Lecture 20: Input-output norms; Pseudospectra

- Singular Value Decomposition of the frequency response operator

- Measures of input-output amplification (across frequency)
  - ⋆ Largest singular value
  - ⋆ Hilbert-Schmidt norm (power spectral density)

- Systems with one spatial variable
  - ⋆ Two point boundary value problems

- Input-output norms
  - ⋆ H_∞ norm: $\begin{cases} \text{worst-case amplification of deterministic disturbances} \\ \text{measure of robustness} \end{cases}$
  - ⋆ H_2 norm: $\begin{cases} \text{energy of the impulse response} \\ \text{variance amplification} \end{cases}$

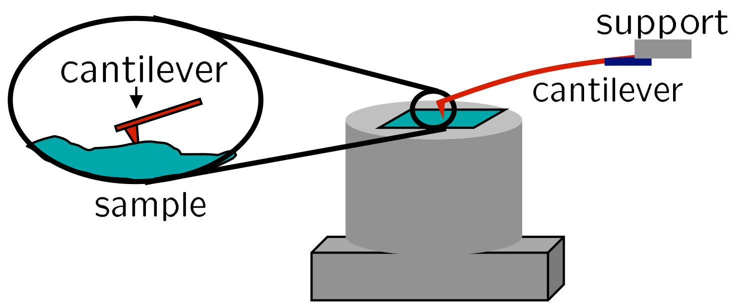
- Pseudospectra of linear operators

Example: cantilever beam

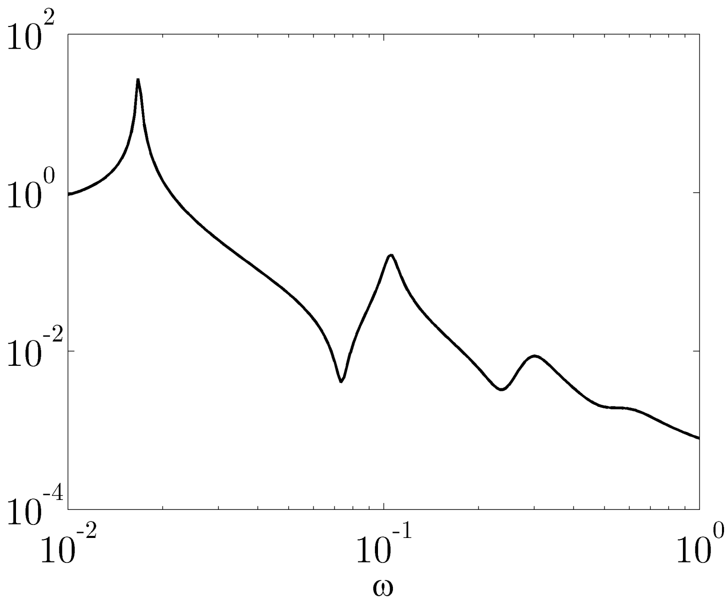
$$
\left\{
\begin{array}{rcl}
\mu\,\psi_{tt} \, + \, \alpha EI\,\psi_{txxxx} \, + \, EI\,\psi_{xxxx} & = & 0 \\
\psi(0,t) & = & 0, \qquad \psi_x(0,t) \; = \; 0 \\
\alpha EI\,\psi_{txxx}(l,t) \, + \, EI\psi_{xxx}(l,t) & = & u(t), \quad \psi_{xx}(l,t) \; = \; 0 \\
\psi(l,t) & = & y(t)
\end{array}
\right.
$$

input: $u(t)$

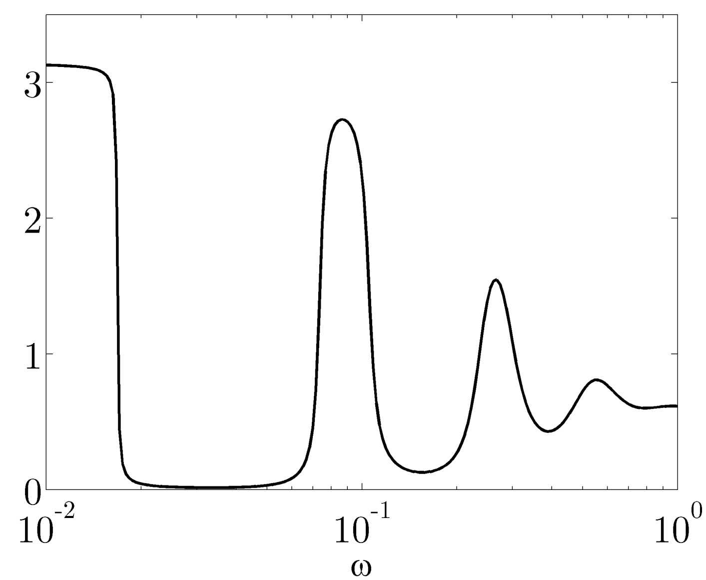
output: $y(t)$



magnitude:



phase:



Example: diffusion equation on $L_2\,[-1,1]$

- Distributed input and output fields

$$
\begin{aligned}
\phi_t(y,t) &= \phi_{yy}(y,t) \; + \; d(y,t) \\
\phi(y,0) &= 0 \\
\phi(\pm 1,t) &= 0
\end{aligned}
$$

Frequency response operator

$$
\begin{aligned}
\phi(y,\omega) &= \left[\, \mathcal{T}(\omega)\, d(\,\cdot\,,\omega) \,\right](y) \\
&= \left[\, (\mathrm{j}\omega I \; - \; \partial_{yy})^{-1}\, d(\,\cdot\,,\omega) \,\right](y) \\
&= \int_{-1}^{1} T_{\mathrm{ker}}(y,\eta;\omega)\, d(\eta,\omega)\, \mathrm{d}\eta
\end{aligned}
$$

Two point boundary value realizations of $\mathcal{T}(\omega)$

- Input-output differential equation

$$\mathcal{T}(\omega) : \left\{ \begin{array}{rcl} \phi''(y,\omega) \, - \, \mathrm{j}\omega \, \phi(y,\omega) & = & -d(y,\omega) \\ \phi(\pm 1,\omega) & = & 0 \end{array} \right.$$

- Spatial state-space realization

$$\mathcal{T}(\omega) : \left\{ \begin{array}{rcl} \begin{bmatrix} x_1'(y,\omega) \\ x_2'(y,\omega) \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ \mathrm{j}\omega & 0 \end{bmatrix} \begin{bmatrix} x_1(y,\omega) \\ x_2(y,\omega) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} d(y,\omega) \\ \phi(y,\omega) & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(y,\omega) \\ x_2(y,\omega) \end{bmatrix} \\ 0 & = & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(-1,\omega) \\ x_2(-1,\omega) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(1,\omega) \\ x_2(1,\omega) \end{bmatrix} \end{array} \right.$$

Frequency response operator

- Evolution equation

$$\begin{aligned} \mathcal{E}\,\boldsymbol{\phi}_t(y,t) &= \mathcal{F}\,\boldsymbol{\phi}(y,t) \,+\, \mathcal{G}\,\mathbf{d}(y,t) \\ \boldsymbol{\varphi}(y,t) &= \mathcal{C}\,\boldsymbol{\phi}(y,t) \end{aligned}$$

  ⋆ Spatial differential operators

$$\mathcal{F} \;=\; [\,\mathcal{F}_{ij}\,] \;=\; \left[\,\sum_{k\,=\,0}^{n_{ij}} f_{ij,k}(y)\,\frac{\mathrm{d}^k}{\mathrm{d}y^k}\,\right]$$

- Frequency response operator

$$\mathcal{T}(\omega) \;=\; \mathcal{C}\,(\mathrm{j}\omega\mathcal{E} \,-\, \mathcal{F})^{-1}\,\mathcal{G}$$

Singular Value Decomposition of $\mathcal{T}(\omega)$

- compact operator $\mathcal{T}(\omega)$: $\mathbb{H}_{\mathrm{in}} \longrightarrow \mathbb{H}_{\mathrm{out}}$

$$\boldsymbol{\varphi}(y,\omega) \;=\; [\,\mathcal{T}(\omega)\,\mathbf{d}(\,\cdot\,,\omega)\,]\,(y) \;=\; \sum_{n\,=\,1}^{\infty} \sigma_n(\omega)\,\mathbf{u}_n(y,\omega)\,\langle \mathbf{v}_n, \mathbf{d}\rangle$$

$$\left[\,\mathcal{T}(\omega)\,\mathcal{T}^{\dagger}(\omega)\,\mathbf{u}_n(\,\cdot\,,\omega)\right](y) \;=\; \sigma_n^2(\omega)\,\mathbf{u}_n(y,\omega) \;\Rightarrow\; \{\mathbf{u}_n\} \text{ orthonormal basis of } \mathbb{H}_{\mathrm{out}}$$

