Lectures 25 & 26: Consensus and vehicular formation problems

- Consensus
  - ⋆ Make subsystems (agents, nodes) reach agreement
  - ⋆ Distributed decision making

- Vehicular formations
  - ⋆ How does performance scale with size?
  - ⋆ Are there any fundamental limitations?
  - ⋆ Is it enough to only look at neighbors?
  - ⋆ Should information be broadcast to all?

Collective behavior in nature

SNOW GEESE STRING FORMATION

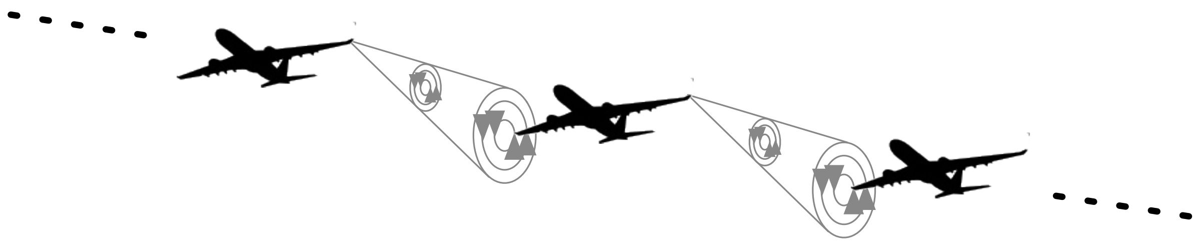
WILDEBEEST HERD MIGRATION

COLLECTIVE MOTION IN 3D

Coordinated control of formations

FORMATION FLIGHT FOR AERODYNAMIC ADVANTAGE

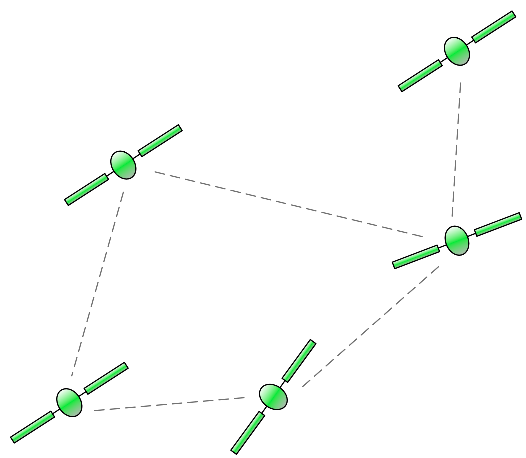
e.g. additional lift in V-formations



precise control needed

MICRO-SATELLITE FORMATIONS

e.g. for synthetic aperture



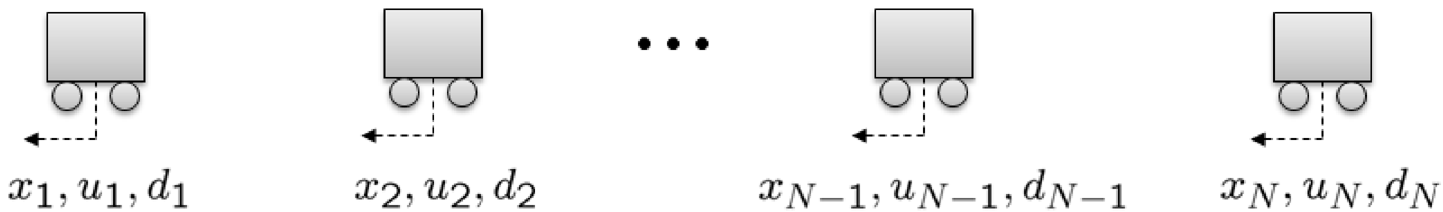
MAKE VEHICLES SMALLER AND CHEAPER $\Rightarrow$ USE MANY

cooperative control becomes a major issue

Vehicular strings

AUTOMATED CONTROL OF EACH VEHICLE

tight spacing at highway speeds



KEY ISSUES (**also in: control of swarms, flocks, formation flight**)

⋆ Is it enough to only look at neighbors?

⋆ How does performance scale with size?

⋆ Are there any fundamental limitations?

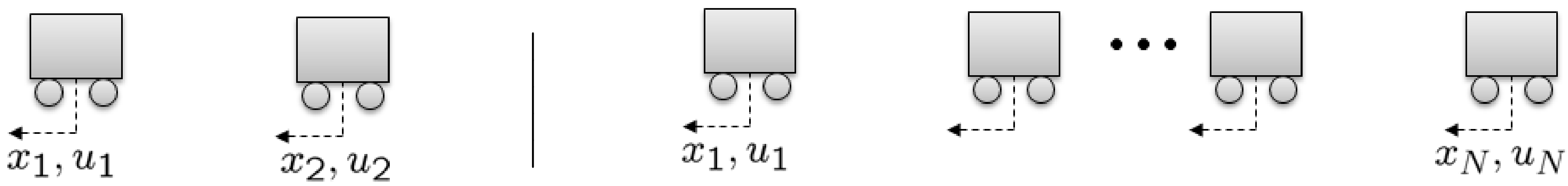
FUNDAMENTALLY DIFFICULT PROBLEM (**scales poorly**)

⋆ *Jovanović & Bamieh, IEEE TAC '05*

⋆ *Bamieh, Jovanović, Mitra, Patterson, IEEE TAC '11 (to appear)*

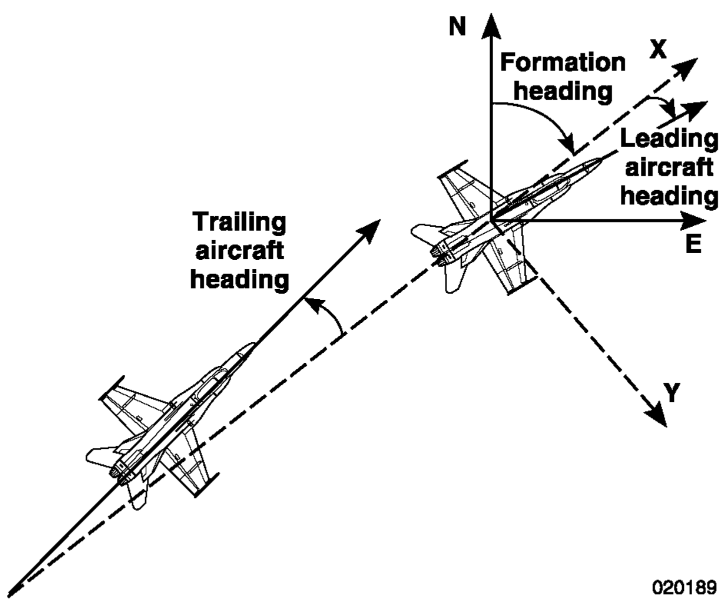
String instability

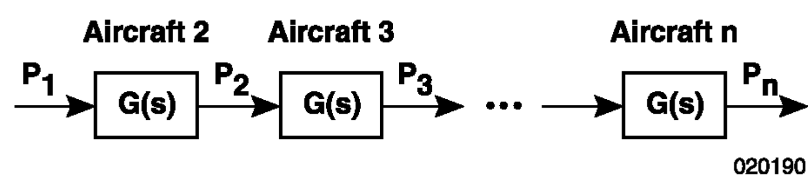
ONE APPROACH: design a follower cruise control $\Rightarrow$ chain into a formation



PROBLEM: STRING INSTABILITY

FLIGHT FORMATION EXAMPLE (Allen et al., 2002)

