Lecture 15: Systems with inputs

- Input types
  - ⋆ Additive inputs
  - ⋆ Boundary inputs
- Input-output mappings
  - ⋆ Transfer function
  - ⋆ Frequency response
  - ⋆ Impulse response
- Abstract evolution equation for boundary control systems
  - ⋆ Objective: bring system into a form that resembles standard formulation
- Two point boundary value problems

Additive inputs

- Example: diffusion equation on $L_2\,[-1,1]$ with Dirichlet BCs

$$\begin{aligned}\phi_t(x,t) &= \phi_{xx}(x,t) \,+\, u(x,t)\\ \phi(x,0) &= \phi_0(x)\\ \phi(\pm 1,t) &= 0\end{aligned}$$

- Abstract evolution equation

$$\begin{aligned}\psi_t(t) &= \mathcal{A}\,\psi(t) \,+\, u(t)\\ \mathcal{A} &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \mathcal{D}(\mathcal{A}) \,=\, \{f \,\in\, L_2\,[-1,\,1],\; f'' \,\in\, L_2\,[-1,\,1],\; f(\pm 1) \,=\, 0\}\end{aligned}$$

- Solution

$$\psi(t) \,=\, \mathcal{T}(t)\,\psi(0) \,+\, \int_0^t \mathcal{T}(t\,-\,\tau)\,u(\tau)\,\mathrm{d}\tau$$

$\mathcal{T}(t)$: C_0-semigroup generated by $\mathcal{A}$

Input-output maps

$$\begin{aligned} \psi_t(t) &= \mathcal{A}\,\psi(t) \,+\, \mathcal{B}\,u(t) \\ \phi(t) &= \mathcal{C}\,\psi(t) \end{aligned}$$

- Underlying operators: $\begin{cases} \mathcal{A}: & \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H} \\ \mathcal{B}: & \mathbb{U} \longrightarrow \mathbb{H} \\ \mathcal{C}: & \mathbb{H} \longrightarrow \mathbb{Y} \end{cases}$

- Input-output mapping

$$\phi(t) \;=\; [\,\mathcal{H}\,u\,]\,(t) \;=\; \int_0^t \mathcal{C}\,\mathcal{T}(t \,-\, \tau)\,\mathcal{B}\,u(\tau)\,\mathrm{d}\tau$$

⋆ Impulse response

$$\mathcal{H}(t) \;=\; (\mathcal{C}\,\mathcal{T}(t)\,\mathcal{B})\,\mathbb{1}(t)$$

⋆ Transfer function

$$\mathcal{H}(s) \;=\; \mathcal{C}\,(sI \,-\, \mathcal{A})^{-1}\,\mathcal{B}$$

⋆ Frequency response

$$\mathcal{H}(\mathrm{j}\omega) \;=\; \mathcal{C}\,(\mathrm{j}\omega I \,-\, \mathcal{A})^{-1}\,\mathcal{B}$$

An example

$$\left.\begin{array}{rcl} \phi_t(x,t) & = & \phi_{xx}(x,t) \,+\, u(x,t) \\ \phi(\pm 1,t) & = & 0 \end{array}\right\} \xrightarrow[\text{transform}]{\text{Laplace}} \left\{\begin{array}{rcl} \phi''(x,s) & = & s\,\phi(x,s) \,-\, u(x,s) \\ \phi(\pm 1,s) & = & 0 \end{array}\right.$$

- Spatial realization of $\mathcal{H}(s)$ (with $\psi_1 = \phi$, $\psi_2 = \phi'$)

$$\begin{aligned} \begin{bmatrix} \psi_1'(x,s) \\ \psi_2'(x,s) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u(x,s) \\ \phi(x,s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} \\ 0 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(-1,s) \\ \psi_2(-1,s) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(1,s) \\ \psi_2(1,s) \end{bmatrix} \end{aligned}$$