$$\left[\,\mathcal{T}^{\dagger}(\omega)\,\mathcal{T}(\omega)\,\mathbf{v}_n(\,\cdot\,,\omega)\right](y) \;=\; \sigma_n^2(\omega)\,\mathbf{v}_n(y,\omega) \;\Rightarrow\; \{\mathbf{v}_n\} \text{ orthonormal basis of } \mathbb{H}_{\mathrm{in}}$$

$$\sigma_1(\omega) \;\geq\; \sigma_2(\omega) \;\geq\; \cdots \;>\; 0$$

$$\mathbf{d}(y,\omega) \;=\; \mathbf{v}_m(y,\omega) \;\Rightarrow\; \boldsymbol{\varphi}(y,\omega) \;=\; \sigma_m(\omega)\,\mathbf{u}_m(y,\omega)$$

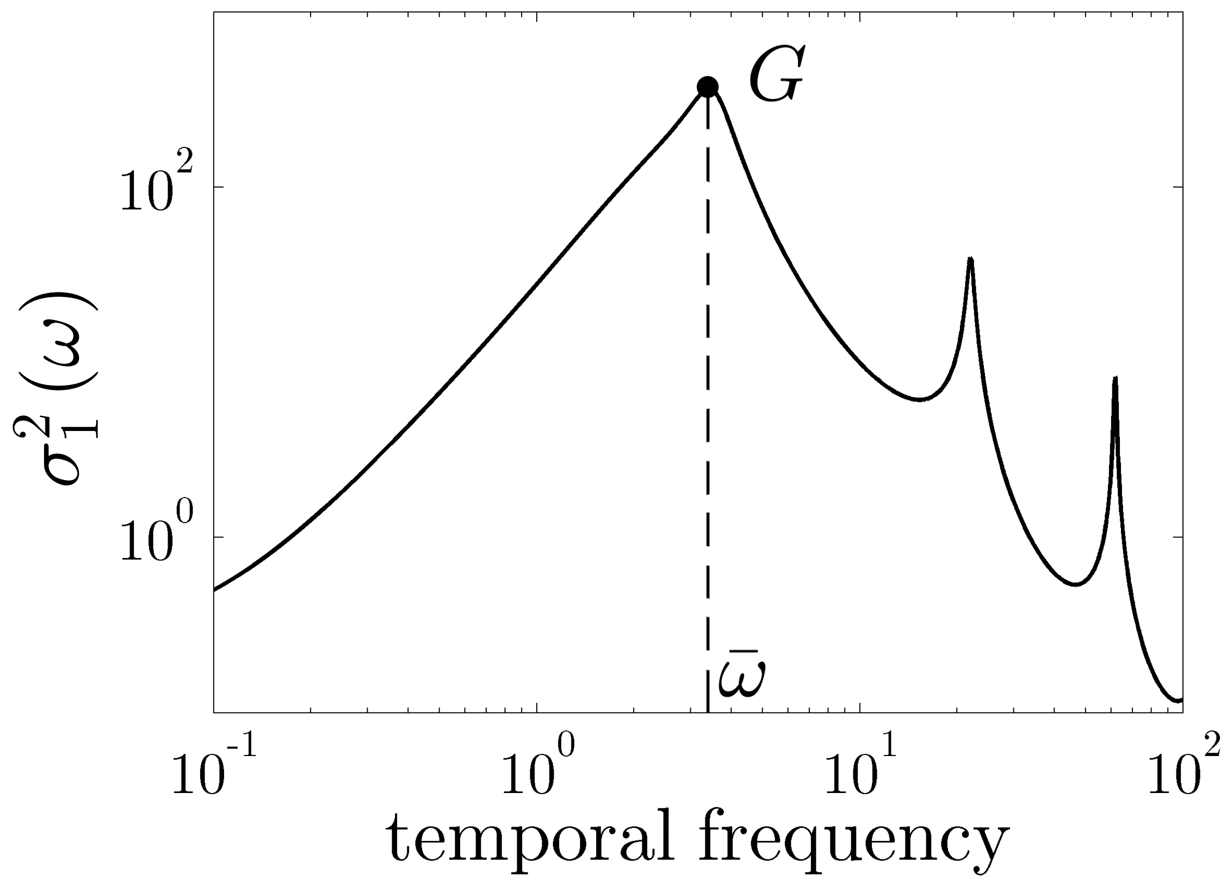
$\sigma_1(\omega)$: the largest amplification at any frequency