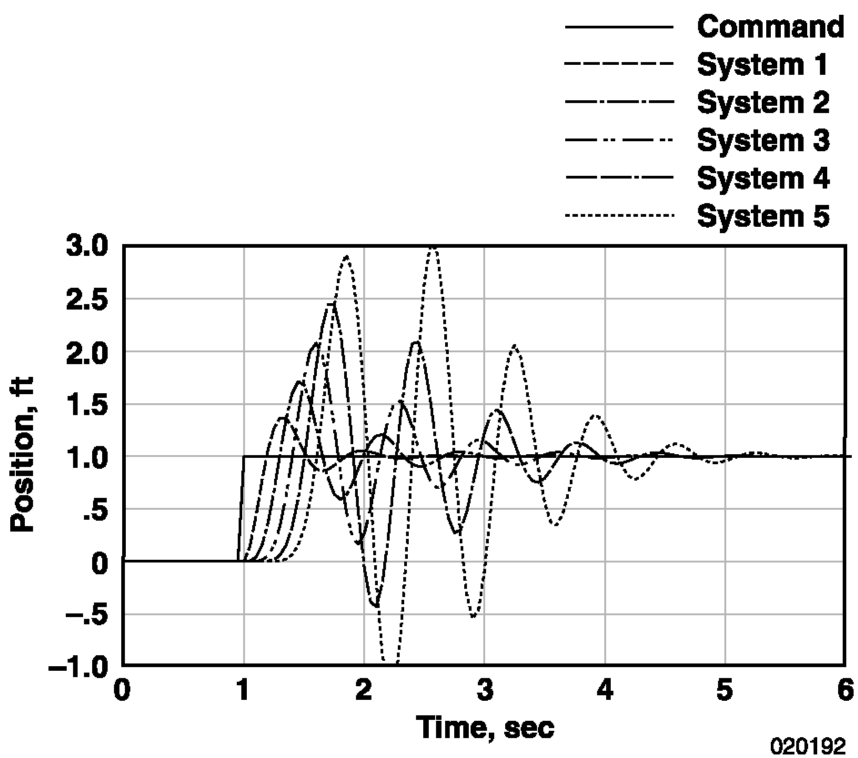




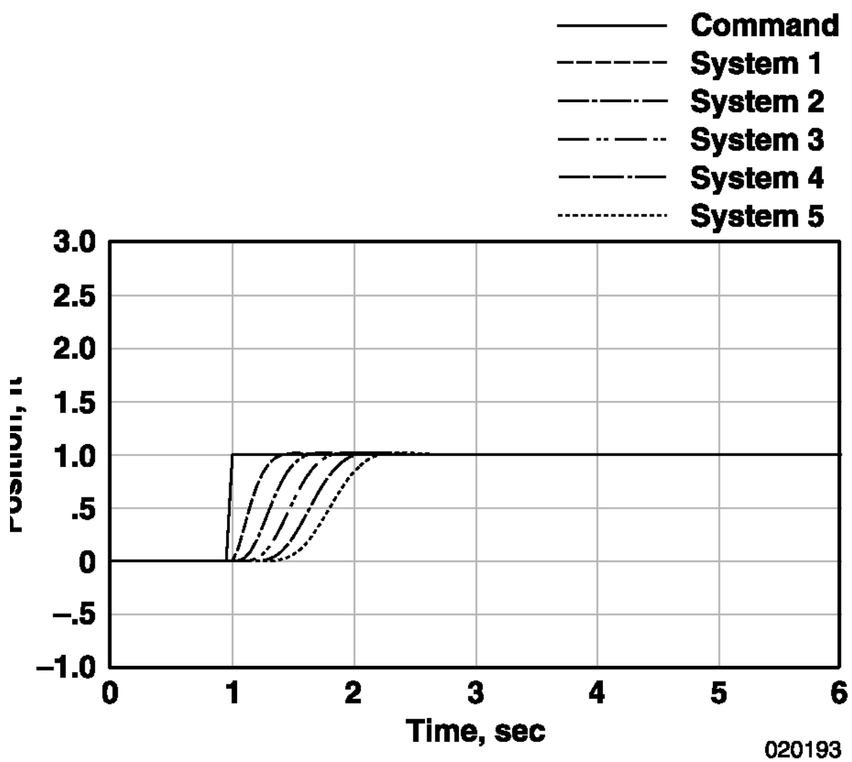
CHAINING OF A FOLLOWER CONTROLLER ⇒ STRING INSTABILITY

Allen et al., 2002

STRING INSTABILITY:



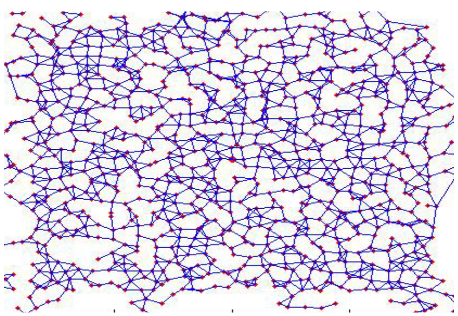
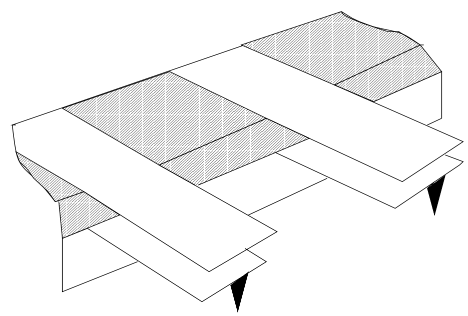
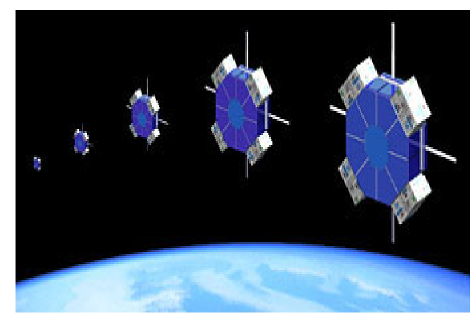
BETTER DESIGN:



Control of vehicular platoons

- ACTIVE RESEARCH AREA FOR $\approx$ 40 YEARS
  (Levine & Athans, Melzer & Kuo, Chu, Ioannou, Varaiya, Hedrick, Swaroop, . . .)

- SPATIO-TEMPORAL SYSTEMS
  signals depend on time & discrete spatial variable n

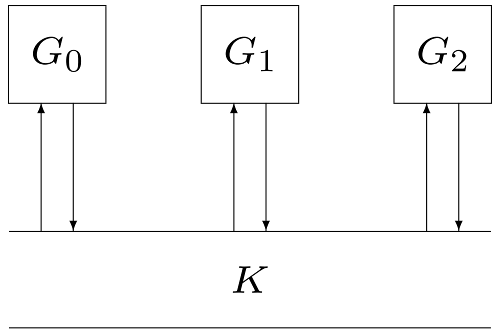
sensor networks	arrays of micro-cantilevers arrays of micro-mirrors	UAV formations satellite constellations
		

- INTERACTIONS CAUSE COMPLEX BEHAVIOR
  'string instability' in vehicular platoons

- SPECIAL STRUCTURE
  every unit has sensors and actuators

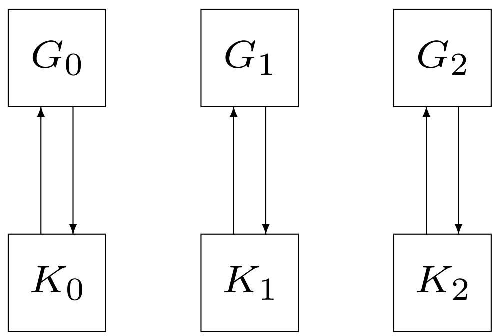
Controller architectures: platoons

CENTRALIZED:



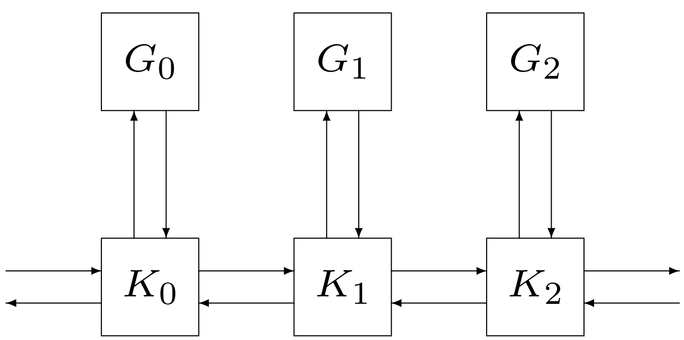
best performance
excessive communication

FULLY DECENTRALIZED:



not safe!

LOCALIZED:

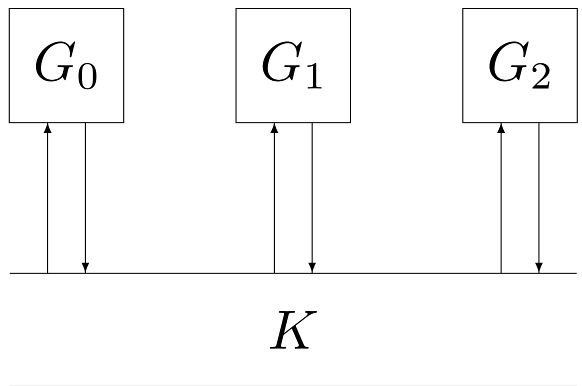


many possible architectures

- FUNDAMENTAL LIMITATIONS

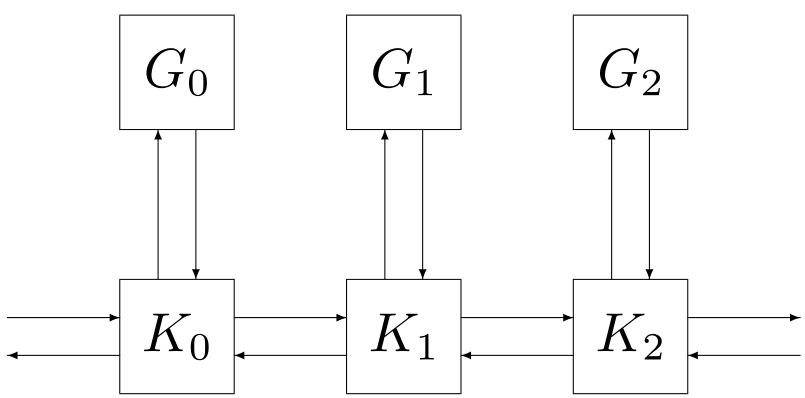
  ⋆ spatially invariant theory

CENTRALIZED:



performance vs. size

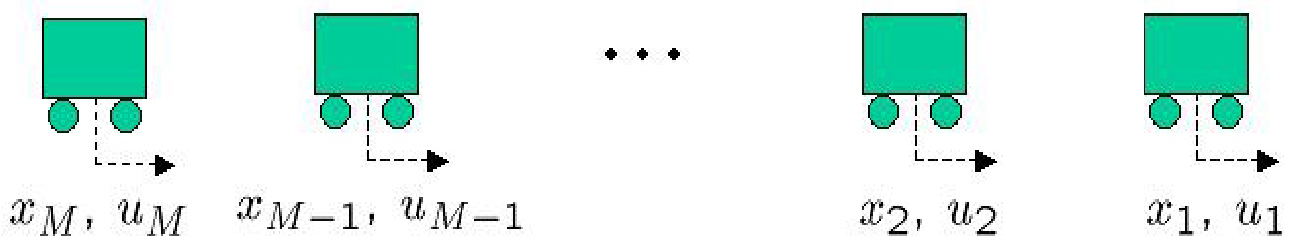
LOCALIZED:



is it enough to look only at nearest neighbors?

