Lecture 24: LQR for spatially invariant systems

- Structure of optimal distributed controllers
  - ⋆ Also spatially invariant
  - ⋆ Feedback gains decay exponentially with spatial distance
  - ⋆ Obtained from solving parameterized family of AREs

- Examples
  - ⋆ Systems on lattices
  - ⋆ PDEs
  - ⋆ Vehicular formations

Spatially invariant systems

$$\psi_t(x,t) \;=\; [\,\mathcal{A}\,\psi(\cdot,t)\,]\,(x) \;+\; [\,\mathcal{B}\,u(\cdot,t)\,]\,(x)$$

spatial coordinate: $x \in \mathbb{G}$

translation invariant operators: $\mathcal{A}$, $\mathcal{B}$

Spatial Fourier transform

$$\dot{\hat{\psi}}(\kappa,t) \;=\; \hat{\mathcal{A}}(\kappa)\,\hat{\psi}(\kappa,t) \;+\; \hat{\mathcal{B}}(\kappa)\,\hat{u}(\kappa,t)$$

spatial frequency: $\kappa \in \hat{\mathbb{G}}$

multiplication operators: $\hat{\mathcal{A}}(\kappa)$, $\hat{\mathcal{B}}(\kappa)$

$\mathbb{G}$	$\mathbb{R}$	$\mathbb{S}$	$\mathbb{Z}$	$\mathbb{Z}_N$
$\hat{\mathbb{G}}$	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{S}$	$\mathbb{Z}_N$

- $\mathbb{R}$ reals
- $\mathbb{Z}$ integers
- $\mathbb{S}$ unit circle
- $\mathbb{Z}_N$ integers modulo N

LQR for spatially invariant systems over $\mathbb{Z}_N$

$$\text{minimize} \quad J \;=\; \int_0^\infty \Big(\psi^*(t)\, Q\, \psi(t) \;+\; u^*(t)\, R\, u(t) \Big)\, \mathrm{d}t$$

$$\text{subject to} \quad \dot{\psi}(t) \;=\; A\, \psi(t) \;+\; B\, u(t)$$

- Circulant matrices: A, B, Q, R

  ⋆ Jointly unitarily diagonalizable by DFT Matrix V

$$\begin{aligned} \dot{\hat{\psi}}(t) &= A_d\, \hat{\psi}(t) \;+\; B_d\, \hat{u}(t) \\ A_d &= \mathrm{diag}\left(\hat{A}(\kappa) \right) \;=\; V\, A\, V^* \\ \psi^*\, Q\, \psi &= \hat{\psi}^*\, Q_d\, \hat{\psi} \end{aligned}$$

  ⋆ Entries into ARE – diagonal matrices

$$A_d^*\, P_d \;+\; P_d\, A_d \;+\; Q_d \;-\; P_d\, B_d\, R_d^{-1} B_d^* P_d \;=\; 0$$

$$\Updownarrow$$

$$\hat{A}^*(\kappa)\, \hat{P}(\kappa) \;+\; \hat{P}(\kappa)\, \hat{A}(\kappa) \;+\; \hat{Q}(\kappa) \;-\; \hat{P}(\kappa)\, \hat{B}(\kappa)\, \hat{R}^{-1}(\kappa)\, \hat{B}^*(\kappa)\, \hat{P}(\kappa) \;=\; 0, \quad \kappa \in \mathbb{Z}_N$$

Example: mass-spring system on a circle

$$\dot{\psi}(x,t) \;=\; \begin{bmatrix} 0 & 1 \\ S_{-1} \,-\, 2 \,+\, S_1 & 0 \end{bmatrix} \psi(x,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(x,t), \quad x \,\in\, \mathbb{Z}_N$$

$\downarrow$ **discrete Fourier transform**

block-diagonal family of 2nd order systems:

$$\dot{\hat{\psi}}(\kappa,t) \;=\; \begin{bmatrix} 0 & 1 \\ \hat{a}_{21}(\kappa) & 0 \end{bmatrix} \hat{\psi}(\kappa,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(\kappa,t), \quad \kappa \,\in\, \mathbb{Z}_N$$

$$\hat{a}_{21}(\kappa) \;=\; -\,2\left(1 \,-\, \cos\tfrac{2\,\pi\,\kappa}{N}\right)$$

