Systems with inputs

Two input types:

* Additive inputs
* Boundary inputs

Heat equation example for additive inputs

$$\phi_t(x,t) = \phi_{xx}(x,t) + u(x,t)$$
$$\phi(x,0) = \phi_0(x)$$
$$\phi(\pm 1, t) = 0$$

Abstract evolution equation

$$\psi_t(t) = A\psi(t) + u(t)$$

$$A = \frac{d^2}{dx^2}, \quad D(A) = \{ f \in L_2[-1,1],\ f'' \in L_2[-1,1],\ f(\pm 1) = 0 \}$$

Solution $\quad \psi(t) = T(t)\psi(0) + \int_0^t T(t-\tau)u(\tau)\,d\tau$

$$\phi(x,t) = \sum_{n=1}^{+\infty} a_n(t) v_n(x)$$

$$[T(t)\phi(0)](x) = \sum_{n=1}^{+\infty} e^{\lambda_n t} v_n(x) \langle v_n, \phi(0) \rangle$$

$$\dot{a}_n(t) = \lambda_n a_n(t)$$
$$\lambda_n \to -\left(\frac{n\pi}{2}\right)^2$$

$$\int_0^t T(t-\tau)u(\tau)\,d\tau = \sum_{n=1}^{+\infty} \int_0^t e^{\lambda_n(t-\tau)} v_n(x) \langle v_n, u(\tau) \rangle\,d\tau$$

Input – Output maps

$$\psi_t(t) = A\psi(t) + Bu(t)$$
$$\phi(t) = C\psi(t)$$

Input-output mapping

$$\phi(t) = [\mathcal{H}u](t) = \int_0^t CT(t-\tau)B\,d\tau$$

* Impulse response

$$\mathcal{H}(t) = (\mathcal{C}\,\mathcal{T}(t)\,\mathcal{B})\,\mathbb{1}(t)$$

* Transfer function

$$\mathcal{H}(s) = C(sI - A)^{-1}B$$

* Frequency response

$$\mathcal{H}(j\omega) = C(j\omega I - A)^{-1}B$$

An example

$$\left.\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) + d(t) \\ \phi(\pm 1, t) &= 0 \end{aligned}\right\} \xrightarrow{\text{Laplace transform}} \left\{\begin{aligned} \phi''(x,s) &= s\,\phi(x,s) - d(x,s) \\ \phi(\pm 1, s) &= 0 \end{aligned}\right.$$

$$\begin{bmatrix} \psi_1'(x,s) \\ \psi_2'(x,s) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} d(x,s)$$

$$\phi(x,s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix}$$

$$0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(-1,s) \\ \psi_2(-1,s) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(1,s) \\ \psi_2(1,s) \end{bmatrix}$$

Two point boundary value problems

$$\begin{aligned} \psi'(x) &= A(x)\,\psi(x) + B(x)\,d(x) \\ \psi(x) &= C(x)\,\psi(x) \end{aligned}$$

$$0 = N_a\,\psi(a) + N_b\,\psi(b)$$

For boundary inputs

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) + d(x,t) \\ \phi(-1,t) &= u(t) \\ \phi(+1,t) &= 0 \end{aligned}$$

Problem : control does not enter additively into the equation

Define :

$$\psi(x,t) = \phi(x,t) - f(x)\,u(t)$$

$$\phi(x=-1,t) = u(t)$$

$$\phi(x=+1,t) = 0$$

determine $f(x)$ s.t.

$$\psi(x=\pm 1, t) = 0$$

$$\psi(-1,t) = \phi(-1,t) - f(-1)\,u(t) = u(t) - f(-1)\,u(t) \overset{\text{want}}{=} 0 \iff f(-1) = 1$$

$$\psi(1,t) = \phi(1,t) - f(1)u(t) = 0 - f(1)u(t) \overset{\text{want}}{=} 0 \Leftarrow f(1) = 0$$

Many choices for $f(x)$, for example, $\boxed{f(x) = \frac{1-x}{2}}$

In new coordinates :

$$\psi_t(x,t) + f(x)\,\dot{u}(t) = \psi_{xx}(x,t) + f''(x)\,u(t) + d(x,t)$$

$$\psi(\pm 1,t) = 0$$

New input : $v(t) = \dot{u}(t)$

$$\frac{d}{dt}\begin{bmatrix} \psi(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_0 & f'' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \psi(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} d(t) + \begin{bmatrix} -f \\ I \end{bmatrix} v(t)$$

$$\phi(t) = [\, I \quad f \,]\begin{bmatrix} \psi(t) \\ u(t) \end{bmatrix}$$

Note : Curtain's book (More general)

$$\phi_t = A\phi$$

$\mathcal{E}\phi(t) = u(t)$, point evaluation functional

$$[\quad \mathcal{E} \quad]\begin{bmatrix} \\ \phi(t) \\ \\ \end{bmatrix} = u(t)$$

Right inverse of $\mathcal{E}$

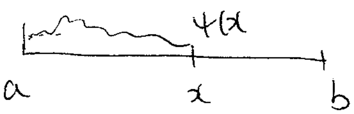
$\mathcal{E}F = I$, in our example, we select $F = f(x)$, $f(-1) = 1$, $f(+1) = 0$

$$\mathcal{E}F \cdot u(t) = u(t)$$

Two point boundary value problems

$$\psi' = A\psi + Bd \qquad \text{--- (1)}$$

$$r = N_a \psi(a) + N_b \psi(b) \qquad \text{--- (2)}$$



$$\psi(x) = e^{A(x-a)} \underbrace{\psi(a)}_{\uparrow \text{ unknown!}} + \int_a^x e^{A(x-\xi)} B d(\xi)\, d\xi \qquad \text{--- (3)}$$

evaluation (3) at $x = b$ and plug into (2)

$$\Rightarrow \quad r = \underbrace{\left[N_a + N_b e^{A(b-a)} \right]}_{\text{If invertible}} \psi(a) + N_b \int_a^b e^{A(b-\xi)} B d(\xi) . d\xi$$

If invertible, then $\psi(a) = f(r, d)$