- Two point boundary value problem

$$\begin{aligned} \psi'(x) &= A(x)\,\psi(x) \,+\, B(x)\,u(x) \\ \phi(x) &= C(x)\,\psi(x) \\ 0 &= N_a\,\psi(a) \,+\, N_b\,\psi(b) \end{aligned}$$

Boundary control

- Example: diffusion equation on $L_2\,[-1,1]$

$$\left.\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) + d(x,t) \\ \phi(-1,t) &= u(t) \\ \phi(+1,t) &= 0 \end{aligned}\right\} \xrightarrow[\text{transform}]{\text{Laplace}} \left\{\begin{aligned} \phi''(x,s) &= s\,\phi(x,s) - d(x,s) \\ \phi(-1,s) &= u(s) \\ \phi(+1,s) &= 0 \end{aligned}\right.$$

- Spatial realization of $\mathcal{H}(s)$ (with $\psi_1 = \phi$, $\psi_2 = \phi'$)

$$\begin{aligned} \begin{bmatrix} \psi_1'(x,s) \\ \psi_2'(x,s) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} d(x,s) \\ \phi(x,s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} \\ \begin{bmatrix} u(s) \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(-1,s) \\ \psi_2(-1,s) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(1,s) \\ \psi_2(1,s) \end{bmatrix} \end{aligned}$$

- Two point boundary value problem

$$\begin{aligned} \psi'(x) &= A(x)\,\psi(x) + B(x)\,d(x) \\ \phi(x) &= C(x)\,\psi(x) \\ \nu &= N_a\,\psi(a) + N_b\,\psi(b) \end{aligned}$$

Abstract evolution equation for systems with boundary inputs

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) + d(x,t) \\ \phi(-1,t) &= u(t) \\ \phi(+1,t) &= 0 \end{aligned}$$

- Problem: control doesn't enter additively into the equation
- Coordinate transformation

$$\psi(x,t) = \phi(x,t) - f(x)\,u(t)$$

  - ⋆ Choose $f(x)$ to obtain homogeneous boundary conditions $\psi(\pm 1,t) = 0$
  - ⋆ Many possible choices

Conditions for selection of f:

$$\{f(-1) = 1,\ f(1) = 0\} \xrightarrow{\text{simple option}} f(x) = \frac{1 - x}{2}$$

- In new coordinates:

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) + d(x,t) \\ \phi(-1,t) &= u(t) \\ \phi(+1,t) &= 0 \end{aligned}$$

$$\Big\downarrow \; \phi(x,t) = \psi(x,t) + f(x)\, u(t)$$

$$\begin{aligned} \psi_t(x,t) + f(x)\,\dot{u}(t) &= \psi_{xx}(x,t) + f''(x)\,u(t) + d(x,t) \\ \psi(\pm 1,t) &= 0 \end{aligned}$$

- New input: $v(t) = \dot{u}(t)$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \psi(t) \\ u(t) \end{bmatrix} &= \begin{bmatrix} \mathcal{A}_0 & f'' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \psi(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} d(t) + \begin{bmatrix} -f \\ I \end{bmatrix} v(t) \\ \phi(t) &= \begin{bmatrix} I & f \end{bmatrix}\begin{bmatrix} \psi(t) \\ u(t) \end{bmatrix} \end{aligned}$$

$$\mathcal{A}_0 = \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \;\; \mathcal{D}(\mathcal{A}_0) = \{f \in L_2[-1,\,1],\; f'' \in L_2[-1,\,1],\; f(\pm 1) = 0\}$$

Two point boundary value problems

$$
\begin{aligned}
\psi'(x) &= A(x)\,\psi(x) \;+\; B(x)\,d(x) \\
\phi(x) &= C(x)\,\psi(x) \\
\nu &= N_a\,\psi(a) \;+\; N_b\,\psi(b)
\end{aligned}
$$