- State and control weights

$$\left\{ Q \;=\; \left[\begin{array}{cc} Q_p & 0 \\ 0 & Q_v \end{array} \right];\; R \right\} \;\Rightarrow\; \left\{ \hat{Q}(\kappa) \;=\; \left[\begin{array}{cc} \hat{q}_p(\kappa) & 0 \\ 0 & \hat{q}_v(\kappa) \end{array} \right];\; \hat{R}(\kappa) \;=\; \hat{r}(\kappa) \right\}$$

- Solution to ARE

$$\hat{P}(\kappa) \;=\; \left[\begin{array}{cc} \hat{p}_1(\kappa) & \hat{p}_0^*(\kappa) \\ \hat{p}_0(\kappa) & \hat{p}_2(\kappa) \end{array} \right] \;\Rightarrow\; \left\{ \begin{array}{rcl} \hat{a}_{21}\,(\hat{p}_0 \,+\, \hat{p}_0^*) \,+\, \hat{q}_p \,-\, \dfrac{\hat{p}_0\,\hat{p}_0^*}{\hat{r}} & = & 0 \\ \hat{p}_0 \,+\, \hat{p}_0^* \,+\, \hat{q}_v \,-\, \dfrac{\hat{p}_2^2}{\hat{r}} & = & 0 \\ \hat{a}_{21}\,\hat{p}_2 \,+\, \hat{p}_1 \,-\, \dfrac{\hat{p}_2\,\hat{p}_0^*}{\hat{r}} & = & 0 \\ \hat{a}_{21}\,\hat{p}_2 \,+\, \hat{p}_1 \,-\, \dfrac{\hat{p}_2\,\hat{p}_0}{\hat{r}} & = & 0 \end{array} \right.$$

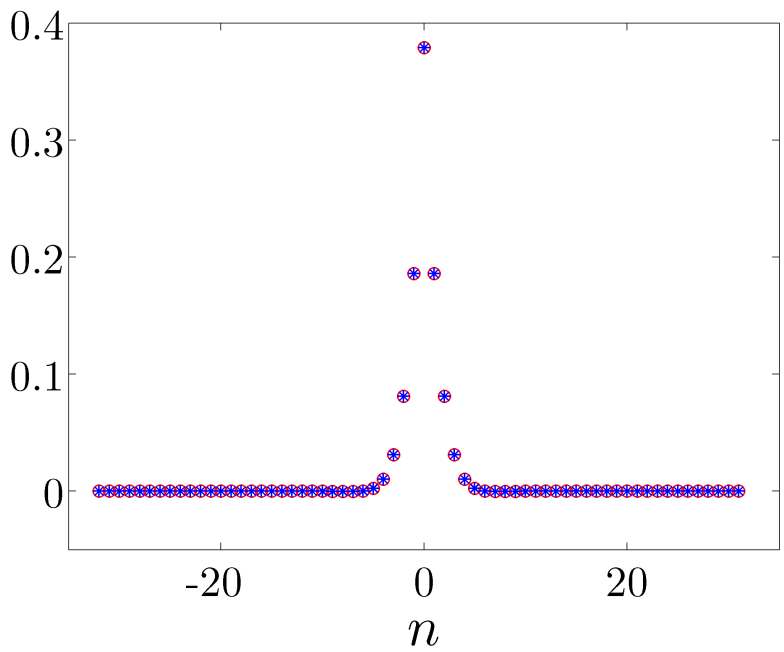
A bit of algebra yields

$$\begin{array}{rcl} \hat{p}_0(\kappa) & = & \hat{r}(\kappa) \left(\hat{a}_{21}(\kappa) \,+\, \sqrt{\hat{a}_{21}^2(\kappa) \,+\, \hat{q}_p(\kappa)/\hat{r}(\kappa)} \right) \\ \hat{p}_2(\kappa) & = & \sqrt{\hat{r}(\kappa)\,(\hat{q}_v(\kappa) \,+\, 2\,\hat{p}_0(\kappa))} \\ \hat{p}_1(\kappa) & = & \hat{p}_2(\kappa)\,(\hat{p}_0(\kappa)/\hat{r}(\kappa) \,-\, \hat{a}_{21}(\kappa)) \end{array}$$

