Exponential stability:

$$(*) \qquad \|\mathcal{T}\| < M e^{-\alpha t}, \qquad M, \alpha > 0$$

$\|\mathcal{T}\|$: induced operator norm

$\mathcal{T}$: $\mathcal{C}_0$-semigroup generated by $\mathcal{A}$.

if $(*)$ is satisfied, solutions of the system decay exponentially with time, because:

$$\|\mathcal{T}\| = \sup_{0 < \|\psi_0\| \leq 1} \frac{\|\mathcal{T}(t)\psi_0\|}{\|\psi_0\|} < M e^{-\alpha t}$$

$$\Rightarrow \underbrace{\|\mathcal{T}(t)\psi_0\|}_{\substack{\text{norm of the solution, } \psi(t), \\ \text{starting at } \psi(0) = \psi_0}} < M \|\psi_0\| e^{-\alpha t}$$

Lyapunov-based Characterization

$\mathcal{T}(t)$ on $\mathcal{H}$ is exponentially stable

$$\Updownarrow$$

$\exists$ bounded positive operator $\mathcal{P}$ s.t.

$$\langle \mathcal{A}\psi, \mathcal{P}\psi \rangle + \langle \mathcal{P}\psi, \mathcal{A}\psi \rangle = -\langle \psi, \psi \rangle$$

for all $\psi \in \mathcal{D}(\mathcal{A})$

equivalently:

$$\mathcal{A}^{+}\mathcal{P} + \mathcal{P}\mathcal{A} = -\mathcal{C} \quad \text{on} \quad \mathcal{D}(\mathcal{A})$$

$\downarrow$

observability Gramian on an infinite time horizon.

$$\mathcal{P}\psi_0 = \int_0^{\infty} \mathcal{T}^{+}(t) \cdot \mathcal{C} \cdot \mathcal{T}(t)\psi_0 \, dt \quad : \text{observability Gramian}$$

$\boxed{\Rightarrow}$ proof

"Boundedness"

$$\langle \psi_0, \mathcal{P}\psi_0 \rangle = \int_0^{\infty} \psi_0^{*} \mathcal{T}^{*}(t)\mathcal{T}(t)\psi_0 \, dt$$

$$= \int_0^{\infty} \| \mathcal{T}(t)\psi_0 \|^2 \, dt \leqslant \gamma_{\psi} < \infty$$

$\downarrow$

Datko's Lemma

Also, $\langle \psi_0, \mathcal{P}\psi_0 \rangle \geqslant 0$: "Positivity"

Note

$$\langle \psi_0, \mathcal{P}\psi_0 \rangle = 0 \Leftrightarrow \| \mathcal{T}(t)\psi_0 \| = 0 \quad \underbrace{\text{a.e.}}_{\text{almost everywhere}}$$

From strong continuity of $\mathcal{T}(t) \Rightarrow \| \mathcal{T}(t)\psi_0 \| = 0$ (a.e)

$\boxed{\Leftarrow}$ proof $\qquad \psi_0 = 0 \Rightarrow \mathcal{P} > 0.$

Lyapunov functional candidate:

$$V(\psi(t)) = \langle \psi(t), \mathcal{P}\psi(t) \rangle$$

Note: $\boxed{\mathcal{P} \neq \mathcal{P}(t)}$

$$\boxed{\frac{dV(\psi(t))}{dt}} = \langle \psi_t(t), \mathcal{P}\psi(t)\rangle + \langle \psi(t), \mathcal{P}\psi_t(t)\rangle$$

$$= \langle \mathcal{A}\psi(t), \mathcal{P}\psi(t)\rangle + \langle \psi(t), \mathcal{P}\mathcal{A}\psi(t)\rangle$$

$$= \langle \mathcal{A}\psi, \mathcal{P}\psi\rangle + \langle \mathcal{P}\psi, \mathcal{A}\psi\rangle$$

$$= -\langle \psi, \psi\rangle = -\|\psi(t)\|^2$$

$$\boxed{= -\|\mathcal{T}(t)\psi_0\|^2}$$

$$V(\psi(t)) = V(\psi(0)) - \int_0^t \|\mathcal{T}(t)\psi_0\|^2\, dt$$

$$0 \leq V(\psi(t)) = V(\psi(0)) - \int_0^t \|\mathcal{T}(t)\psi_0\|^2\, dt$$

$$\int_0^t \|\mathcal{T}(t)\psi_0\|^2\, dt \leq V(\psi_0) \quad \text{on } \mathcal{D}(\mathcal{A})$$

Example: Diffusion on $L_2[-1,1]$

$$\phi'' = \frac{-1}{2}\psi, \quad \phi(\pm 1) = 0$$

$$\begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}\psi$$

$$\phi = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1(-1) \\ \phi_2(-1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \phi_1(1) \\ \phi_2(1) \end{bmatrix}$$

$$\phi(x) = \int_{-1}^{1} P_{ker}(x, \xi)\psi(\xi)\, d\xi$$