Lecture 16: Controllability and observability

- Controllability
  - ⋆ Ability to steer state
- Observability
  - ⋆ Ability to estimate state
- Topics:
  - ⋆ Connections and differences with finite-dimensional case
  - ⋆ Exact vs. approximate controllability/observability
  - ⋆ Conditions for controllability/observability
  - ⋆ Gramians
  - ⋆ Operator Lyapunov equations

An example

- Diffusion equation on $L_2\,[-1, 1]$ with point actuation and sensing

$$
\begin{aligned}
\psi_t(x,t) &= \psi_{xx}(x,t) \;+\; b(x)\,u(t)\\
\phi(t) &= \int_{-1}^{1} c(x)\,\psi(x,t)\,\mathrm{d}x\\
\psi(x,0) &= \psi_0(x)\\
\psi(\pm 1,t) &= 0
\end{aligned}
$$

Control and sensing points x_c and x_s

$$
\begin{aligned}
b(x) &= \frac{1}{2\epsilon}\,\mathbb{1}_{[x_c-\epsilon,\,x_c+\epsilon]}\,(x)\\
c(x) &= \frac{1}{2\delta}\,\mathbb{1}_{[x_s-\delta,\,x_s+\delta]}\,(x)\\
\mathbb{1}_{[a,\,b]}\,(x) &= \begin{cases} 1, & x \in [a,\,b]\\ 0, & \text{otherwise}\end{cases}
\end{aligned}
$$

Controllability operator and Gramian

$$\begin{aligned} \psi_t(t) &= \mathcal{A}\,\psi(t) \,+\, \mathcal{B}\,u(t) \\ \mathcal{A}: &\quad \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H} \\ \mathcal{B}: &\quad \mathbb{U} \longrightarrow \mathbb{H} \end{aligned}$$

- Controllability operator

$$\mathcal{R}_t : L_2\left([0,\, t]; \mathbb{U}\right) \;\longrightarrow\; \mathbb{H}$$

$$\psi(t) \;=\; [\,\mathcal{R}_t\, u\,]\,(t) \;=\; \int_0^t \mathcal{T}(t \,-\, \tau)\,\mathcal{B}\,u(\tau)\,\mathrm{d}\tau$$

  - ⋆ Adjoint

$$\left[\mathcal{R}_t^\dagger\,\psi\right](\tau) \;=\; \mathcal{B}^\dagger\,\mathcal{T}^\dagger(t \,-\, \tau), \;\; \tau \,\in\, [0,\, t]$$

- Controllability Gramian

$$\mathcal{P}_t \;=\; \mathcal{R}_t\,\mathcal{R}_t^\dagger \;=\; \int_0^t \mathcal{T}(\tau)\,\mathcal{B}\,\mathcal{B}^\dagger\,\mathcal{T}^\dagger(\tau)\,\mathrm{d}\tau$$

Exact vs. approximate controllability

- Exact controllability on $[0,\, t]$

$$\mathrm{range}\,(\mathcal{R}_t) \;=\; \mathbb{H}$$

  - ⋆ rarely satisfied by infinite-dimensional systems
  - ⋆ never satisfied for systems with finite-dimensional $\mathbb{U}$

- Approximate controllability on $[0,\, t]$

$$\overline{\mathrm{range}\,(\mathcal{R}_t)} \;=\; \mathbb{H}$$

  - ⋆ reasonable notion of controllability for infinite-dimensional systems
  - ⋆ easily checkable conditions for Riesz-spectral systems

approximate controllability on $[0,\, t]$

$$\Updownarrow$$

$$\mathcal{P}_t \,>\, 0 \;\;\Leftrightarrow\;\; \{\langle \psi, \mathcal{P}_t\,\psi\rangle \,>\, 0, \text{ for all } 0 \neq \psi \in \mathbb{H}\}$$

or

$$\mathrm{null}\left(\mathcal{R}_t^\dagger\right) \;=\; 0 \;\;\Leftrightarrow\;\; \{\mathcal{B}^\dagger\,\mathcal{T}^\dagger(\tau)\,\psi \;=\; 0 \text{ on } [0,\, t] \;\Rightarrow\; \psi = 0\}$$