Input-output gains

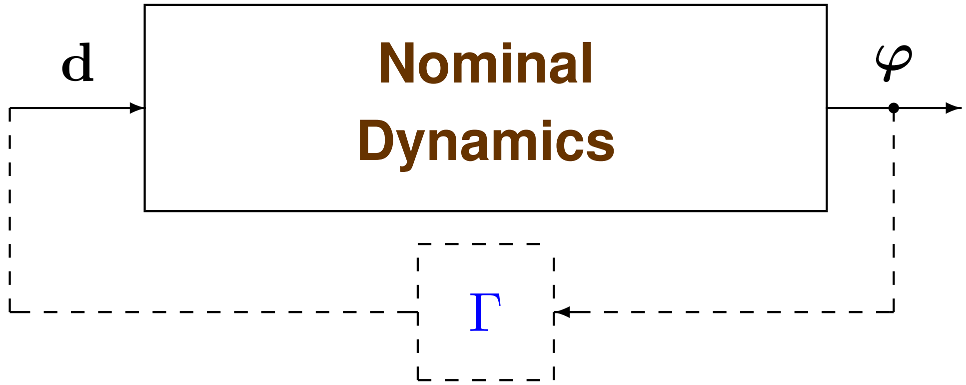
- **Determined by singular values of $\mathcal{T}(\omega)$**
  - $\star$ H_∞ norm: an induced L_2 gain (of a system)

worst case amplification:

$$\|\mathcal{T}\|_\infty^2 \;=\; \sup \frac{\text{output energy}}{\text{input energy}} \;=\; \sup_{\omega} \sigma_1^2(\omega)$$



- Robustness interpretation



modeling uncertainty
(can be nonlinear or time-varying)

small-gain theorem:

stability for all Γ with $\|\Gamma\|_\infty \leq \gamma$	$\Leftrightarrow$	$\gamma < \dfrac{1}{\|\mathcal{T}\|_\infty}$

LARGE worst case amplification $\Rightarrow$ small stability margins

- Hilbert-Schmidt norm of $\mathcal{T}(\omega)$

power spectral density:

$$\|\mathcal{T}(\omega)\|_{\mathrm{HS}}^2 \;=\; \mathrm{trace}\left(\mathcal{T}(\omega)\,\mathcal{T}^\dagger(\omega)\right) \;=\; \sum_{n\,=\,1}^{\infty} \sigma_n^2\,(\omega)$$

☞ Both $\sigma_1(\omega)$ and $\|\mathcal{T}(\omega)\|_{\mathrm{HS}}^2$ can be computed efficiently using Chebfun

⋆ Enabling tool: TPBVRs of $\mathcal{T}(\omega)$ and $\mathcal{T}^\dagger(\omega)$

$\|\mathcal{T}(\omega)\|_{\mathrm{HS}}^2$: *Jovanović & Bamieh, Syst. Control Lett. '06*

$\sigma_1(\omega)$: *Lieu & Jovanović, J. Comput. Phys. '11* (submitted; also: arXiv:1112.0579v1)

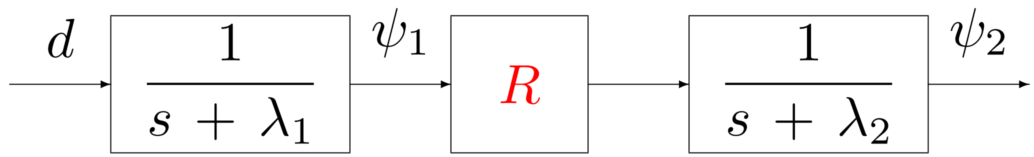
software: Frequency Responses of PDEs in Chebfun

- H_2 norm: variance amplification

$$\|\mathcal{T}\|_2^2 \;=\; \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \|\mathcal{T}(\omega)\|_{\mathrm{HS}}^2 \,\mathrm{d}\omega$$

A toy example

$$\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 \\ R & -\lambda_2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d$$



