Two-point boundary value problems: $\boxed{10/27/11}$

$$\Psi'(x) = A(x)\Psi(x) + B(x)u(x),$$
$$\phi(x) = C(x)\Psi(x)$$
$$\nu = N_a \Psi(a) + N_b \Psi(b)$$

Aside: Curtain and Morris Automatica 2009 transfer function for infinite dimensional systems. (spatially distributed)

→ Solution:

$$\phi(x) = C(x)\Phi(x,a)\left(N_a + N_b\Phi(b,a)\right)^{-1}\nu +$$
$$+ C(x)\int_a^x \Phi(x,\xi)B(\xi)d(\xi)\,d\xi -$$
$$- C(x)\Phi(x,a)\left(N_a + N_b\Phi(b,a)\right)^{-1} N_b \int_a^b \Phi(b,\xi)B(\xi)\,d(\xi)\,d\xi$$

* formula is useful when we have analytical solutions. That is, it useful for symbolic computation using Mathematica.
* Naive approach: compute $\Phi(x,\xi)$ numerically using marching algorithms.

  This approach may give numerical junk.
* another way: bvp4c and chebfun.

bvp4c: two point boundary value problem solver in Matlab

chebfun: coming soon: powerful numerical solvers, for boundary value problems, and more

Controlability and Observability

ability to steer states

ability to estimate states

important:

- grammians
- operator Lyapunov equations

__An example:__

$$\psi_t(x,t) = \psi_{xx}(x,t) + b(x)u(t)$$

$$\phi(t) = \int_{-1}^{1} c(x)\,\psi(x,t)\,dx$$

$$\psi(x,0) = \psi_0(x)$$

$$\psi(\pm 1, t) = 0$$

diffusion eq on $L_2[-1,1]$ with point actuation and sensing.

$$b(x) = \frac{1}{2\varepsilon}\,\mathbb{1}_{[x_c-\varepsilon,\, x_c+\varepsilon]}(x)$$

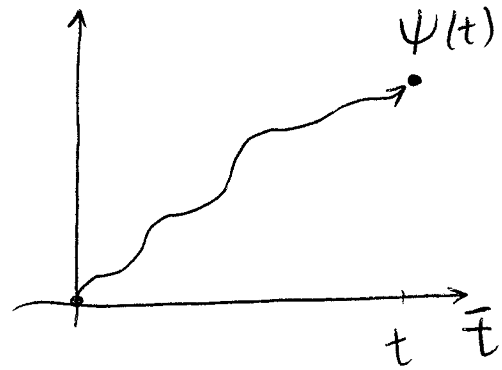
$$c(x) = \frac{1}{2\delta}\,\mathbb{1}_{[x_s-\delta,\, x_s+\delta]}(x)$$

$$\mathbb{1}_{[a,b]}(x) = \begin{cases} 1, & x \in [a,b] \\ 0, & \text{otherwise} \end{cases}$$

Controllability operator and grammian

$$\left.\begin{aligned} \psi_t &= A\psi + Bu \\ \psi(0) &= 0 \end{aligned}\right\}$$ want to study influence of control on $\psi(t)$

$$\psi(t) = \int_0^t T(t-\tau)Bu(\tau)\,d\tau$$



Q: Can we choose $u[0,t]$ such that we bring our system from $\psi(0)=0$ to a given $\psi_f = \psi(t)$?

abstractly: $\psi(t) = [R_t\, u](t) = \int_0^t T(t-\tau)Bu(\tau)\,d\tau$

$$R_t : L_2([0,t];\mathcal{U}) \longrightarrow \mathbb{H}$$

$$\downarrow$$

$$L_2[0,t] = \left\{ f ;\ \int_0^t f^*(\tau)f(\tau)\,d\tau < +\infty \right\}$$

$$\downarrow$$

$$L_2([0,t];\mathbb{C}^n)$$

In general:

$$L_2([0,t];\mathcal{U}) = \left\{ u ;\ \int_0^t \underbrace{\langle u(\tau), u(\tau)\rangle_{\mathcal{U}}}_{\text{e.g. } \int_{-1}^{1} u^*(x,\tau)\,u(x,\tau)\,dx} d\tau < +\infty \right\}$$

