Lecture 23: Optimal control of distributed systems

- Linear Quadratic Regulator (LQR)
  - ⋆ Linear: plant
  - ⋆ Quadratic: performance index
  - ⋆ Infinite horizon problem
  - ⋆ Algebraic Riccati Equation (ARE)
- Spatially invariant systems
  - ⋆ LQR: also spatially invariant
  - ⋆ Feedback gains decay exponentially with spatial distance
- Examples
  - ⋆ Distributed control
  - ⋆ Boundary control

Linear Quadratic Regulator

$$\text{minimize} \quad J \;=\; \int_0^\infty \Big(\langle \psi(t), \mathcal{Q}\, \psi(t) \rangle \;+\; \langle u(t), \mathcal{R}\, u(t) \rangle \Big)\, \mathrm{d}t$$

$$\text{subject to} \quad \psi_t(t) \;=\; \mathcal{A}\, \psi(t) \;+\; \mathcal{B}\, u(t), \;\; \psi(0) \in \mathbb{H}$$

- Finite dimensional problems

  ⋆ Optimal controller determined by

$$\begin{aligned} u(t) \;&=\; -K\, \psi(t) \\ K \;&=\; R^{-1} B^T P \end{aligned}$$

  ⋆ $P \,=\, P^*$ – non-negative solution to ARE

$$A^* P \;+\; P A \;+\; Q \;-\; P B R^{-1} B^* P \;=\; 0$$

  ⋆ ARE – quadratic equation in the elements of P

- Infinite dimensional problems

  ⋆ Optimal controller determined by

$$\begin{aligned} u(t) &= -\mathcal{K}\,\psi(t) \\ \mathcal{K} &= \mathcal{R}^{-1}\mathcal{B}^{\dagger}\,\mathcal{P} \end{aligned}$$

  ⋆ $\mathcal{P} = \mathcal{P}^{\dagger}$ – bounded non-negative operator that solves ARE

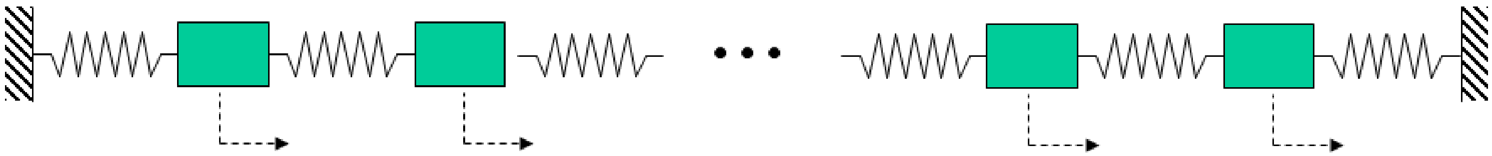
$$\langle \mathcal{A}\,\psi_1, \mathcal{P}\,\psi_2 \rangle + \langle \mathcal{P}\,\psi_1, \mathcal{A}\,\psi_2 \rangle + \left\langle \mathcal{Q}^{\frac{1}{2}}\,\psi_1, \mathcal{Q}^{\frac{1}{2}}\,\psi_2 \right\rangle - \langle \mathcal{B}^{\dagger}\,\mathcal{P}\,\psi_1, \mathcal{R}^{-1}\mathcal{B}^{\dagger}\,\mathcal{P}\,\psi_2 \rangle = 0$$

$$\psi_1, \psi_2 \in \mathcal{D}(\mathcal{A})$$

  ⋆ ARE – operator-valued equation in the unknown $\mathcal{P}$

An example

- Mass-spring system on a line



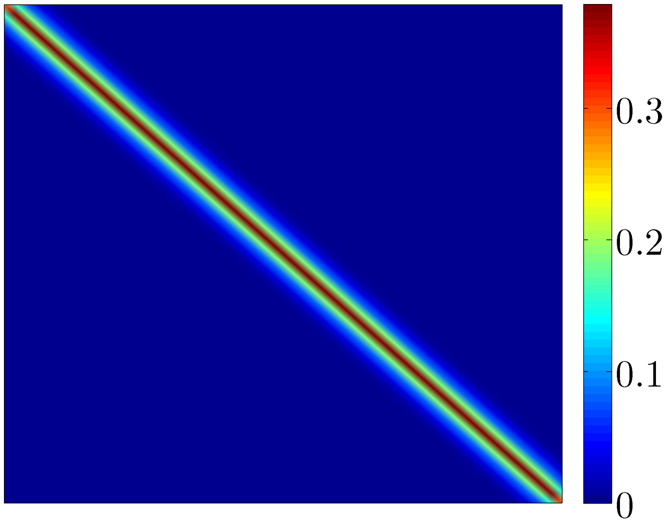
$$\begin{bmatrix} \dot{p} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ T & 0 \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