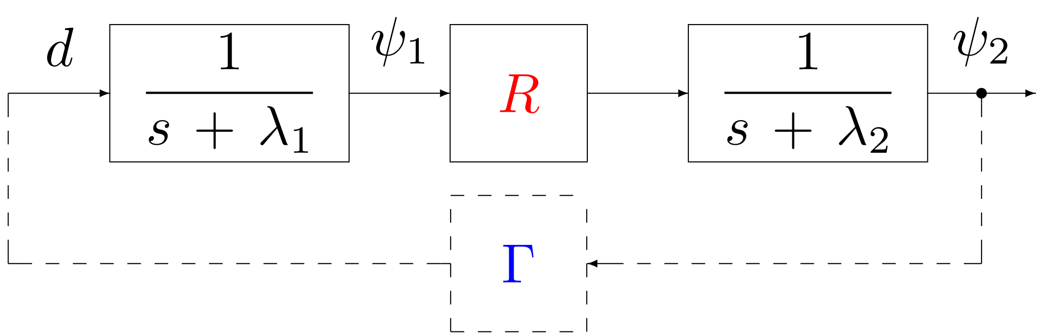
Worst case amplification

$$\sup \frac{\text{energy of } \psi_2}{\text{energy of } d} = \sup_{\omega} |T(\mathrm{j}\omega)|^2 = \frac{R^2}{(\lambda_1 \lambda_2)^2}$$

Variance amplification

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |T(\mathrm{j}\omega)|^2 \, \mathrm{d}\omega = \frac{R^2}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)}$$

Robustness



modeling uncertainty
(can be nonlinear or time-varying)

small-gain theorem:

stability for all Γ with

$$\|\Gamma\|_\infty \le \gamma$$

$$\Updownarrow$$

$$\gamma < \lambda_1 \lambda_2 / R$$

A note on computation of H_2 and H_∞ norms

$$\begin{aligned} \boldsymbol{\phi}_t(y,t) &= \mathcal{A}\,\boldsymbol{\phi}(y,t) \;+\; \mathcal{B}\,\mathbf{d}(y,t) \\ \boldsymbol{\varphi}(y,t) &= \mathcal{C}\,\boldsymbol{\phi}(y,t) \end{aligned}$$

- H_2 norm
  - ⋆ Operator Lyapunov equation

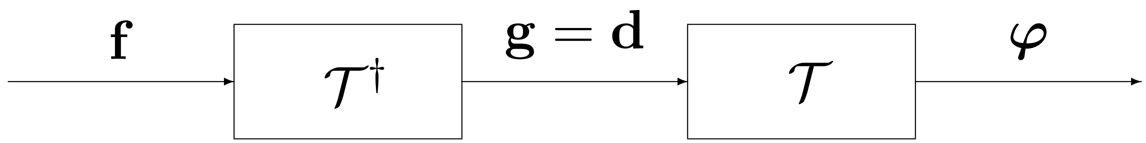
$$\begin{aligned} \|\mathcal{T}\|_2^2 &= \operatorname{trace}\left(\mathcal{C}\,\mathcal{X}\,\mathcal{C}^\dagger\right) \\ \mathcal{A}\,\mathcal{X} \;+\; \mathcal{X}\,\mathcal{A}^\dagger &= -\,\mathcal{B}\,\mathcal{B}^\dagger \end{aligned}$$

- H_∞ norm
  - ⋆ E-value decomposition of Hamiltonian in conjunction with bisection

$$\|\mathcal{T}\|_\infty \geq \gamma \;\;\Leftrightarrow\;\; \left[\begin{array}{cc} \mathcal{A} & \frac{1}{\gamma}\,\mathcal{B}\,\mathcal{B}^\dagger \\ -\frac{1}{\gamma}\,\mathcal{C}^\dagger\,\mathcal{C} & -\mathcal{A}^\dagger \end{array}\right] \text{ has at least one imaginary e-value}$$

Spatial state-space realization of $\mathcal{T}(\omega)$

- Cascade connection of $\mathcal{T}^\dagger$ and $\mathcal{T}$



- Realization of $\mathcal{T}$

$$\mathcal{T} : \left\{ \begin{array}{rcl} \mathbf{x}'(y) & = & \mathbf{A}_0(y)\,\mathbf{x}(y) \;+\; \mathbf{B}_0(y)\,\mathbf{d}(y) \\ \boldsymbol{\varphi}(y) & = & \mathbf{C}_0(y)\,\mathbf{x}(y) \\ 0 & = & \mathbf{N}_a\,\mathbf{x}(a) \;+\; \mathbf{N}_b\,\mathbf{x}(b) \end{array} \right.$$

