Optimal Control of Distributed Systems

LQR :
$$\begin{cases} \min & J = \int_0^\infty \left(\langle \psi(t), Q\psi(t) \rangle + \langle u(t), Ru(t) \rangle \right) dt \\ \text{s.t.} & \psi_t = A\psi + Bu \ ; \ \psi(0) \in \mathbb{R} \end{cases}$$

$\boxed{\text{finite dimensions:}}$

$$u(t) = -K\psi(t)$$

$$K = R^{-1}B^*P$$

$$P = P^* \ ; \quad A^*P + PA + Q - PBR^{-1}B^*P = 0$$

ARE : $$A^*P + PA + Q - PB\underbrace{R^{-1}B^*P}_{K} = 0$$

Add & subtract $PBR^{-1}B^*P$ to ARE

$$(A - BK)^*P + P(A - BK) = -(Q + K^*RK)$$

$$A_{cl}^*P + PA_{cl} = -(Q + K^*RK)$$

So P is observability Gramian with respect to an appropriate output

$$A_{cl}^*P + PA_{cl} = -C^*C \qquad \begin{cases} \dot{x} = Ax \\ z = Cx \end{cases}$$

$$z = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u$$

$$J = \int_0^\infty z^*(t)\, z(t)\, dt$$

$$\begin{cases} \dot{\psi} = (A - BK)\psi \\ z = \begin{bmatrix} Q^{1/2} \\ -R^{1/2}K \end{bmatrix} \psi \\ K = R^{-1}B^*P \end{cases}$$

Infinite dimensions

$$u(t) = -\mathcal{K}\,\psi(t)$$

$$\mathcal{K} = \mathcal{R}^{-1}\mathcal{B}^{\dagger}P$$

$$\langle \mathcal{A}\psi_1, P\psi_2\rangle + \langle P\psi_1, \mathcal{A}\psi_2\rangle + \underbrace{\langle \quad \rangle}_{\langle Q^{+1/2}\psi_1,\, Q^{1/2}\psi_2\rangle} + \langle \mathcal{B}^{\dagger}P\psi_1, \mathcal{R}^{-1}\mathcal{B}^{\dagger}P\psi_2\rangle = 0$$

- optimal Controller that is obtained for <u>spatially-invarient</u> systems is <u>centralized</u>.

$$\psi_t(x,t) = [\mathcal{A}\psi(\cdot,t)](x) + [\mathcal{B}u(\cdot,t)](x)$$

translation-invarient operators $\mathcal{A}$, $\mathcal{B}$.

<u>Spatial Fourier Transform</u>

$$\dot{\hat{\psi}}(k,t) = \hat{\mathcal{A}}(k)\,\hat{\psi}(k,t) + \hat{\mathcal{B}}(k)\,\hat{u}(k,t)$$

$\downarrow$ spatial frequency

multiplication operators: $\hat{\mathcal{A}}(k)$, $\hat{\mathcal{B}}(k)$

- The appropriate Fourier Transforms in space, effectively block-diagonalize the system.
- Similarly, the appropriate Fourier Transform can decouple the ARE associated with an infinite-dimensional system. The requirement is that $\mathcal{A}$, $\mathcal{B}$, Q, $\mathcal{R}$ are jointly, unitarily block diagonalizable.