- Solution:

$$
\begin{aligned}
\phi(x) \;=\; & C(x)\,\Phi(x,a)\,(N_a \,+\, N_b\,\Phi(b,a))^{-1}\,\nu \,+\, C(x)\int_a^x \Phi(x,\xi)\,B(\xi)\,d(\xi)\,\mathrm{d}\xi \;- \\
& C(x)\,\Phi(x,a)\,(N_a \,+\, N_b\,\Phi(b,a))^{-1}\,N_b\int_a^b \Phi(b,\xi)\,B(\xi)\,d(\xi)\,\mathrm{d}\xi
\end{aligned}
$$

$\Phi(x,\xi)$: the state transition matrix of $A(x)$

$$
\frac{\mathrm{d}\Phi(x,\xi)}{\mathrm{d}x} \;=\; A(x)\,\Phi(x,\xi), \;\; \Phi(\xi,\xi) \;=\; I
$$

For systems with $A \neq A(x)$:

$$
\Phi(x,\xi) \;=\; \mathrm{e}^{A(x-\xi)}
$$

Examples

- Heat equation with boundary actuation

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) \\ \phi(-1,t) &= u(t) \\ \phi(+1,t) &= 0 \end{aligned}$$

$$\Big\downarrow \text{Laplace trasform}$$

$$\begin{aligned} \begin{bmatrix} \psi_1'(x,s) \\ \psi_2'(x,s) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} \\ \phi(x,s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} \\ \begin{bmatrix} u(s) \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(-1,s) \\ \psi_2(-1,s) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(1,s) \\ \psi_2(1,s) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \phi(x,s) &= C\,\mathrm{e}^{A(s)(x-a)} \left(N_a + N_b\,\mathrm{e}^{A(s)(b-a)}\right)^{-1} \nu(s) \\ &= \frac{\sinh\left(\sqrt{s}\,(1-x)\right)}{\sinh\left(2\sqrt{s}\right)}\, u(s) \\ &= \left(\frac{1-x}{2} - \sum_{n=1}^{\infty} \frac{2}{n\pi} \frac{s}{s + (n\pi/2)^2}\, v_n(x)\right) u(s) \end{aligned}$$

- Eigenvalue problem for streamwise constant linearized NS equations

Orr-Sommerfeld:
$$\left\{\begin{array}{rclrcl} \mathcal{L}\,\psi_{os} & = & \lambda_{os}\,\psi_{os}, & \psi_{os}(\pm 1) & = & \psi'_{os}(\pm 1) \; = \; 0 \\ \mathcal{S}\,u_{os} & = & \lambda_{os}\,u_{os} \; - \; \mathcal{C}_p\,\psi_{os}, & u_{os}(\pm 1) & = & 0 \end{array}\right.$$

$$\downarrow$$

$$\left\{\begin{array}{rclrcl} \Delta^2\,\psi_{os} & = & \lambda_{os}\,\Delta\,\psi_{os}, & \psi_{os}(\pm 1) & = & \psi'_{os}(\pm 1) \; = \; 0 \\ \Delta\,u_{os} & = & \lambda_{os}\,u_{os} \; - \; \mathrm{j}k_z U'(y)\,\psi_{os}, & u_{os}(\pm 1) & = & 0 \end{array}\right.$$

Two point boundary value problem for u_{os}:

$$\begin{aligned} \begin{bmatrix} x'_1(y,k_z) \\ x'_2(y,k_z) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \lambda_{os} + k_z^2 & 0 \end{bmatrix} \begin{bmatrix} x_1(y,k_z) \\ x_2(y,k_z) \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathrm{j}k_z U'(y) \end{bmatrix} \psi_{os}(y,k_z) \\ u_{os}(y,k_z) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(y,k_z) \\ x_2(y,k_z) \end{bmatrix} \\ 0 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(-1,k_z) \\ x_2(-1,k_z) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(1,k_z) \\ x_2(1,k_z) \end{bmatrix} \end{aligned}$$