Optimal control of vehicular platoons

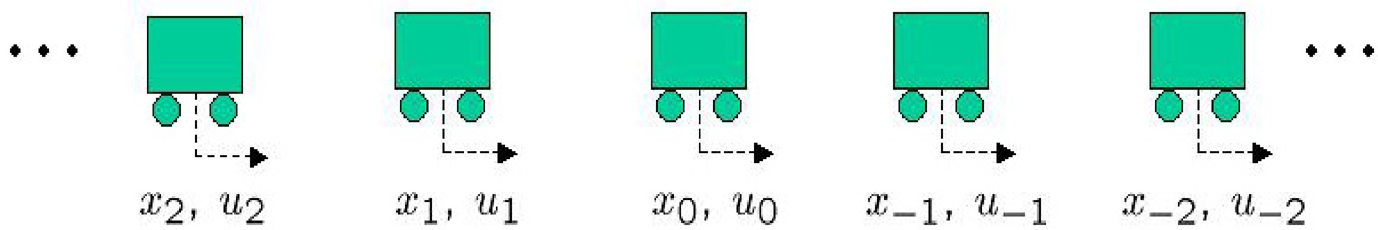
- FINITE PLATOONS



Levine & Athans, IEEE TAC '66

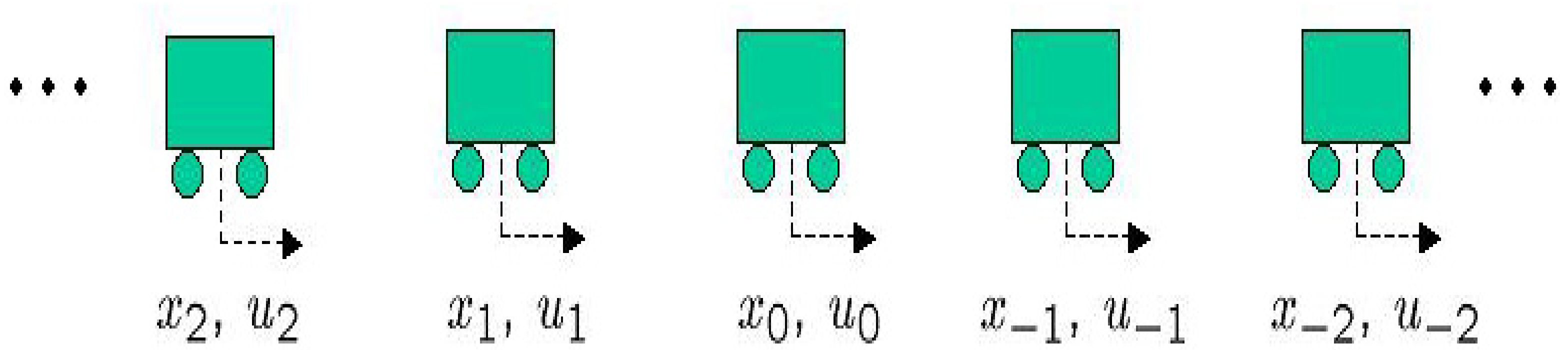
Melzer & Kuo, IEEE TAC '71

- INFINITE PLATOONS



Melzer & Kuo, Automatica '71

Control objective



DYNAMICS OF n-TH VEHICLE: $\ddot{x}_n = u_n$

CONTROL OBJECTIVE:

desired cruising velocity $v_d := \text{const.}$

inter-vehicular distance $L := \text{const.}$

☞ COUPLING ONLY THROUGH FEEDBACK CONTROLS

ABSOLUTE DESIRED TRAJECTORY

$$x_{nd}(t) := v_d t - nL$$

Optimal control of finite platoons

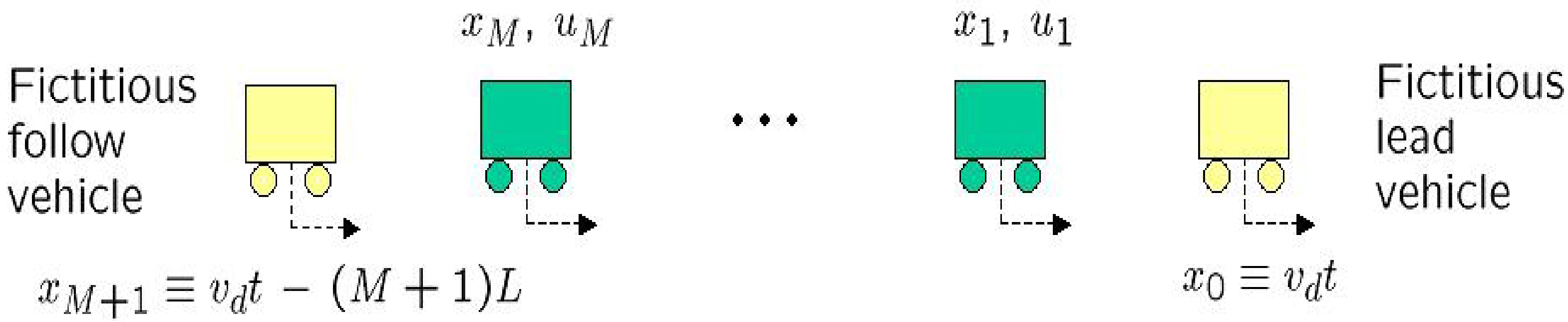
absolute position error: $p_n(t) \;:=\; x_n(t) \,-\, v_d t \,+\, nL$

absolute velocity error: $v_n(t) \;:=\; \dot{x}_n(t) \,-\, v_d$

$$\Downarrow$$

$$\begin{bmatrix} \dot{p}_n \\ \dot{v}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n, \quad n \in \{1, \ldots, M\}$$

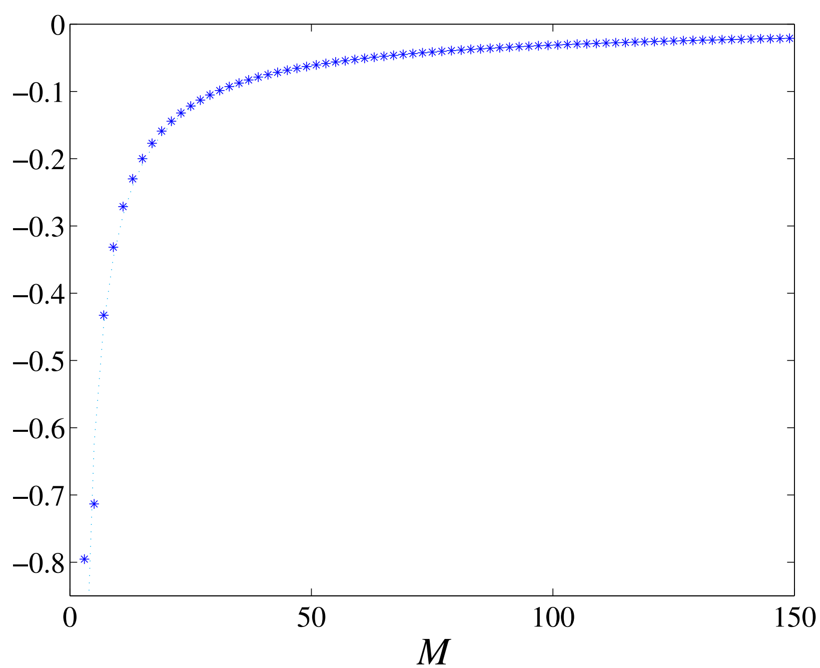
$$J \;:=\; \int_0^\infty \left(\sum_{n=1}^{M+1} (p_n(t) \,-\, p_{n-1}(t))^2 \,+\, \sum_{n=1}^{M} (v_n^2(t) \,+\, u_n^2(t)) \right) \mathrm{d}t$$



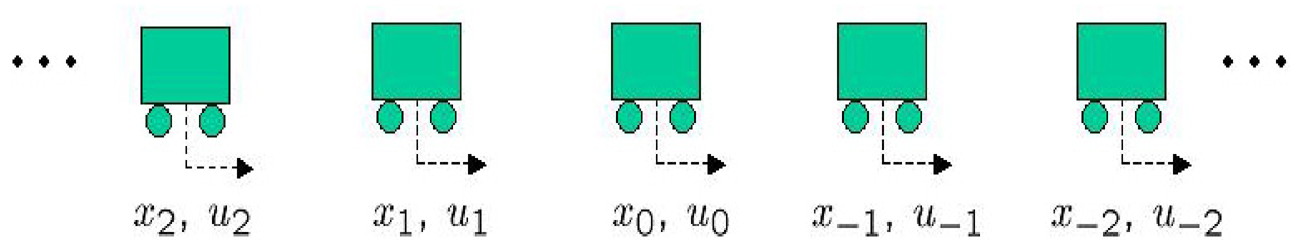
$$\begin{bmatrix} \dot{p}_n \\ \dot{v}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n, \quad n \in \{1, \ldots, M\}$$

$$J := \int_0^\infty \left(\sum_{n=1}^{M+1} (p_n(t) - p_{n-1}(t))^2 + \sum_{n=1}^{M} (v_n^2(t) + u_n^2(t)) \right) \mathrm{d}t$$

