Lecture 22: Stability of infinite dimensional systems

- Exponential stability
  - ⋆ Definition
  - ⋆ Conditions
  - ⋆ Lyapunov-based characterization
  - ⋆ Examples

Exponential stability

$$\psi_t(t) \; = \; \mathcal{A}\,\psi(t), \;\; \psi(0) \; = \; \psi_0 \in \mathbb{H}$$

- Exponential stability of a C_0-semigroup $\mathcal{T}(t)$ generated by $\mathcal{A}$

  there exist $M > 0$, $\alpha > 0$ s.t. $\|\mathcal{T}(t)\| \leq M\mathrm{e}^{-\alpha\, t}$ for all $t \geq 0$

- Consequence
  - ⋆ exponential convergence to zero of solutions to $\psi_t(t) = \mathcal{A}\,\psi(t)$

$$\|\psi(t)\| \; \leq \; M\,\|\psi_0\|\,\mathrm{e}^{-\alpha\, t}$$

Conditions for exponential stability

Datko's Lemma:

Exponential stability of $\mathcal{T}(t)$ on $\mathbb{H}$

$$\Updownarrow$$

for every $\psi_0 \in \mathbb{H}$ there exists positive $\gamma_\psi < \infty$ s.t.

$$\int_0^\infty \|\mathcal{T}(t)\,\psi_0\|^2 \, \mathrm{d}t \;\leq\; \gamma_\psi$$

Lyapunov-based characterization

Exponential stability of $\mathcal{T}(t)$ on $\mathbb{H}$

$$\Updownarrow$$

there exists a bounded positive operator $\mathcal{P}$ s.t.

$$\langle \mathcal{A}\,\psi, \mathcal{P}\,\psi \rangle \; + \; \langle \mathcal{P}\,\psi, \mathcal{A}\,\psi \rangle \; = \; -\,\langle \psi, \psi \rangle \quad \text{for all } \psi \in \mathcal{D}(\mathcal{A})$$

- $\mathcal{P}$ – infinite horizon observability Gramian of system with $\mathcal{C} = I$

$$\mathcal{P}\,\psi_0 \; = \; \int_0^\infty \mathcal{T}^\dagger(t)\,\mathcal{T}(t)\,\psi_0\,\mathrm{d}t, \quad \psi_0 \in \mathbb{H}$$

- Lyapunov functional

$$V(\psi(t)) \; = \; \langle \psi(t), \mathcal{P}\,\psi(t) \rangle \; = \; \langle \mathcal{T}(t)\,\psi(0), \mathcal{P}\,\mathcal{T}(t)\,\psi(0) \rangle$$

Example: diffusion equation on $L_2\,[-1, 1]$

$$
\begin{aligned}
\psi_t(x,t) &= \psi_{xx}(x,t) \\
\psi(x,0) &= \psi_0(x) \\
\psi(\pm 1, t) &= 0
\end{aligned}
$$

- Lyapunov equation

$$
\mathcal{A}^\dagger \mathcal{P} + \mathcal{P}\mathcal{A} = -I \quad \text{on} \quad \mathcal{D}(\mathcal{A})
$$

$$
\mathcal{A}^\dagger = \mathcal{A} \;\Rightarrow\; \phi = \mathcal{P}\psi = -\frac{1}{2}\mathcal{A}^{-1}\psi
$$

- Lyapunov functional

$$
\left.
\begin{aligned}
V(\psi) &= \langle \psi, \mathcal{P}\psi \rangle = \langle \psi, \phi \rangle \\
\phi''(x) &= -\frac{1}{2}\psi(x), \;\; \phi(\pm 1) = 0
\end{aligned}
\right\}
$$

$$
\Downarrow
$$

$$
V(\psi(t)) = \int_{-1}^{1}\int_{-1}^{1} \psi^*(x,t)\, P_{\text{ker}}(x,\xi)\, \psi(\xi,t)\, \mathrm{d}\xi\, \mathrm{d}x
$$

- Alternative approach

$$V(\psi) \;=\; \frac{1}{2}\,\langle \psi, \psi \rangle \quad \Rightarrow \quad \left\{ \begin{array}{l} \dfrac{\mathrm{d}\,V(\psi(t))}{\mathrm{d}\,t} \;=\; \langle \psi(t), \partial_{xx}\,\psi(t) \rangle \;\leq\; -\,\epsilon_Q\,\|\psi(t)\|^2 \\[2ex] \dfrac{\mathrm{d}\,\|\psi(t)\|^2}{\mathrm{d}\,t} \;\leq\; -2\,\epsilon_Q\,\|\psi(t)\|^2 \end{array} \right.$$

- In class:
  - ⋆ Use $V(\psi) \;=\; \frac{1}{2}\,\langle \psi, \psi \rangle$ to show exponential stability of

$$\begin{aligned} \psi_t(x,t) &= \psi_{xx}(x,t) \;-\; \mathrm{j}\kappa\,U(x)\,\psi(x,t) \\ \psi(x,0) &= \psi_0(x) \\ \psi(\pm 1,t) &= 0 \end{aligned}$$