$$T \sim \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

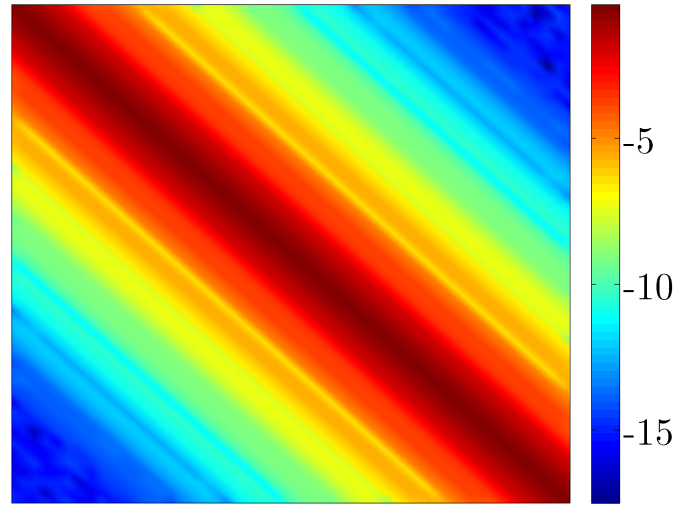
In class: use Matlab to illustrate structure of optimal feedback gains

Structure of optimal solution

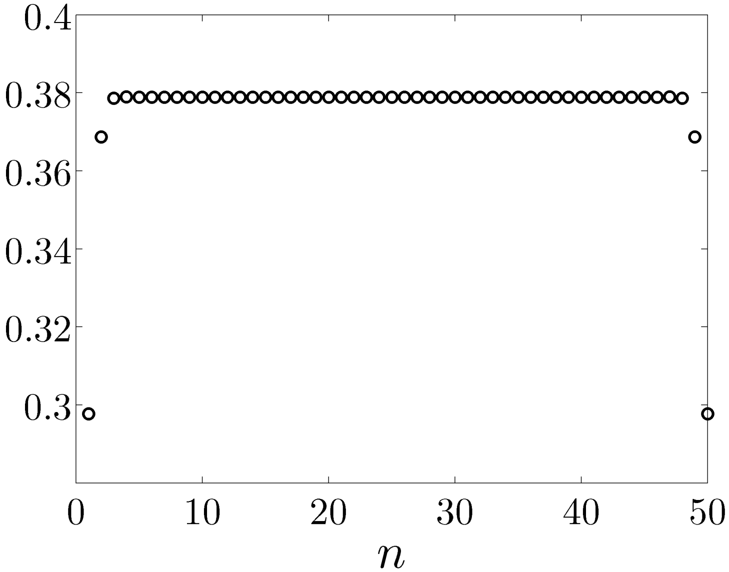
K_p:



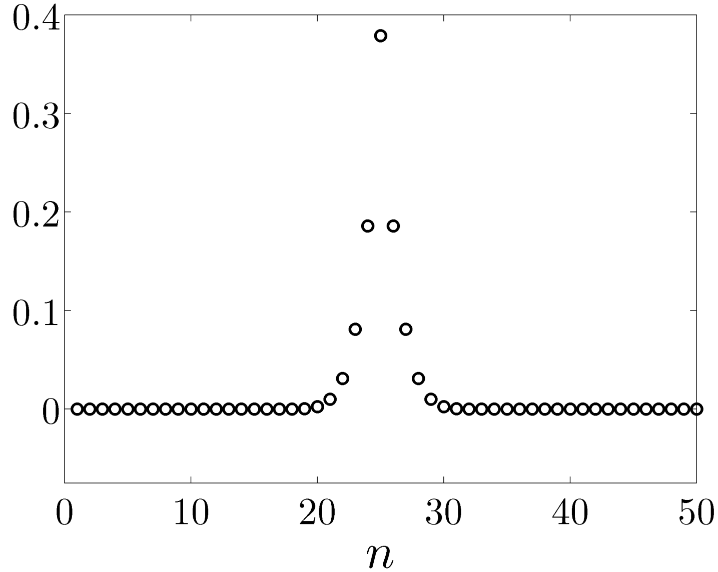
$\log_{10}(|K_p|)$:

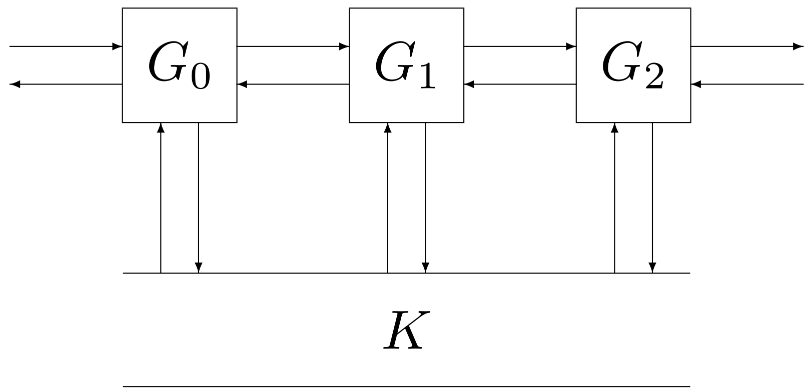


$\mathrm{diag}(K_p)$:



$K_p(25,:)$:





$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = -\underbrace{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}}_{K_p} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix} - \underbrace{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}}_{K_v} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \end{bmatrix}$$

- Observations:
  - ⋆ LQR – centralized controller
  - ⋆ Diagonals almost constant (modulo edges)
  - ⋆ Off-diagonal decay of centralized gain

Spatially invariant systems

$$\psi_t(x,t) \;=\; [\,\mathcal{A}\,\psi(\cdot,t)\,]\,(x) \;+\; [\,\mathcal{B}\,u(\cdot,t)\,]\,(x)$$

spatial coordinate: $x \in \mathbb{G}$

translation invariant operators: $\mathcal{A}$, $\mathcal{B}$

SPATIAL FOURIER TRANSFORM

$$\dot{\hat{\psi}}(\kappa,t) \;=\; \hat{\mathcal{A}}(\kappa)\,\hat{\psi}(\kappa,t) \;+\; \hat{\mathcal{B}}(\kappa)\,\hat{u}(\kappa,t)$$

spatial frequency: $\kappa \in \hat{\mathbb{G}}$

multiplication operators: $\hat{\mathcal{A}}(\kappa)$, $\hat{\mathcal{B}}(\kappa)$

$\mathbb{G}$	$\mathbb{R}$	$\mathbb{S}$	$\mathbb{Z}$	$\mathbb{Z}_N$
$\hat{\mathbb{G}}$	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{S}$	$\mathbb{Z}_N$

- $\mathbb{R}$ reals
- $\mathbb{Z}$ integers
- $\mathbb{S}$ unit circle
- $\mathbb{Z}_N$ integers modulo N

- Partial Differential Equations

  ⋆ Constant coefficients $+$ Infinite spatial extent

$$\psi_t(x,t) \;=\; \psi_{xx}(x,t) \;+\; u(x,t), \;\; x \,\in\, \mathbb{R}$$

$$\Big\downarrow \textbf{Fourier transform}$$

$$\dot{\hat{\psi}}(\kappa,t) \;=\; -\,\kappa^2\,\hat{\psi}(\kappa,t) \;+\; \hat{u}(\kappa,t), \;\; \kappa \,\in\, \mathbb{R}$$

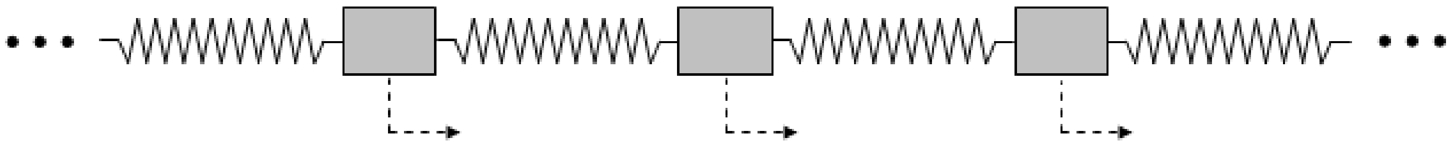
  ⋆ Constant coefficients $+$ Periodic domain

$$\psi_t(x,t) \;=\; \psi_{xx}(x,t) \;+\; u(x,t), \;\; x \,\in\, \mathbb{S}$$

$$\Big\downarrow \textbf{Fourier series}$$

$$\dot{\hat{\psi}}(\kappa,t) \;=\; -\,\kappa^2\,\hat{\psi}(\kappa,t) \;+\; \hat{u}(\kappa,t), \;\; \kappa \,\in\, \mathbb{Z}$$