Observability operator and Gramian

$$\begin{aligned} \psi_t(t) &= \mathcal{A}\,\psi(t) \\ \phi(t) &= \mathcal{C}\,\psi(t) \end{aligned}$$

$$\begin{aligned} \mathcal{A}: &\quad \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H} \\ \mathcal{C}: &\quad \mathbb{H} \longrightarrow \mathbb{Y} \end{aligned}$$

- Observability operator

$$\begin{aligned} \mathcal{O}_t : \mathbb{H} &\longrightarrow L_2\left([0,\, t]; \mathbb{Y}\right) \\ \phi(t) \;=\; \left[\,\mathcal{O}_t\,\psi(0)\,\right](t) &= \mathcal{C}\,\mathcal{T}(t)\,\psi(0) \end{aligned}$$

  - ⋆ Adjoint

$$\left[\mathcal{O}_t^\dagger\,\phi\right](t) \;=\; \int_0^t \mathcal{T}^\dagger(\tau)\,\mathcal{C}^\dagger\,\phi(\tau)\,\mathrm{d}\tau$$

- Observability Gramian

$$\mathcal{V}_t \;=\; \mathcal{O}_t^\dagger\,\mathcal{O}_t \;=\; \int_0^t \mathcal{T}^\dagger(\tau)\,\mathcal{C}^\dagger\,\mathcal{C}\,\mathcal{T}(\tau)\,\mathrm{d}\tau$$

Exact vs. approximate observability

- Exact observability on $[0,\, t]$
  - $\star$ $\mathcal{O}_t$ one-to-one and $\mathcal{O}_t^{-1}$ bounded on the range of $\mathcal{O}_t$
- Approximate observability on $[0,\, t]$
  - $\star$ $\mathrm{null}\,(\mathcal{O}_t) = 0$
- $(\mathcal{A},\, \cdot\,, \mathcal{C})$ approximately obsv on $[0,\, t]$ $\Leftrightarrow$ $(\mathcal{A}^\dagger, \mathcal{C}^\dagger,\, \cdot\,)$ approximately ctrb on $[0,\, t]$

approximate observability on $[0,\, t]$

$$\Updownarrow$$

$$\mathcal{V}_t \;>\; 0 \quad \Leftrightarrow \quad \{\langle \psi, \mathcal{V}_t\, \psi \rangle \;>\; 0, \text{ for all } 0 \neq \psi \in \mathbb{H}\}$$

or

$$\mathrm{null}\,(\mathcal{O}_t) \;=\; 0 \quad \Leftrightarrow \quad \{\mathcal{C}\,\mathcal{T}(\tau)\,\psi \;=\; 0 \text{ on } [0,\, t] \;\Rightarrow\; \psi = 0\}$$

Infinite horizon Gramians

- Exponentially stable C_0-semigroup $\mathcal{T}(t)$

$$\exists\, M,\, \alpha \;>\; 0 \quad\Rightarrow\quad \|\mathcal{T}(t)\| \;\leq\; M\,\mathrm{e}^{-\alpha t}$$

- Extended (i.e., infinite horizon) Gramians

$$\begin{aligned}
\mathcal{P} \;&=\; \mathcal{R}_\infty\,\mathcal{R}_\infty^\dagger \;&=\; \int_0^\infty \mathcal{T}(\tau)\,\mathcal{B}\,\mathcal{B}^\dagger\,\mathcal{T}^\dagger(\tau)\,\mathrm{d}\tau \\
\mathcal{V} \;&=\; \mathcal{O}_\infty^\dagger\,\mathcal{O}_\infty \;&=\; \int_0^\infty \mathcal{T}^\dagger(\tau)\,\mathcal{C}^\dagger\,\mathcal{C}\,\mathcal{T}(\tau)\,\mathrm{d}\tau
\end{aligned}$$

- Approximate controllability

$$\mathcal{P} > 0 \quad\Leftrightarrow\quad \mathrm{null}\left(\mathcal{R}_\infty^\dagger\right) \;=\; 0$$

- Approximate observability

$$\mathcal{V} > 0 \quad\Leftrightarrow\quad \mathrm{null}\left(\mathcal{O}_\infty\right) \;=\; 0$$

