Power spectral density:

$$\| \mathcal{T}(\omega) \|^2_{HS} = \text{trace}\left(\mathcal{T}(\omega) \mathcal{T}^{+}(\omega) \right) = \sum_{n=1}^{\infty} \sigma_n^2(\omega)$$

What is trace of an operator P?

if P is a matrix $[P_{ij}]$, then

$$\text{trace}(P) = \sum_{i=1}^{n} P_{ii}$$

if P is an operator:

$$g(x) = [Pf](x)$$

$$= \int_a^b P_{ker}(x, \xi) f(\xi) \, d\xi$$

$$= \int_a^b \left[P_{ker}(x,\xi) \right] \left[f \right] d\xi$$

$$\text{trace}(P) = \int_a^b P_{ker}(x,x) \, dx$$

→ if $f(x) \in \mathbb{C}$, $g(x) \in \mathbb{C}$

Now, if $f(x) \in \mathbb{C}^m$; $g(x) \in \mathbb{C}^m$

$$\boxed{\underbrace{\text{trace}(P)}_{\text{operator trace}} = \int_a^b \underbrace{\text{tr}\left(P_{ker}(x,x) \right)}_{\text{matrix trace}} dx}$$

here, $P(\omega) = \mathcal{T}(\omega) \mathcal{T}^{+}(\omega)$

$\boxed{H_2 \text{ norm}}$

$$\| \mathcal{T} \|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \| \mathcal{T}(\omega) \|_{HS}^2 \, d\omega \quad \leftarrow \text{definition}$$

Computation : $\| \mathcal{T} \|_2^2 = \text{trace} (\mathcal{C} \mathcal{X} \mathcal{C}^{+})$

$$\underbrace{\mathcal{A} \mathcal{X} + \mathcal{X} \mathcal{A}^{+} = - \mathcal{B} \mathcal{B}^{+}}$$

operator Lyapunov equation: in general it is difficult to solve explicitely.

If $\mathcal{A} \mathcal{A}^{+} = \mathcal{A}^{+} \mathcal{A}$; $\mathcal{B} \mathcal{B}^{+} = I$

Then $\mathcal{X} = - (\mathcal{A} + \mathcal{A}^{+})^{-1}$

Example Heat equation :

$$\mathcal{A} = \Delta = \begin{cases} \frac{d^2}{dy^2} , & 1D \\ \frac{d^2}{dy^2} - k_z^2 , & 2D \end{cases}$$

$+$

Dirichlet BCs

$$X = \frac{-1}{2} \Delta^{-1} .$$

Example $\mathcal{A} \mathcal{X} + \mathcal{X} \mathcal{A}^{+} = - Q$

$\mathcal{A} \mathcal{A}^{+} = \mathcal{A}^{+} \mathcal{A}$

$$\| \mathcal{T} \|_2^2 = \text{trace} (\mathcal{X}) = - \text{trace} ((\mathcal{A} + \mathcal{A}^{+})^{-1} Q) .$$