- Spatially discrete systems (Interconnected ODEs)
  - ⋆ Constant coefficients $+$ Infinite lattices

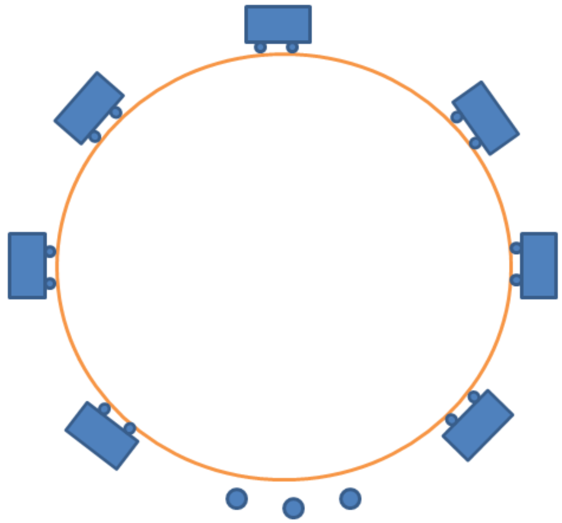


$$\dot{\psi}(x,t) \;=\; \begin{bmatrix} 0 & 1 \\ S_{-1} - 2 + S_1 & 0 \end{bmatrix} \psi(x,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(x,t), \quad x \in \mathbb{Z}$$

$\downarrow$ **$\mathcal{Z}$-transform evaluated at $z = \mathrm{e}^{\mathrm{j}\kappa}$**

$$\dot{\hat{\psi}}(\kappa,t) \;=\; \begin{bmatrix} 0 & 1 \\ 2\,(\cos\,\kappa - 1) & 0 \end{bmatrix} \hat{\psi}(\kappa,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(\kappa,t), \quad \kappa \in \mathbb{S}$$

⋆ Constant coefficients $+$ Circular lattices



Example: Mass-spring system on a circle

$$\dot{\psi}(x,t) \;=\; \begin{bmatrix} 0 & 1 \\ S_{-1} \,-\, 2 \,+\, S_1 & 0 \end{bmatrix} \psi(x,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(x,t), \;\; x \,\in\, \mathbb{Z}_N$$

$\downarrow$ **discrete Fourier transform**

$$\dot{\hat{\psi}}(\kappa,t) \;=\; \begin{bmatrix} 0 & 1 \\ 2\left(\cos \frac{2\,\pi\,\kappa}{N} \,-\, 1\right) & 0 \end{bmatrix} \hat{\psi}(\kappa,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(\kappa,t), \;\; \kappa \,\in\, \mathbb{Z}_N$$

LQR for spatially invariant system over $\mathbb{Z}_N$

$$\text{minimize} \quad J \;=\; \int_0^\infty \Big(\psi^*(t)\, Q\, \psi(t) \;+\; u^*(t)\, R\, u(t) \Big)\, \mathrm{d}t$$

$$\text{subject to} \quad \dot{\psi}(t) \;=\; A\, \psi(t) \;+\; B\, u(t)$$

- Circulant matrices: A, B, Q, R

  ⋆ Jointly unitarily diagonalizable by DFT Matrix V

$$\begin{aligned} \dot{\hat{\psi}}(t) &= A_d\, \hat{\psi}(t) \;+\; B_d\, \hat{u}(t) \\ A_d &= \mathrm{diag}\left(\hat{A}(\kappa)\right) \;=\; V A V^* \\ \psi^*\, Q\, \psi &= \hat{\psi}^*\, Q_d\, \hat{\psi} \end{aligned}$$

  ⋆ Entries into ARE – diagonal matrices

$$A_d^* P_d \;+\; P_d A_d \;+\; Q_d \;-\; P_d B_d R_d^{-1} B_d^* P_d \;=\; 0$$

$$\Updownarrow$$

$$\hat{A}^*(\kappa)\, \hat{P}(\kappa) \;+\; \hat{P}(\kappa)\, \hat{A}(\kappa) \;+\; \hat{Q}(\kappa) \;-\; \hat{P}(\kappa)\, \hat{B}(\kappa)\, \hat{R}^{-1}(\kappa)\, \hat{B}^*(\kappa)\, \hat{P}(\kappa) \;=\; 0, \quad \kappa \in \mathbb{Z}_N$$