Adjoint: $[R_t^+ \psi](\tau) = B^+ T^+(t-\tau)$, $\tau \in [0,t]$

Controlability grammian:

$$P_t = R_t R_t^+ = \int_0^t T(\tau) B B^+ T^+(\tau) \, d\tau .$$

* exact contrölability : $\text{range}(R_t) = \mathbb{H}$
  - rarely-satisfied by infinite dimensional systems
  - never satisfied for systems with finite dimensional $\mathbb{U}$
* approximate controlability : $\overline{\text{range}(R_t)} = \mathbb{H}$

Observability operator and grammian:

$$O_t : \mathbb{H} \longrightarrow L_2([0,t]; \mathbb{Y})$$

$$\phi(t) = [O_t \psi(0)](t) = C T(t) \psi(0)$$

grammian:

$$\mathcal{V}_t = O_t^+ O_t = \int_0^t T^+(\tau) C^+ C T(\tau) \, d\tau . \quad \text{(finite horizon grammian)}$$

* $(A, \cdot, C)$ is approximately obser. on $[0,t]$
  $\iff (A^+, C^+, \cdot)$ is approx. contr. on $[0,t]$.

approx contr on $[0,t] \iff$

1) $P_t > 0 \iff \{ \langle \psi, P_t \psi \rangle > 0, \ \forall \psi \neq 0 \in \mathbb{H} \}$

or

2) $\text{null}(R_t^+) = 0 \iff \{ B^+ T^+(\tau) \psi = 0 \text{ on } [0,t] \implies \psi = 0 \}$

Idea: $P_t = R_t R_t^\dagger$

$$\langle \psi, P_t \psi \rangle > 0$$

$$\langle \psi, R_t R_t^\dagger \psi \rangle = \langle \underbrace{R_t^\dagger \psi}_{\phi}, \underbrace{R_t^\dagger \psi}_{\phi} \rangle > 0$$

$\rightarrow$ This shows: $\operatorname{null}(R_t^\dagger) = \{0\} \iff P_t > 0$

$\rightarrow$ $(\operatorname{null}(R_t^\dagger))^\perp = \overline{\operatorname{range}(R_t)} = \mathbb{H}$

$$R_t : L_2([0,t]; \mathbb{U}) \longrightarrow \mathbb{H}$$

$$R_t^\dagger : \mathbb{H} \longrightarrow L_2([0,t]; \mathbb{U})$$

$$\begin{tikzcd} \mathbb{H} & L_2 \arrow[l, bend left, "R_t"] & \mathbb{H} \arrow[l, bend left, "R_t^\dagger"] \arrow[ll, bend right, "P_t"'] \end{tikzcd}$$

Infinite horizon grammians:

$$P = R_\infty R_\infty^\dagger = \int_0^\infty T(\tau) B B^\dagger T^\dagger(\tau)\, d\tau$$

$$\mathcal{V} = \mathcal{O}_\infty^\dagger \mathcal{O}_\infty = \int_0^\infty T^\dagger(\tau) C^\dagger C\, T(\tau)\, d\tau.$$

Lyapunov equations:

$$\langle A^\dagger \psi_1, P\psi_2 \rangle + \langle P\psi_1, A^\dagger \psi_2 \rangle = -\langle B^\dagger \psi_1, B^\dagger \psi_2 \rangle$$

for all $\psi_1, \psi_2 \in D(A^\dagger)$

$$\left.\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx\end{array}\right\} \quad \begin{array}{ll} AP + PA^* = -BB^* & \ldots (1) \\ A^*V + VA = -C^*C & \ldots (2)\end{array}$$

Operator versions of (1) and (2) given by

$$AP + PA^\dagger = -BB^\dagger \quad \ldots (3)$$

$$A^\dagger V + VA = -C^\dagger C \quad \ldots (4)$$

where $A^\dagger$, $B^\dagger$, $C^\dagger$ are determined using proper inner products.

Discretization of (3) and (4) does not necessarily give you problem of the form:

$$A_d P_d + P_d A_d^* = -B_d B_d^*.$$