$\max Re\,(\lambda\{A_{cl}\})$:



Optimal control of infinite platoons



MAIN IDEA: EXPLOIT SPATIAL INVARIANCE

$$\begin{bmatrix} \dot{p}_n \\ \dot{v}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n, \quad n \in \mathbb{Z}$$

$$J := \int_0^\infty \sum_{n \in \mathbb{Z}} \left((p_n(t) - p_{n-1}(t))^2 + v_n^2(t) + u_n^2(t) \right) \mathrm{d}t$$

$$\Big\downarrow \text{ SPATIAL } \mathcal{Z}_\theta\text{-TRANSFORM}$$

$$A_\theta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q_\theta = \begin{bmatrix} 2(1 - \cos\theta) & 0 \\ 0 & 1 \end{bmatrix}, \quad 0 \le \theta < 2\pi$$

⋆ **pair (Q_θ, A_θ) not detectable at $\theta = 0$**

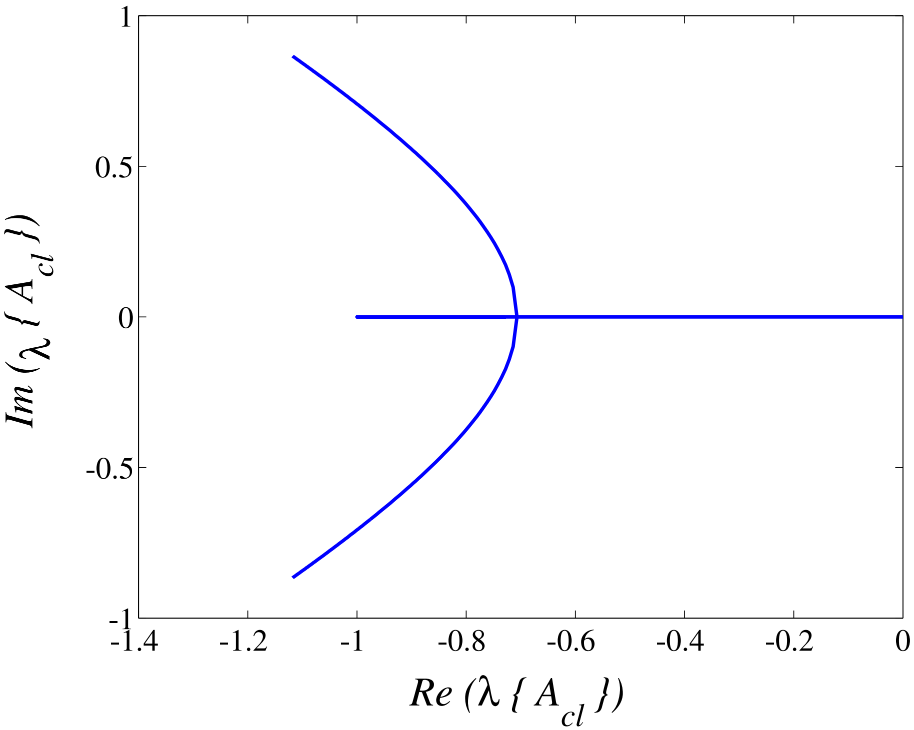
POSSIBLE FIX: PENALIZE ABSOLUTE POSITION ERRORS IN J

$$\begin{bmatrix} \dot{p}_n \\ \dot{v}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n, \quad n \in \mathbb{Z}$$

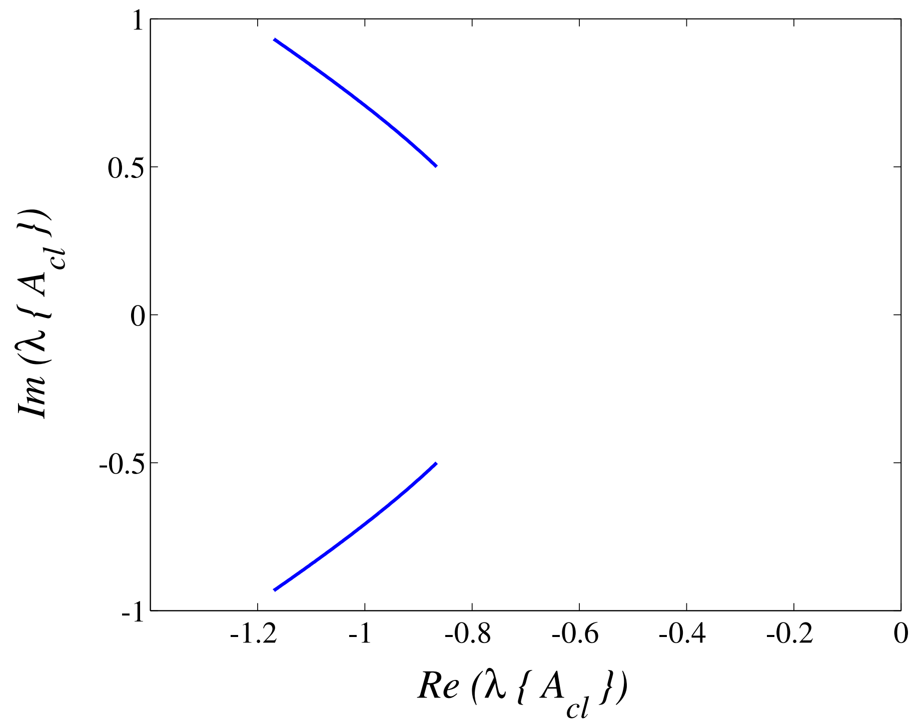
$$J := \int_0^\infty \sum_{n \in \mathbb{Z}} \left(q\, p_n^2(t) + (p_n(t) - p_{n-1}(t))^2 + v_n^2(t) + u_n^2(t) \right) \mathrm{d}t$$

CLOSED-LOOP SPECTRUM:

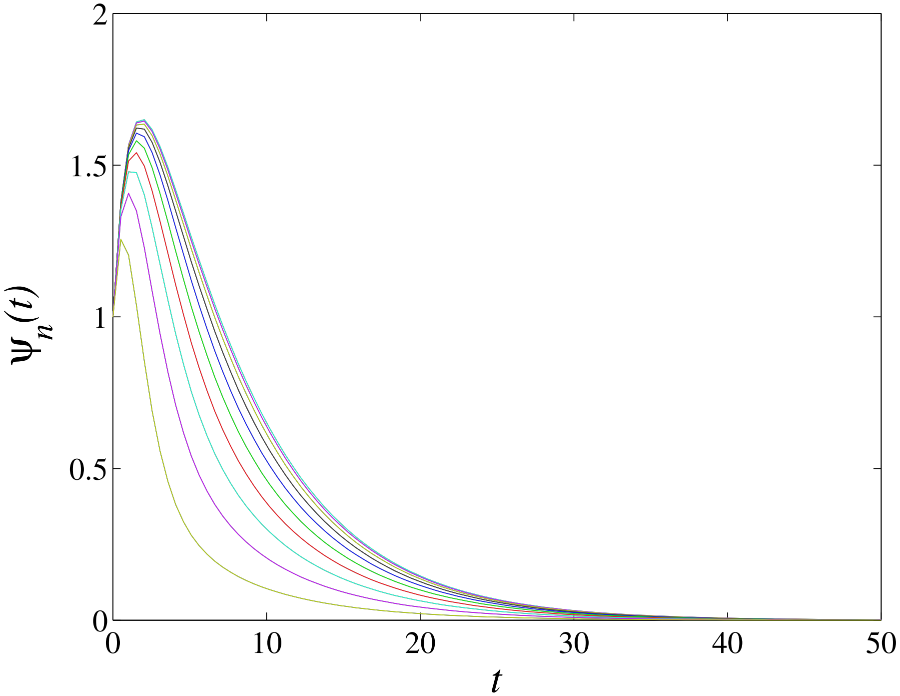
$q = 0$:



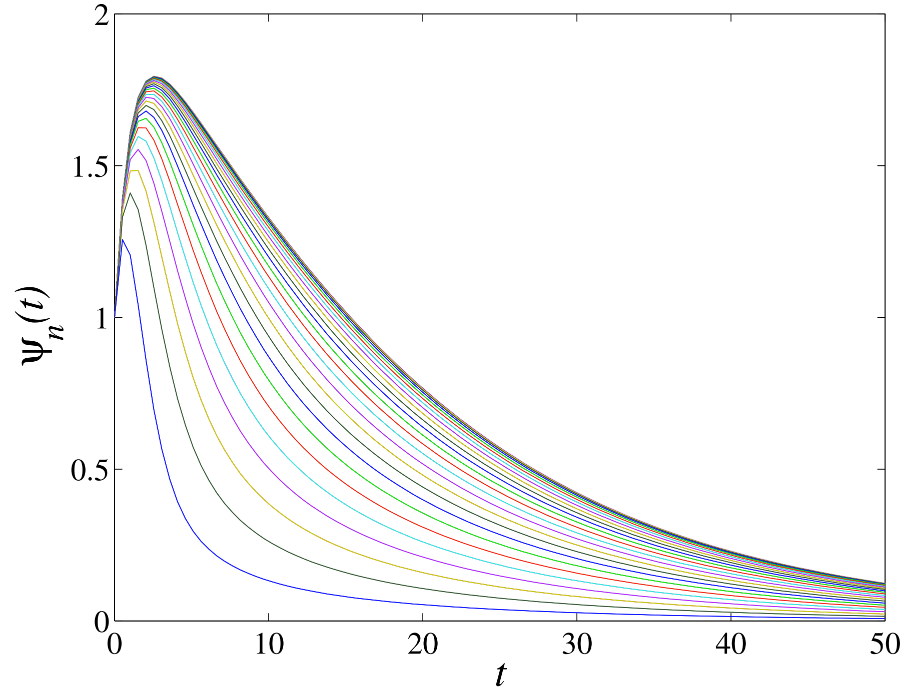
$q = 1$:



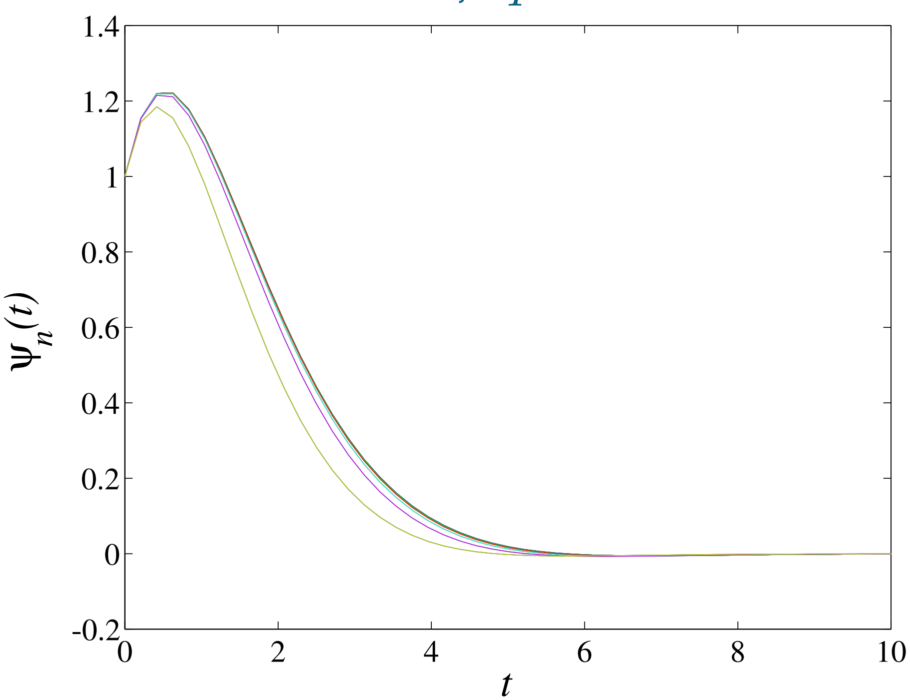
$M = 20,\ q = 0$:



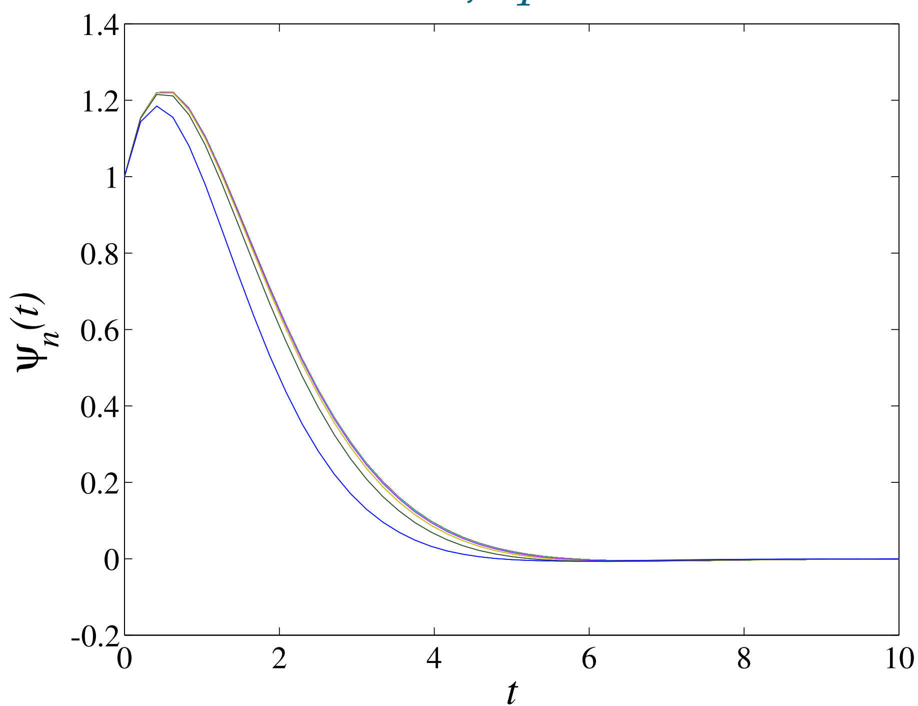
$M = 50,\ q = 0$:



$M = 20,\ q = 1$:



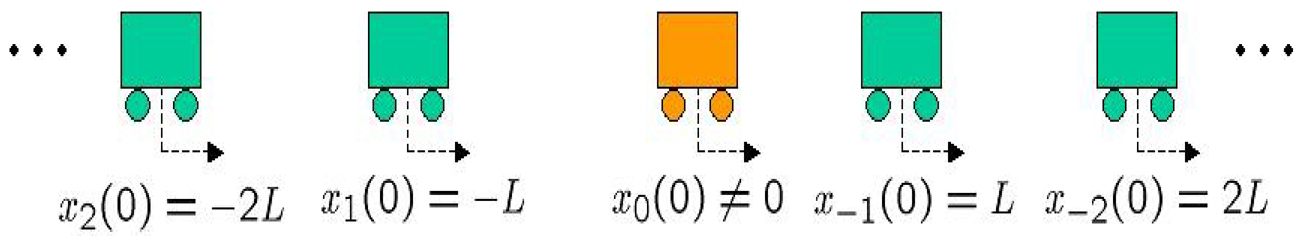
$M = 50,\ q = 1$:



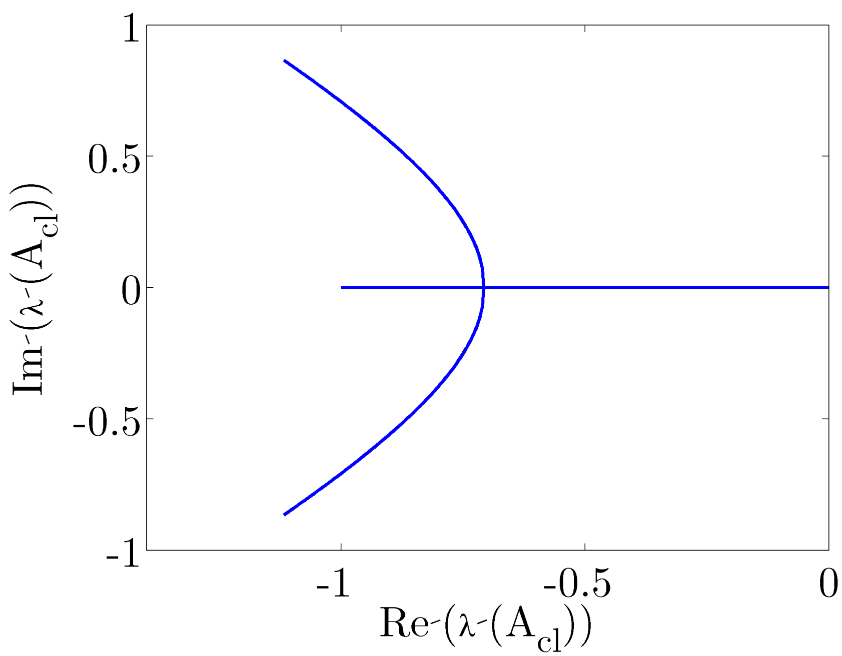
‘Problematic’ initial conditions

- INFINITE PLATOONS:
  non-zero mean initial conditions cannot be driven to zero

$$\sum_{n \in \mathbb{Z}} p_n(0) \neq 0 \;\Rightarrow\; \lim_{t \to \infty} \sum_{n \in \mathbb{Z}} p_n(t) \neq 0$$