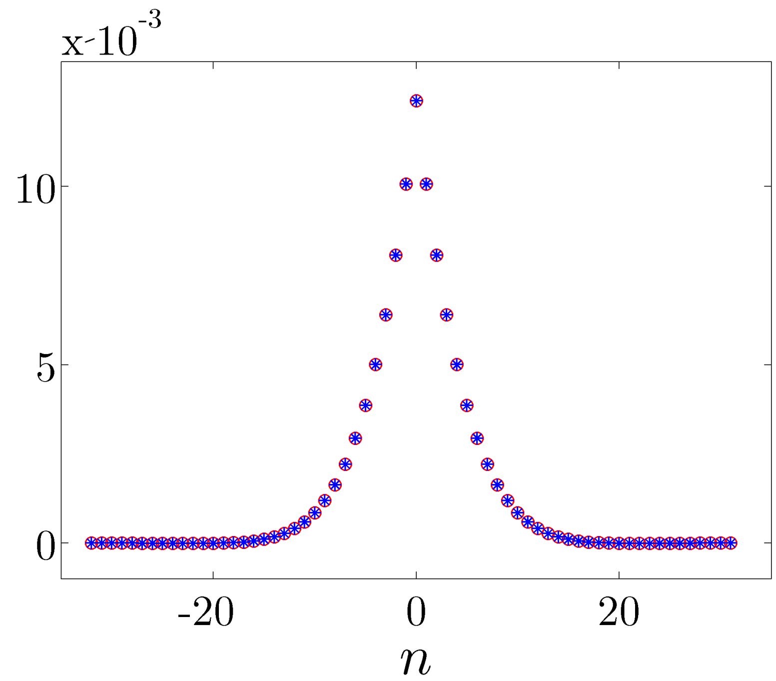
Structure of optimal solution

- Optimal position gain

$Q = I, R = I$:



$Q = I, R = 100\, I$:



- General trends

  ⋆ High actuation authority $\xrightarrow{\text{cheap control}}$ Less communication

  ⋆ Low actuation authority $\xrightarrow{\text{expensive control}}$ More communication

Bamieh, Paganini, Dahleh, IEEE TAC '02

LQR for systems with standard L_2 (or l_2) inner product

- Optimal controller determined by

$$\begin{aligned} u(x,t) &= -\left[\,\mathcal{K}\,\psi(\,\cdot\,,t)\,\right](x), \quad x \in \mathbb{G} \\ \mathcal{K} &= \mathcal{R}^{-1}\mathcal{B}^\dagger\,\mathcal{P} \end{aligned}$$

  - ⋆ $\mathcal{P} = \mathcal{P}^\dagger$ – bounded non-negative operator that solves ARE

$$\langle \mathcal{A}\,\psi_1, \mathcal{P}\,\psi_2\rangle + \langle \mathcal{P}\,\psi_1, \mathcal{A}\,\psi_2\rangle + \left\langle \mathcal{Q}^{\frac{1}{2}}\,\psi_1, \mathcal{Q}^{\frac{1}{2}}\,\psi_2\right\rangle - \langle \mathcal{B}^\dagger\,\mathcal{P}\,\psi_1, \mathcal{R}^{-1}\mathcal{B}^\dagger\,\mathcal{P}\,\psi_2\rangle = 0$$

$$\psi_1, \psi_2 \in \mathcal{D}(\mathcal{A})$$

- For standard L_2 (or l_2) inner product $\langle\,\cdot\,,\,\cdot\,\rangle$

$$\begin{aligned} \hat{u}(\kappa,t) &= -\hat{K}(\kappa)\,\hat{\psi}(\kappa,t), \quad \kappa \in \hat{\mathbb{G}} \\ \hat{K}(\kappa) &= \hat{R}^{-1}(\kappa)\,\hat{B}^*(\kappa)\,\hat{P}(\kappa) \\ 0 &= \hat{A}^*(\kappa)\,\hat{P}(\kappa) + \hat{P}(\kappa)\,\hat{A}(\kappa) + \hat{Q}(\kappa) - \hat{P}(\kappa)\,\hat{B}(\kappa)\,\hat{R}^{-1}(\kappa)\,\hat{B}^*(\kappa)\,\hat{P}(\kappa) \end{aligned}$$

In class: diffusion equation on $L_2(-\infty,\infty)$