12-01-11

Penalty term in physical & frequency domains.

Example:

circulant

$$Q_p = I + C$$

$$C_{4\times4} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$\hat{q}_p(k) = 1 + 2\left(1 - \cos\frac{2\pi k}{N}\right)$$

here, $N = 4$

Position penalty

$$Q = \begin{bmatrix} Q_p & 0 \\ 0 & Q_v \end{bmatrix}$$

velocity penalty

example: $\hat{K}(k) = \hat{R}^{-1}(k)\,\hat{B}^{*}(k)\,\hat{P}(k)$

$$= \frac{1}{\hat{r}(k)}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{P}_1(k) & \hat{P}_0^{*}(k) \\ \hat{P}_0(k) & \hat{P}_2(k) \end{bmatrix}$$

$$= \begin{bmatrix} \underbrace{\frac{1}{\hat{r}(k)}\hat{P}_0(k)}_{\text{Position feedback gain}} & \underbrace{\frac{1}{\hat{r}(k)}\hat{P}_2(k)}_{\text{velocity feedback gain}} \end{bmatrix}$$

PDEs on $L_2(-\infty,\infty)$:

ex: heat equation

$$\psi_t(x,t) = \psi_{xx}(x,t) + u(x,t)$$

$$u(x,t) = -\left[K\psi(\cdot,t)\right](x) = -\int_{-\infty}^{\infty} K_{ker}(x,\xi)\,\psi(\xi,t)\,d\xi$$

For spatially-invariant systems:

$$u(x,t) = -\int_{-\infty}^{\infty} K_{ker}(x-\xi)\,\psi(\xi,t)\,d\xi$$

$$\Big\downarrow \text{ F.T.}$$

$$\hat{u}(k,t) = -\hat{K}(k)\,\hat{\psi}(k,t)$$

$$K_{ker}(x) = \mathcal{F}^{-1}\left\{ \hat{K}(k) \right\}$$

Heat equation:

$$\dot{\hat{\psi}}(k,t) = -k^2\,\hat{\psi}(k,t) + \hat{u}(k,t)$$

$$\begin{cases} \hat{A}(k) = -k^2 \\ \hat{B}(k) = 1 \end{cases} \qquad \begin{cases} \hat{Q}(k) = \hat{q}(k) \\ \hat{R}(k) = \hat{r}(k) \end{cases}$$

$$\underbrace{-2k^2\hat{P}}_{\hat{A}^*\hat{P} + \hat{P}\hat{A}} + \underbrace{\hat{q}}_{\hat{Q}} - \frac{1}{\hat{r}}\hat{P}^2 = 0 \qquad \leftarrow \text{ARE}$$

$$\hat{P} = \hat{r}\left(-k^2 \pm \sqrt{k^4 + \frac{\hat{q}}{\hat{r}}}\right)$$

$\downarrow$

Choose (+) to get $\hat{P} > 0$.

① $$\boxed{\hat{K} = \frac{1}{\hat{r}}\hat{P} = -k^2 + \sqrt{k^4 + \frac{\hat{q}}{\hat{r}}}}$$ feedback gain

Check boundedness of $\hat{K}$

② $$\hat{K} = \frac{-\hat{q}/\hat{r}}{k^2 + \sqrt{k^4 + \frac{\hat{q}}{\hat{r}}}}$$

even though $\hat{K}$ written in form ① looks like a 2nd derivative operator and may indicate unboundedness of $\hat{K}$, when written in form ②, it is clear that it can be written in terms of integral operators.