Lyapunov equations

Controllability Gramian $\mathcal{P}$ – unique self-adjoint solution to:

$$\left\langle \mathcal{A}^\dagger \psi_1, \mathcal{P}\,\psi_2 \right\rangle \;+\; \left\langle \mathcal{P}\,\psi_1, \mathcal{A}^\dagger \psi_2 \right\rangle \;=\; -\left\langle \mathcal{B}^\dagger \psi_1, \mathcal{B}^\dagger \psi_2 \right\rangle \quad \text{for } \psi_1,\, \psi_2 \in \mathcal{D}\left(\mathcal{A}^\dagger\right)$$

$$\Updownarrow$$

$$\mathcal{P}\,\mathcal{D}\left(\mathcal{A}^\dagger\right) \subset \mathcal{D}\left(\mathcal{A}\right) \;\text{ and }\; \mathcal{A}\,\mathcal{P}\,\psi \;+\; \mathcal{P}\,\mathcal{A}^\dagger\,\psi \;=\; -\,\mathcal{B}\,\mathcal{B}^\dagger\,\psi \quad \text{for } \psi \in \mathcal{D}\left(\mathcal{A}^\dagger\right)$$

Observability Gramian $\mathcal{V}$ – unique self-adjoint solution to:

$$\left\langle \mathcal{A}\,\psi_1, \mathcal{V}\,\psi_2 \right\rangle \;+\; \left\langle \mathcal{V}\,\psi_1, \mathcal{A}\,\psi_2 \right\rangle \;=\; -\left\langle \mathcal{C}\,\psi_1, \mathcal{C}\,\psi_2 \right\rangle \quad \text{for } \psi_1,\, \psi_2 \in \mathcal{D}\left(\mathcal{A}\right)$$

$$\Updownarrow$$

$$\mathcal{V}\,\mathcal{D}\left(\mathcal{A}\right) \subset \mathcal{D}\left(\mathcal{A}^\dagger\right) \;\text{ and }\; \mathcal{A}^\dagger\,\mathcal{V}\,\psi \;+\; \mathcal{V}\,\mathcal{A}\,\psi \;=\; -\,\mathcal{C}^\dagger\,\mathcal{C}\,\psi \quad \text{for } \psi \in \mathcal{D}\left(\mathcal{A}\right)$$

Controllability of Riesz-spectral systems

$$\psi_t(x,t) \;=\; [\mathcal{A}\,\psi(\cdot,t)]\,(x) \,+\, \sum_{i\,=\,1}^{m} b_i(x)\,u_i(t)$$

modal controllability $\Leftrightarrow$ approximate controllability

$\mathcal{A}$ – Riesz-spectral operator with e-pair $\{(\lambda_n, v_n)\}_{n\,\in\,\mathbb{N}}$

$\{w_n\}_{n\,\in\,\mathbb{N}}$ – e-functions of $\mathcal{A}^\dagger$ s.t. $\langle w_n, v_m\rangle \;=\; \delta_{nm}$

$$[\mathcal{A}\,f]\,(x) \;=\; \sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\,\langle w_n, f\rangle$$

$$\downarrow$$

approximate controllability $\Leftrightarrow$ $\mathrm{rank}\left(\left[\;\langle w_n, b_1\rangle \;\;\cdots\;\; \langle w_n, b_m\rangle\;\right]\right) \;=\; 1$

- Necessary condition for controllability
  - ⋆ Number of controls $\geq$ maximal multiplicity of e-vectors of $\mathcal{A}$

Example (to be done in class)

- Diffusion equation on $L_2\,[-1,1]$ with Dirichlet BCs

$$\begin{aligned} \psi_t(x,t) &= \psi_{xx}(x,t) \;+\; b(x)\,u(t) \\ \psi(x,0) &= \psi_0(x) \\ \psi(\pm 1,t) &= 0 \end{aligned}$$

Diagonal coordinate form

$$\dot{\alpha}_n(t) \;=\; -\left(\frac{n\pi}{2}\right)^2 \alpha_n(t) \;+\; \underbrace{\langle v_n, b\rangle}_{b_n}\, u(t), \quad n \in \mathbb{N}$$

approximate/modal controllability $\;\Leftrightarrow\;$ $\{b_n \neq 0$, for all $n \in \mathbb{N}\}$