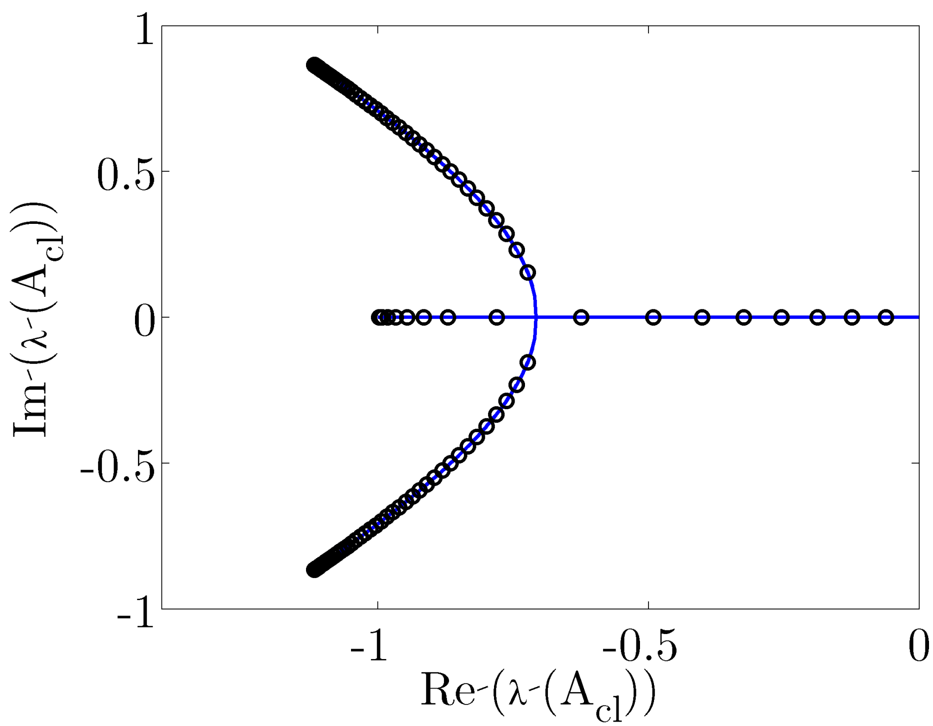


☞ **many modes have very slow rates of convergence**



$$
\begin{aligned}
\begin{bmatrix} \dot{p} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \\
J &= \int_0^\infty \left(p^T(t)\, Q_p\, p(t) \;+\; q_v\, v^T(t)\, v(t) \;+\; r\, u^T(t)\, u(t) \right) \mathrm{d}t \\
Q_p &= Q_p^T \;=\; V \Lambda V^* \;>\; 0, \;\; q_v \;\geq\; 0, \;\; r \;>\; 0
\end{aligned}
$$

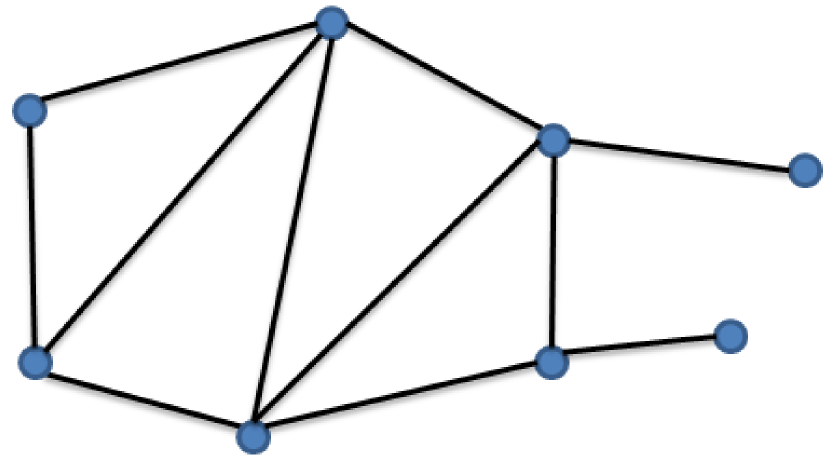
- spectrum of large-but-finite platoon dense in the spectrum of infinite platoon



- Key: entries into ARE jointly unitarily diagonalizable by V

Jovanović & Bamieh, IEEE TAC '05

Consensus by distributed computation



- Relative information exchange with neighbors
  - ⋆ Simple distributed averaging algorithm

$$\dot{x}_i(t) \;=\; -\sum_{j\,\in\,\mathcal{N}_i}\Big(x_i(t) \;-\; x_j(t)\Big)$$

- Questions
  - ⋆ Will the network asymptotically equilibrate?

$$\lim_{t\,\to\,\infty} x_n(t) \;\stackrel{?}{=}\; \bar{x}(t) \;:=\; \frac{1}{N}\sum_{n\,=\,1}^{N} x_n(t)$$

  - ⋆ Quantify performance (e.g., rate of convergence, response to disturbances)

Convergence to deviation from average

- Write dynamics as

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_N(t) \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix} + \begin{bmatrix} d_1(t) \\ \vdots \\ d_N(t) \end{bmatrix}$$

$$\dot{x}(t) = A\,x(t) + d(t)$$

- Let A be such that
  - ⋆ All rows and columns sum to zero

$$\begin{aligned} A\,\mathbb{1} &= 0 \cdot \mathbb{1} \\ \mathbb{1}^T A &= 0 \cdot \mathbb{1}^T \end{aligned}$$

  - ⋆ $\mathbb{1} := \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$ is an equilibrium point, $A\,\mathbb{1} = 0$
  - ⋆ All other eigenvalues of A have negative real parts

$$\bar{x}(t) \;:=\; \frac{1}{N}\left(x_1(t) \,+\, \cdots \,+\, x_N(t)\right) \;=\; \frac{1}{N}\mathbb{1}^T x(t)$$

- Deviation from average

scalar form: $$\tilde{x}_n(t) \;=\; x_n(t) \;-\; \bar{x}(t)$$

vector form: $$\begin{bmatrix} \tilde{x}_1(t) \\ \vdots \\ \tilde{x}_N(t) \end{bmatrix} \;=\; \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix} \;-\; \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \underbrace{\frac{1}{N}\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix}}_{\bar{x}(t)}$$

$$\tilde{x}(t) \;=\; \left(I \,-\, \frac{1}{N}\mathbb{1}\mathbb{1}^T\right) x(t)$$

$$\downarrow$$

$$x(t) \;=\; \underbrace{\tilde{x}(t)}_{\in\, \mathbb{1}^\perp} \;+\; \mathbb{1}\,\bar{x}(t)$$

$$\{u_1, \ldots, u_{N-1}\} \;\; - \text{ orthonormal basis of } \mathbb{1}^\perp$$

- Write $\tilde{x}(t)$ as

$$\tilde{x}(t) \;=\; \psi_1(t)\, u_1 \;+\; \cdots \;+\; \psi_{N-1}(t)\, u_{N-1} \;=\; \underbrace{\left[\begin{array}{ccc} u_1 & \cdots & u_{N-1} \end{array}\right]}_{U} \underbrace{\left[\begin{array}{c} \psi_1(t) \\ \vdots \\ \psi_{N-1}(t) \end{array}\right]}_{\psi(t)}$$

- Coordinate transformation

$$x(t) \;=\; \tilde{x}(t) \;+\; \mathbb{1}\, \bar{x}(t) \;=\; \left[\begin{array}{cc} U & \mathbb{1} \end{array}\right] \left[\begin{array}{c} \psi(t) \\ \bar{x}(t) \end{array}\right]$$

$$\Updownarrow$$

$$\left[\begin{array}{c} \psi(t) \\ \bar{x}(t) \end{array}\right] \;=\; \left[\begin{array}{c} U^* \\ \frac{1}{N}\, \mathbb{1}^T \end{array}\right] x(t)$$

$$\dot{x}(t) \; = \; A\, x(t) \; + \; d(t)$$

- In new coordinates

$$\left[\begin{array}{cc} U & \mathbb{1} \end{array}\right] \left[\begin{array}{c} \dot{\psi}(t) \\ \dot{\bar{x}}(t) \end{array}\right] \; = \; A \left[\begin{array}{cc} U & \mathbb{1} \end{array}\right] \left[\begin{array}{c} \psi(t) \\ \bar{x}(t) \end{array}\right] \; + \; d(t)$$

$$\begin{aligned} \left[\begin{array}{c} \dot{\psi}(t) \\ \dot{\bar{x}}(t) \end{array}\right] \; &= \; \left[\begin{array}{c} U^* \\ \frac{1}{N}\mathbb{1}^T \end{array}\right] A \left[\begin{array}{cc} U & \mathbb{1} \end{array}\right] \left[\begin{array}{c} \psi(t) \\ \bar{x}(t) \end{array}\right] \; + \; \left[\begin{array}{c} U^* \\ \frac{1}{N}\mathbb{1}^T \end{array}\right] d(t) \\ &= \; \left[\begin{array}{cc} U^* A\, U & U^* A\, \mathbb{1} \\ \frac{1}{N}\mathbb{1}^T A\, U & \frac{1}{N}\mathbb{1}^T A\, \mathbb{1} \end{array}\right] \left[\begin{array}{c} \psi(t) \\ \bar{x}(t) \end{array}\right] \; + \; \left[\begin{array}{c} U^* \\ \frac{1}{N}\mathbb{1}^T \end{array}\right] d(t) \end{aligned}$$

- Use structure of A to obtain

$$\begin{aligned} \dot{\psi}(t) \; &= \; U^* A\, U\, \psi(t) \; + \; U^* d(t) \\ \dot{\bar{x}}(t) \; &= \; 0 \cdot \bar{x}(t) \; + \; \frac{1}{N}\,\mathbb{1}^T d(t) \end{aligned}$$

