Evolution equation that we will consider:

$$\mathcal{E}\,\phi_t(y,t) = \mathcal{F}\,\phi(y,t) + \mathcal{G}\,d(y,t)$$
$$\varphi(y,t) = \mathcal{C}\,\phi(y,t)$$

$\mathcal{E}$ can be invertible, it may also be not invertible.

example: Navier-Stokes equations

$$\underbrace{\begin{bmatrix} \Delta & 0 \\ 0 & I \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \phi_{1t} \\ \phi_{2t} \end{bmatrix} = \underbrace{\begin{bmatrix} \Delta^2 & 0 \\ -jk_z U' & \Delta \end{bmatrix}}_{\mathcal{F}} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\Delta^2 = \mathcal{D}^{(4)} - 2k_z^2\,\mathcal{D}^{(2)} + k_z^4$$
$$\Delta = \mathcal{D}^{(2)} - k_z^2$$

$$\mathcal{F} = \begin{bmatrix} k_z^4 & 0 \\ -jk_z U' & -k_z^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{D}^{(1)} + \begin{bmatrix} -2k_z^2 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{D}^{(2)} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{D}^{(3)} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{D}^{(4)}$$

Input-output map: $\mathcal{T}(\omega) : \mathbb{H}_{in} \longrightarrow \mathbb{H}_{out}$

$$\varphi(y,\omega) = [\mathcal{T}(\omega)\, d(\cdot,\omega)](y)$$
$$= \sum_{n=1}^{\infty} \sigma_n(\omega)\, u_n(y,\omega) \langle v_n(y,\omega), d \rangle$$

σ_1 ... Largest eigenvalue of $\mathcal{T}(\omega)\,\mathcal{T}^*(\omega)$

v_1 ... the input direction that yields Largest response

u_1 ... the spatial pattern that is generated by v_1, i.e. the most energetic pattern that one expects to see if the system is forced by a stochastic forcing.

<u>Remark</u> in situation that we don't have a <u>normal</u> operator, eigenvalue decomposition is not a good measure of the system response. (because we cannot obtain eigen-directions that evolve independently.)

For <u>non-normal</u> operators, singular-value decompositi on is the right notion. It is a description of the input-output maps.

<u>The input-output gains</u> are obtained from singular values of $\mathcal{T}(\omega)$.

H_∞ norm: $\phi(y,t)$

$$\| H \|_\infty^2 = \sup_{\left\{\substack{d \neq 0 \\ \int_0^\infty \langle d, d \rangle\, dt \leq 1}\right.} \frac{\int_0^\infty \langle \phi(\cdot,t), \phi(\cdot,t)\rangle\, dt}{\int_0^\infty \langle d(\cdot,t), d(\cdot,t)\rangle\, dt} \quad \Bigg\} \begin{array}{l} L_2\text{-induced} \\ \text{gain of an} \\ \text{LTI} \\ \text{system} \end{array}$$

$$[\text{in } L_2] \quad = \sup \frac{\int_0^\infty \int_{-1}^1 \phi^*(y,t)\,\phi(y,t)\, dy\, dt}{\int_0^\infty \int_{-1}^1 d^*(y,t)\, d(y,t)\, dy\, dt} = \sup_\omega \sigma_1^2(\omega)$$

- H_∞ defined with time-integrals can be interpreted as $\underbrace{\text{worst-case}}_{\text{Largest}}$ energy that can be obtained by arbitrary deterministic inputs.

- H_∞ defined with $\sup_{\omega} \sigma_1^2(\omega)$ can be interpreted as the Largest amplification of persistant sinousoidal inputs.

- Robustness interpretation: (Small-gain theorem)

  if you have a modeling uncertainty Γ,

  $$\|\Gamma\|_\infty \leq \nu \iff \nu < \frac{1}{\|\Gamma\|_\infty}$$

  meaning that system remains stable

  the amount of uncertainty one can handle in the presence of all unstructured uncertainty Γ.

The Larger $\|\Gamma\|_\infty$, the smaller uncertainty can destabilize the system.