- Realization of $\mathcal{T}^\dagger$

$$\mathcal{T}^\dagger : \left\{ \begin{array}{rcl} \mathbf{z}'(y) & = & -\mathbf{A}_0^*(y)\,\mathbf{z}(y) \;-\; \mathbf{C}_0^*(y)\,\mathbf{f}(y) \\ \mathbf{g}(y) & = & \mathbf{B}_0^*(y)\,\mathbf{z}(y) \\ 0 & = & \mathbf{M}_a\,\mathbf{z}(a) \;+\; \mathbf{M}_b\,\mathbf{z}(b) \end{array} \right.$$

$$\begin{bmatrix} \mathbf{M}_a & \mathbf{M}_b \end{bmatrix} \begin{bmatrix} \mathbf{N}_a^* \\ -\mathbf{N}_b^* \end{bmatrix} = 0$$

Integral form of a differential equation

- 1D diffusion equation: differential form

$$\left(D^{(2)} \,-\, \mathrm{j}\omega I\right)\phi(y) \;=\; -\,d(y)$$

$$\left(\left[\begin{array}{c} 1 \\ 0 \end{array}\right] E_{-1} \,+\, \left[\begin{array}{c} 0 \\ 1 \end{array}\right] E_{1}\right)\phi(y) \;=\; \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Auxiliary variable: $\nu(y) \;=\; \left[D^{(2)}\,\phi\right](y)$

Integrate twice

$$\phi'(y) \;=\; \int_{-1}^{y} \nu(\eta_1)\,\mathrm{d}\eta_1 \,+\, k_1 \;=\; \left[J^{(1)}\,\nu\right](y) \,+\, k_1$$

$$\phi(y) \;=\; \int_{-1}^{y}\left(\int_{-1}^{\eta_2} \nu(\eta_1)\,\mathrm{d}\eta_1\right)\mathrm{d}\eta_2 \,+\, \left[\begin{array}{cc} 1 & (y \,+\, 1) \end{array}\right]\left[\begin{array}{c} k_2 \\ k_1 \end{array}\right]$$

$$=\; \left[J^{(2)}\,\nu\right](y) \,+\, K^{(2)}\,\mathbf{k}$$

- 1D diffusion equation: integral form

$$\left(I \,-\, \mathrm{j}\omega J^{(2)}\right)\nu(y) \,-\, \mathrm{j}\omega\, K^{(2)}\,\mathbf{k} \;=\; -\,d(y)$$

$$\left[\begin{array}{cc} 1 & 0 \\ 1 & 2 \end{array}\right]\left[\begin{array}{c} k_2 \\ k_1 \end{array}\right] \;=\; -\,\left(\left[\begin{array}{c} 1 \\ 0 \end{array}\right] E_{-1} \,+\, \left[\begin{array}{c} 0 \\ 1 \end{array}\right] E_1\right) J^{(2)}\nu(y)$$

Eliminate $\mathbf{k}$ from the equations to obtain

$$\left(I \,-\, \mathrm{j}\omega J^{(2)} \,+\, \frac{1}{2}\,\mathrm{j}\omega\,(y \,+\, 1)\,E_1 J^{(2)}\right)\nu(y) \;=\; -d(y)$$

☞ More suitable for numerical computations than differential form
integral operators and point evaluation functionals are well-conditioned

Pseudospectra

- Book
  - ⋆ Trefethen and Embree: Spectra and Pseudospectra

- Online resources
  - ⋆ Talk by Nick Trefethen: Pseudospectra and EigTool
  - ⋆ Software: { Pseudospectra Gateway; EigTool }

perturbed system: $\psi_t \; = \; (\mathcal{A} \, + \, \Gamma) \, \psi$

ϵ-pseudospectrum:

$$
\begin{aligned}
\sigma_\epsilon(\mathcal{A}) \;&=\; \{ s \in \mathbb{C}; \; \|(sI \, - \, \mathcal{A})^{-1}\| \, > \, 1/\epsilon \} \\
&=\; \{ s \in \mathbb{C}; \; s \in \sigma(\mathcal{A} + \Gamma), \; \|\Gamma\| \, < \, \epsilon \}
\end{aligned}
$$

can be converted to an input-output problem