Spatially invariant systems over circle

- Circulant A-matrix

$$\begin{aligned} \dot{x}(t) &= A\, x(t) \;+\; d(t) \\ z(t) &= \left(I \;-\; \frac{1}{N} \mathbb{1}\mathbb{1}^T \right) x(t) \end{aligned}$$

- Use DFT to obtain

$$\begin{aligned} \dot{\hat{x}}_k(t) &= \hat{a}_k\, \hat{x}_k(t) \;+\; \hat{d}_k(t) \\ \hat{z}_k(t) &= (1 \;-\; \delta_k)\, \hat{x}_k(t) \end{aligned}$$

- Variance of the network (i.e., the H_2 norm from d to z)
  - ⋆ solve Lyapunov equation and sum over spatial frequencies

$$\|H\|_2^2 \;=\; -\sum_{k\,=\,1}^{N-1} \frac{1}{(\hat{a}_k \;+\; \hat{a}_k^*)}$$

An example

- Nearest neighbor information exchange

$$\dot{x}_n(t) \;=\; -\,(x_n(t) \,-\, x_{n-1}(t)) \;-\; (x_n(t) \,-\, x_{n+1}(t)) \;+\; d_n(t), \quad n \in \mathbb{Z}_N$$

- Use DFT to obtain

$$\begin{aligned} \dot{\hat{x}}_k(t) &= -2\left(1 \,-\, \cos\left(\tfrac{2\,\pi\,k}{N}\right)\right)\hat{x}_k(t) \,+\, \hat{d}_k(t) \\ \hat{z}_k(t) &= (1 \,-\, \delta_k)\,\hat{x}_k(t) \end{aligned}$$

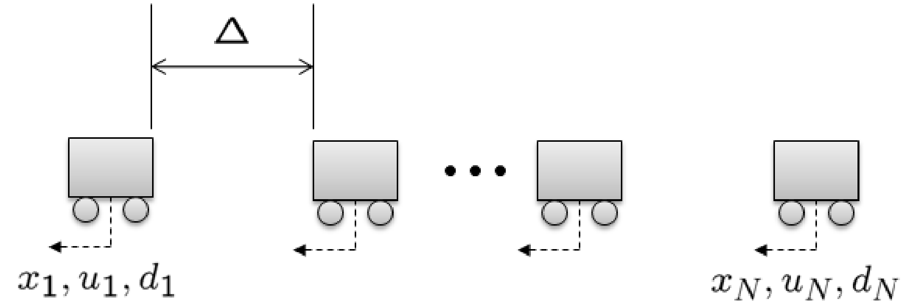
Variance per node

$$\frac{1}{N}\,\|H\|_2^2 \;=\; \frac{1}{N}\sum_{k\,=\,1}^{N-1}\frac{1}{4\left(1 \,-\, \cos\left(\frac{2\,\pi\,k}{N}\right)\right)} \;=\; \frac{1}{N}\sum_{k\,=\,1}^{N-1}\frac{1}{8\,\sin^2\left(\frac{\pi\,k}{N}\right)} \;=\; \frac{N^2 \,-\, 1}{24\,N}$$

- Will the scaling trends change if we
  - use information from more neighbors?
  - work in 2D or 3D?

Problem setup: double-integrator vehicles

$$\ddot{x}_n \;=\; \underset{\text{control}}{\underset{\uparrow}{u_k}} \;+\; \underset{\text{disturbance}}{\underset{\uparrow}{d_n}}$$



- Desired trajectory: $\begin{cases} \bar{x}_n := v_d\, t + n\,\Delta \\ \text{constant velocity} \end{cases}$

- Deviations:

$$p_n := x_n - \bar{x}_n, \qquad v_n := \dot{x}_n - v_d$$

- Controls:

$$u = -K_p\, p - K_v\, v$$

- Closed loop:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K_p & -K_v \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} d(t)$$

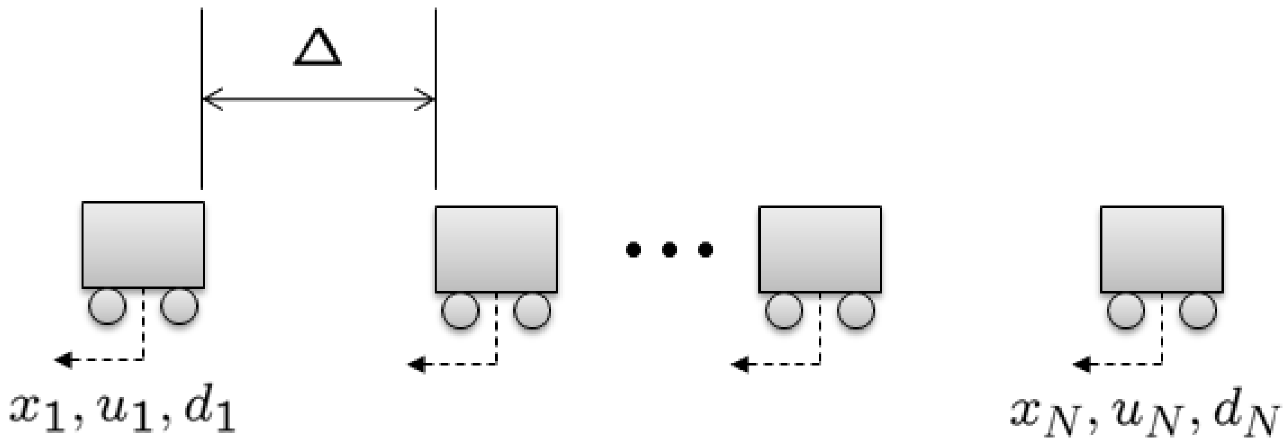
$K_p,\ K_v$: feedback gains

Structured feedback design

Example: design K_p and K_v to use nearest neighbor feedback

e.g. use a simple rule like:

$$\begin{aligned} u_n \;=\; & -K_p^+ \left(x_{n+1} - x_n - \Delta \right) \;-\; K_p^- \left(x_n - x_{n-1} - \Delta \right) \\ & -K_v^+ \left(v_{n+1} - v_n \right) \;-\; K_v^- \left(v_n - v_{n-1} \right) \end{aligned}$$

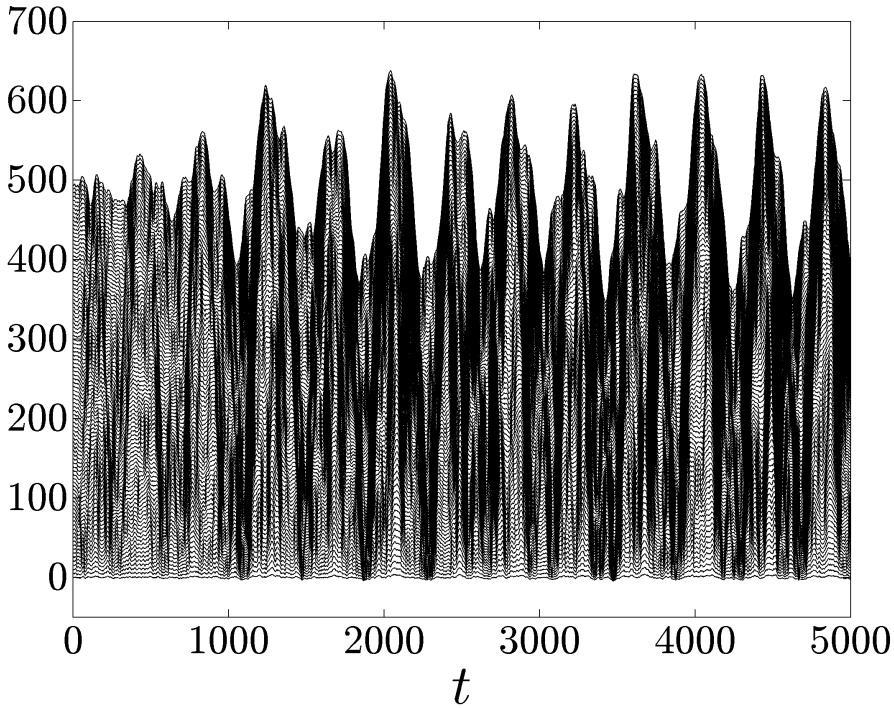


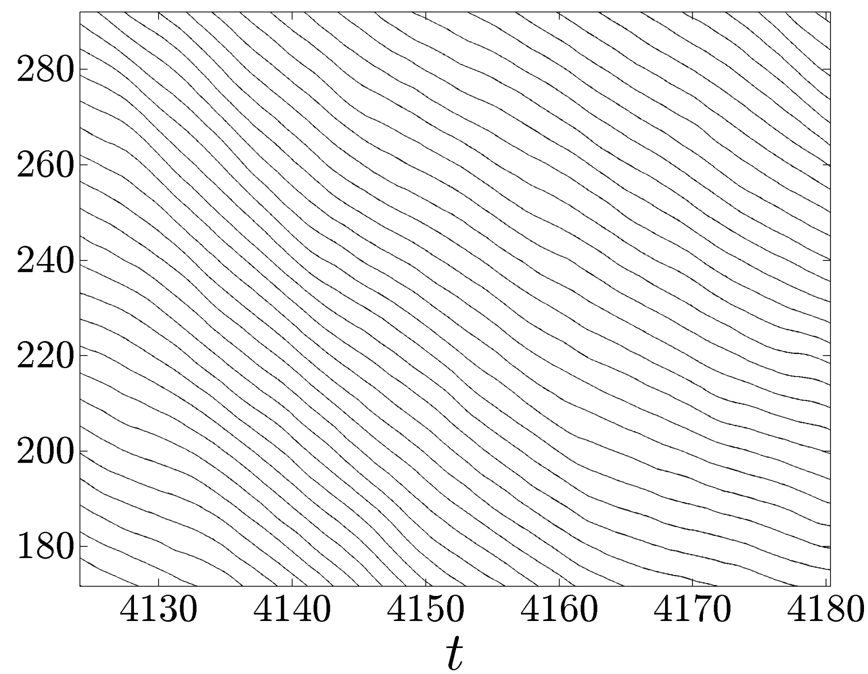
select K_p and K_v to guarantee global stability

