quad P_j = V \Lambda_j V^*$$
$$j = 1, 2, 0$$

$$P = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \Lambda_1 & \Lambda_0 \\ \Lambda_0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}$$

where Λ_j are diagonal matrices with elements determined by $\lambda_n(Q_p)$

$$\boxed{\begin{aligned} \hat{A} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ; \quad \hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hat{Q}_p &= \begin{bmatrix} \lambda_n(Q_p) & 0 \\ 0 & q_v \end{bmatrix} ; \quad \hat{R} = r \end{aligned}} \qquad (*)$$

ARE for the complete system :

$$A^* P + P A^* + Q - P B R^{-1} B^* P = 0$$

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix} ; \quad Q = \begin{bmatrix} Q_p & 0 \\ 0 & q_v I \end{bmatrix} ; \quad R = r I$$

Can write A as :

$$A = \begin{bmatrix} \nu & 0 \\ 0 & \nu \end{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \nu^* & 0 \\ 0 & \nu^* \end{bmatrix}$$

(Since $\nu\nu^* = I$).

"key"

$$\boxed{\text{Choose '}\nu\text{' that diagonalizes } Q. \text{ This brings the Large-size ARE into a set of AREs with size } 2\times 2.}$$

Now, <u>Compare (*)</u> with the following :

$$\hat{A}_\theta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ; \quad \hat{B}_\theta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\hat{Q}_\theta = \begin{bmatrix} 2(1-\cos\theta) & 0 \\ 0 & q_v \end{bmatrix} ; \quad \hat{R}_\theta = r$$

$\boxed{\text{For infinite problem}}$ $\downarrow$

$$J = \int_0^\infty \left(\langle P, Q_p P \rangle + q_v \langle v, v \rangle + r \langle u, u \rangle \right) dt$$

$$P(t) = \begin{bmatrix} \vdots \\ P_{n-1}(t) \\ P_n(t) \\ P_{n+1}(t) \\ \vdots \end{bmatrix} \in \ell_2$$

$$\langle P, Q_p P \rangle_{\ell_2} = \langle \hat{P}(\theta), \widehat{(Q_p P)}(\theta) \rangle_{L^2[0,2\pi]}$$

$$[Q_p P](n) = \sum_{k=-\infty}^{\infty} q_p(n-k) P(k), \quad \begin{cases} q_p(0) = 2 \\ q_p(\pm 1) = -1 \end{cases}$$

$$\widehat{[Q_p P]}(\theta) = q_p(\theta) P(\theta)$$

<u>Note that:</u>

$$q_p(\theta) = \left[q_p(0) z^0 + q_p(1) z^1 + q_p(-1) z^{-1} \right] \Big|_{z = e^{j\theta}}$$

$$= 2(1 - \cos\theta)$$

Then,

$$J = \int_0^\infty \int_0^{2\pi} \Big(\hat{p}^*(\theta,t) q_p(\theta) \hat{p}(\theta,t) + q_v \hat{v}^*(\theta,t) \hat{v}(\theta,t) + r \hat{u}^*(\theta,t) u(\theta,t) \Big) d\theta \, dt$$

<u>Control :</u> $\hat{u}(\theta,t) =$

$$- \hat{K}(\theta) \hat{\psi}(\theta,t)$$

$$= - \hat{R}^{-1}(\theta) \hat{B}^*(\theta) \hat{P}(\theta) \hat{\psi}(\theta,t)$$

where $\hat{K}(\theta) = \left[\hat{k}_p(\theta) \quad \hat{k}_v(\theta) \right]$

$$= \left[\frac{1}{r} \hat{P}_0(\theta) \quad \frac{1}{r} \hat{P}_2(\theta) \right]$$

In the physical space

$$u_n(t) = -\sum_{k=-\infty}^{\infty} k_p(n-k)\, P_k(t) - \sum_{k=-\infty}^{\infty} k_v(n-k)\, v_k(t)$$

Levine & Athans '66 introduced the variable e_n:

$$e_n = P_n - P_{n-1} \quad ; \quad n = 2, \dots, M$$

$$\dot{e}_n = \dot{P}_n - \dot{P}_{n-1} = v_n - v_{n-1}$$

$$\dot{v}_n = u_n$$

$$\begin{bmatrix} \dot{e} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

with $e = \begin{bmatrix} e_1 \\ \vdots \\ e_M \end{bmatrix} \in \mathbb{R}^{M+1}$

$$\begin{cases} \dot{e}_n = v_n - v_{n-1} \\ \dot{v}_n = u_n \end{cases}$$

$$\begin{cases} \dot{e}_\theta = (1 - e^{-j\theta})\, v_\theta \\ \dot{v}_\theta = u_\theta \end{cases}$$

$$\begin{bmatrix} \dot{e}_\theta \\ \dot{v}_\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 - e^{-j\theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_\theta \\ v_\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_\theta$$

$$Q_\theta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} ; \quad R_\theta = r$$

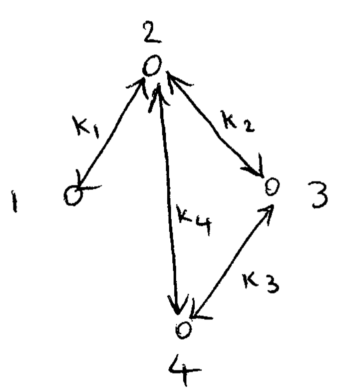
- At the limit of infinite vehicles, Controllability is Lost.

$$L e^{-j\theta} \xrightarrow[\theta \to 0]{} 0$$

This is a consequence of increase in the relative degree of the dynamics of the vehicle located far away from the Leader. In other words, the number of integrators between the vehicles and the Leader increases to infinity which results in Large (close to infinity) delay in the response of the vehicles located infinitely far away from the Leader.

Convergence or deviation from average

<u>Example</u>



$$\left.\begin{aligned}
\dot{x}_1(t) &= -k_1\left(x_1(t) - x_2(t)\right) \\
\dot{x}_2(t) &= -k_1\left(x_2(t) - x_1(t)\right) \\
&\quad - k_2\left(x_2(t) - x_3(t)\right) \\
&\quad - k_4\left(x_2(t) - x_4(t)\right) \\
\dot{x}_3(t) &= -k_2\left(x_3(t) - x_2(t)\right) \\
&\quad - k_3\left(x_3(t) - x_4(t)\right) \\
\dot{x}_4(t) &= -k_3\left(x_4(t) - x_3(t)\right) \\
&\quad - k_4\left(x_4(t) - x_2(t)\right)
\end{aligned}\right\}$$

Iterations that each node take to update their value.

We can show that

$$\dot{x}(t) = -EKE^T x(t) + d(t)$$

incidence matrix

$K = \text{diag}\{k_1, k_2, k_3, k_4\}$

Can check that $A = -EKE^T$

has rows and columns whose sum are 0.

$$A\mathbb{1} = 0.\mathbb{1}$$

$$\mathbb{1}^T A = 0.\mathbb{1}^T$$

example: Let $k_i = 1$; $i = 1, 2, 3, 4$

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

Question:

$$\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t) = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Can all nodes converge to $\bar{x}(t)$?

  Answer: Yes

- How quickly?
- What would be the effect of disturbances on convergence?