Optimal control of Distributed Systems

LQR:
\[
\min_{\mathcal{U}} J = \int_0^\infty \left( v(t)^T Q v(t) + u(t)^T R u(t) \right) dt
\]

s.t. \quad \dot{y}_t = A y_t + B u_t \quad y(0) \in \mathbb{R}

\text{finite dimensions:}
\[
\begin{align*}
U(t) &= -K y(t) \\
K &= R^{-1} B^T P
\end{align*}
\]

\[
P = P^* \quad \Rightarrow \quad A^T P + PA + Q - P BR^{-1} B^T P = 0
\]

ARE:
\[
A^T P + PA + Q - P BR^{-1} B^T P = 0
\]

Add & subtract $P BR^{-1} B^T P$ to ARE
\[
(A - BK)^T P + P (A - BK) = -(Q + K^T RK)
\]

\[
A cl P + PA cl = -(Q + K^T RK)
\]

So $P$ is observability Gramian with respect to an appropriate output

\[
A cl P + PA cl = -C^T C
\]

\[
\begin{align*}
\dot{z} &= \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \\
J &= \int_0^\infty z^T(t) z(t) dt \\
\mathcal{U} &= (A - BK) y \\
2 z &= \begin{bmatrix} Q^{1/2} \\ -R^{1/2} K \end{bmatrix} + y \\
K &= R^{-1} B^T P
\end{align*}
\]
\[ U(t) = -K \dot{y}(t) \]
\[ K = R^{-1} B^T P \]
\[ \langle cd^T y_1, P y_2 \rangle + \langle P y_1, cd^T y_2 \rangle + \cdots + \langle B^T P y_1, R^{-1} B^T P y_2 \rangle \]
\[ \langle Q^{1/2} y_1, Q^{1/2} y_2 \rangle = 0 \]

- Optimal controller that is obtained for \textit{spatially-invariant} systems is \textit{centralized}.

\[ y_t(x, t) = [cd^T q(\cdot, t)](x) + [Bu(\cdot, t)](x) \]

\textit{Translation-invariant} operators \( cd \), \( B \).

\textbf{Spatial Fourier Transform}

\[ \hat{y} (k, t) = \hat{cd} (k) \hat{y}(k, t) + \hat{B}(k) \hat{u}(k, t) \]

\( \hat{cd}(k) \), \( \hat{B}(k) \)

\[ \downarrow \]

\textit{Spatial frequency multiplications operators.}

- The appropriate \textit{Fourier Transforms in space, effectively block-diagonalize the system.}\

- Similarly, the appropriate Fourier Transform can decouple the ARE associated with an infinite-dimensional system. The requirement is that \( cd, B, Q, R \) are jointly, unitarily block diagonalizable.