Incoherence phenomenon

LOCAL FEEDBACK: GLOBAL STABILITY

$N = 100$ VEHICLES



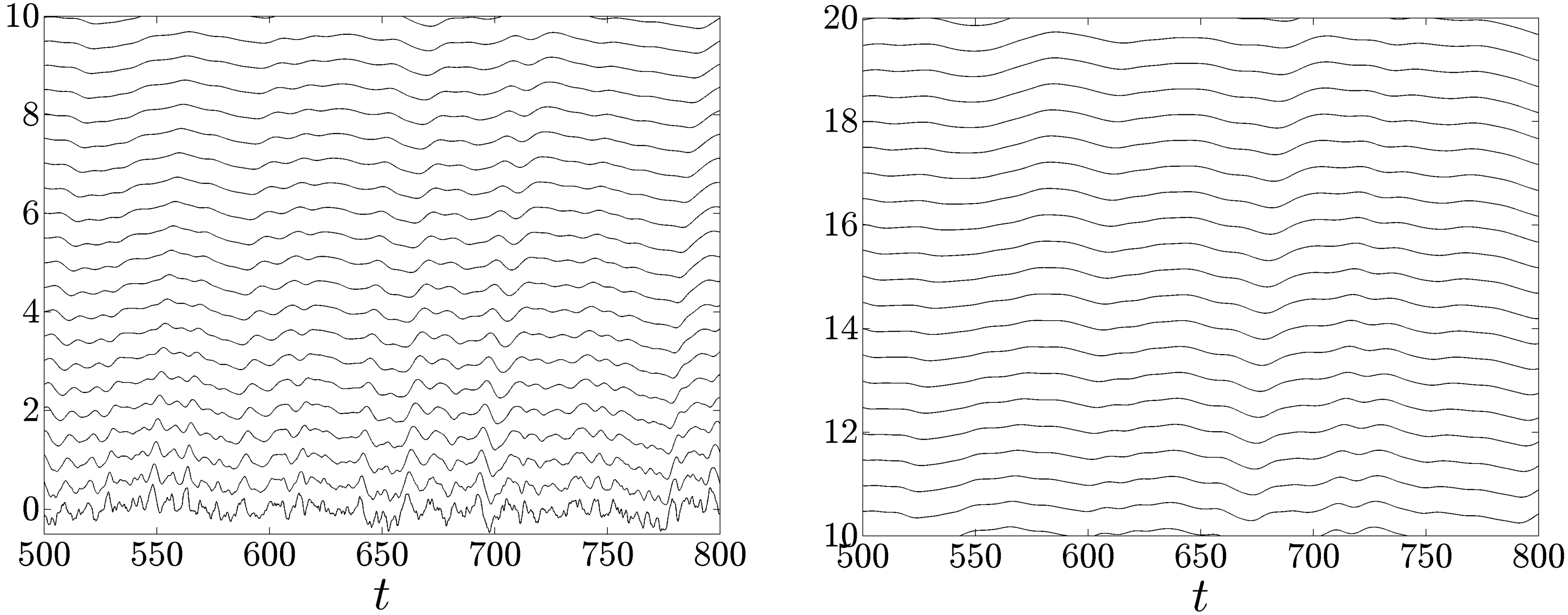


poor macroscopic performance: not string instability!

- ⋆ high frequency disturbance quickly regulated
- ⋆ low frequency disturbance penetrates further into formation

random disturbance acting on lead vehicle

$N = 100$ VEHICLES



Bamieh, Jovanović, Mitra, Patterson, IEEE TAC '11 (to appear)

Role of dimensionality

$M = N^d$ vehicles arranged in d-dimensional torus $\mathbb{Z}_N^d$

$$\ddot{x}_{(n_1,\ldots,n_d)} \;=\; u_{(n_1,\ldots,n_d)} \;+\; w_{(n_1,\ldots,n_d)}, \qquad n_i \in \mathbb{Z}_n$$

desired trajectory: $\bar{x}_k \;:=\; vt \;+\; k\Delta$

- STRUCTURAL FEATURES:
  - ⋆ spatial invariance
  - ⋆ locality
  - ⋆ mirror symmetry

- RELATIVE VS. ABSOLUTE MEASUREMENTS

$$\begin{aligned} u_n \;=\; & -K_p^{+}\,(x_{n+1} - x_n - \Delta) \;-\; K_p^{-}\,(x_n - x_{n-1} - \Delta) \;- \\ & K_v^{+}\,(v_{n+1} - v_n) \;-\; K_v^{-}\,(v_n - v_{n-1}) \;- \\ & K_p^{0}\,(x_n \;-\; (v_d t + n\Delta)) \;-\; K_v^{0}\,(v_n \;-\; v_d) \end{aligned}$$

Performance measures

- Microscopic: local position deviation $(x_{n+1} - x_n - \Delta)$
- Macroscopic: deviation from average or long range deviation

How does variance per vehicle scale with system size?

- relative position & absolute velocity feedback:

MICROSCOPIC ERROR:

bounded for any dimension d

ASYMPTOTIC SCALING OF MACROSCOPIC ERROR:

$d = 1$	M
$d = 2$	$\log M$
$d \geq 3$	bounded!

⋆ Same scaling obtained in standard consensus problem

- relative position & relative velocity feedback:

ASYMPTOTIC SCALING OF MICROSCOPIC ERROR:

$d = 1$	M
$d = 2$	$\log M$
$d = 3$	bounded

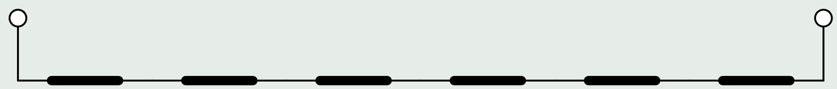
ASYMPTOTIC SCALING OF MACROSCOPIC ERROR:

$d = 1$	M^3
$d = 2$	M
$d = 3$	$M^{1/3}$

Only local feedback: **large 'tight formations' in 1D not possible!**

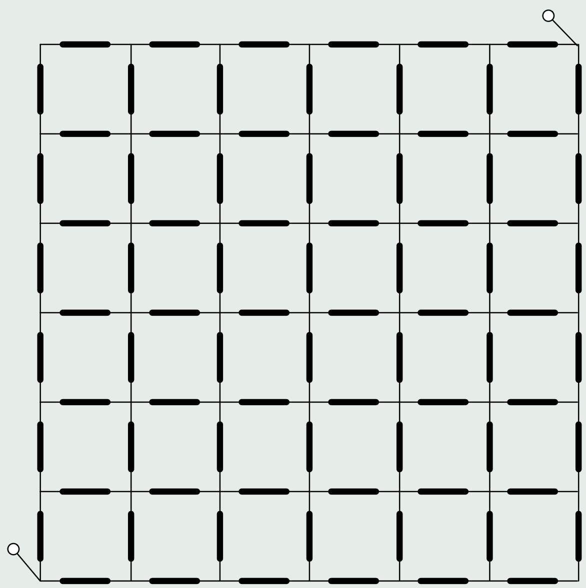
Resistive network analogy

1D :



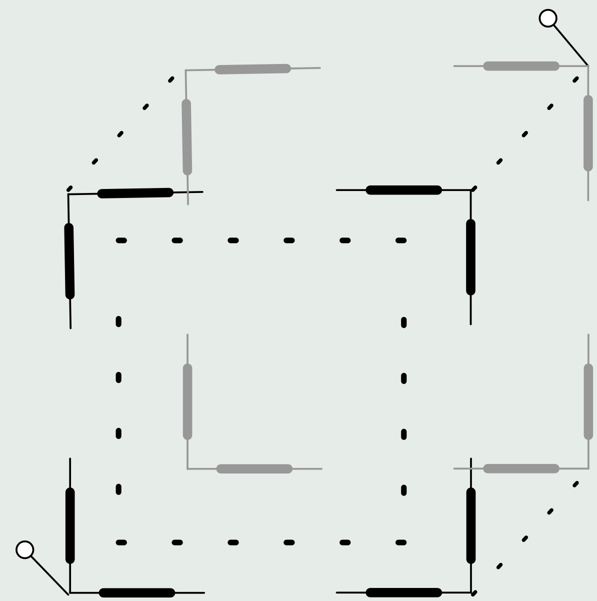
Net resistance $= R\,M$

2D :



Net resistance $= O\left(\log(M)\right)$

3D :



Net resistance is *bounded*!