$$\text{Energy} = \frac{1}{2}\left(\langle u,u\rangle + \langle v,v\rangle + \langle w,w\rangle \right)$$

where: $$\langle u,u\rangle = \int_{-1}^{1} u^{*}(y,k_z,t)\, u(y,k_z,t)\, dy$$

$$\downarrow$$

$$E_u(k_z,t) \quad \ldots \quad \text{energy density at } k_z$$

$$\begin{aligned}
\langle v,v\rangle + \langle w,w\rangle &= \langle \psi_z, \psi_z\rangle + \langle -\psi_y, -\psi_y\rangle \\
&= \langle jk_z\psi, jk_z\psi\rangle + \langle -\psi_y, -\psi_y\rangle \\
&= \langle \psi, k_z^2\psi\rangle + \langle \psi, -\psi_{yy}\rangle \\
&= \langle \psi, -\Delta\psi\rangle
\end{aligned}$$

$$\phi = \begin{bmatrix} \psi \\ u \end{bmatrix}$$

$$\begin{aligned}
\langle u,u\rangle + \langle v,v\rangle + \langle w,w\rangle &= \langle u,u\rangle + \langle \psi, -\Delta\psi\rangle \\
&= \langle \phi, \phi\rangle_e \\
&= \left\langle \begin{bmatrix} \psi \\ u \end{bmatrix}, \begin{bmatrix} -\Delta & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \psi \\ u \end{bmatrix} \right\rangle
\end{aligned}$$

Simple finite-dimensional example

$$\begin{bmatrix} \dot\phi_1 \\ \dot\phi_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}}_{A} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}$$

$k \neq 0 \Rightarrow$ A is not normal .... $AA^T \neq A^TA$

If A has a full set of linearly independent eigenvectors, we can still bring A to diagonal form, <u>But</u> the eigenvectors are not going to be orthonormal.

$$\boxed{A v_i = \lambda_i v_i} \longrightarrow A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$AV = V\Lambda \Rightarrow A = V\Lambda V^{-1}$$

where $V^{-1} \neq V^*$

(not unitary)

Let $V^{-1} = W^*$

$$\boxed{A = V\Lambda W^*}$$

Can show $\underbrace{V^{-1}}_{W^*} A = \Lambda W^*$

$$\begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix} \Rightarrow A^* \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \Lambda^*$$

$$\Rightarrow \boxed{A^* w_i = \bar\lambda_i w_i}$$

So, $$Af = \sum_{i=1}^{n} \lambda_i v_i \langle w_i, f \rangle$$

Action of A on f is determined by a linear combination of the right eigenvectors of A (v_i) with coefficients $\underbrace{\lambda_i \langle w_i, f \rangle}_{\text{modal contribution of } f}$

modal contribution of f ~~in the direction of the i-th left eigenvector of A~~

Important: Eigenvectors are not orthonormal, so even though eigenvalues of A are negative and ~~both~~ all modes decay to zero at large times, there could be large transients ~~[illegible]~~.