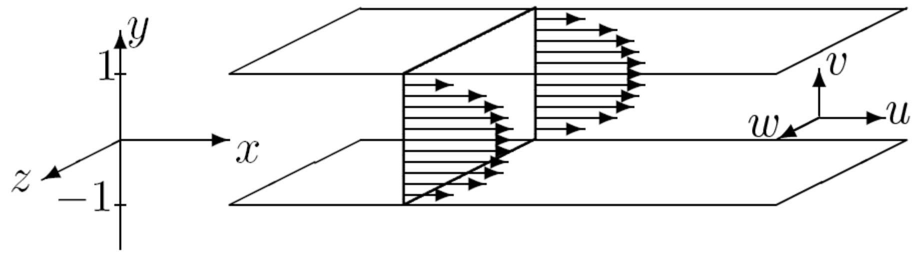
Lectures 13 & 14: . . . and a bit of fluids

- Themes:
  - ⋆ Linearized Navier-Stokes (NS) equations in a channel flow
  - ⋆ Inner product that induces kinetic energy
  - ⋆ Non-normal nature of the dynamical generator
  - ⋆ Riesz spectral basis

- Approach: informal discussion using tools that we've learned so far (more later in the course)

Channel flow



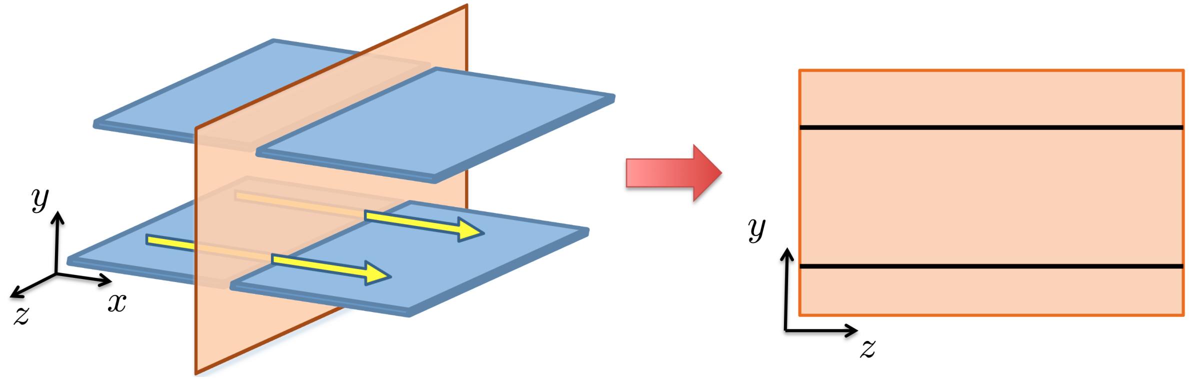
- Steady-state solution: $\begin{bmatrix} U(y) & 0 & 0 \end{bmatrix}^T$
- Linearized NS and continuity equations

$$
\begin{array}{rcccccl}
u_t & + & U(y)\,u_x & + & U'(y)\,v & = & -\,p_x + \dfrac{1}{Re}\,\Delta\,u \\
v_t & + & U(y)\,v_x & & & = & -\,p_y + \dfrac{1}{Re}\,\Delta\,v \\
w_t & + & U(y)\,w_x & & & = & -\,p_z + \dfrac{1}{Re}\,\Delta\,w \\
u_x & + & v_y & + & w_z & = & 0
\end{array}
$$

$$
U(y) \;=\; \begin{cases} 1 - y^2, & \text{pressure driven flow} \\ y, & \text{shear driven flow} \end{cases}
$$

$$
U'(y) \;=\; \frac{\mathrm{d}\,U(y)}{\mathrm{d}\,y} \qquad\qquad \Delta \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
$$

Streamwise constant fluctuations



- Set $(\cdot)_x = 0$

$$
\begin{aligned}
u_t &= -U'(y)\,v + \frac{1}{Re}\Delta\,u \\
v_t &= -p_y + \frac{1}{Re}\Delta\,v \\
w_t &= -p_z + \frac{1}{Re}\Delta\,w \\
0 &= v_y + w_z
\end{aligned}
$$

⋆ Define: stream-function in the (y,z)-plane $\{v = \psi_z,\ w = -\psi_y\}$

⋆ Eliminate pressure from the equations

⋆ Rewrite equations in terms of $\phi = \left[\begin{array}{cc} \psi & u \end{array}\right]^T$

Evolution model

$$\begin{bmatrix} \psi_t(t) \\ u_t(t) \end{bmatrix} = \begin{bmatrix} (1/Re)\,\mathcal{L} & 0 \\ \mathcal{C}_p & (1/Re)\,\mathcal{S} \end{bmatrix} \begin{bmatrix} \psi(t) \\ u(t) \end{bmatrix}$$

Orr-Sommerfeld: $\mathcal{L} = \Delta^{-1}\Delta^2$

Squire: $\mathcal{S} = \Delta$

Coupling: $\mathcal{C}_p = -\,U'(y)\,\partial_z$

- After Fourier transform in z

Laplacian: $\Delta = \partial_{yy} - k_z^2$

"Square of Laplacian": $\Delta^2 = \partial_{yyyy} - 2\,k_z^2\,\partial_{yy} + k_z^4$

Coupling: $\mathcal{C}_p = -\,\mathrm{j}k_z\,U'(y)$

Boundary conditions:

⋆ Dirichlet: $u(y = \pm 1, k_z, t) = 0$

⋆ Dirichlet and Neumann: $\psi(y = \pm 1, k_z, t) = \psi_y(y = \pm 1, k_z, t) = 0$

- Re-scale time: $\tau = t/Re$

$$\begin{bmatrix} \psi_\tau(\tau) \\ u_\tau(\tau) \end{bmatrix} = \begin{bmatrix} \mathcal{L} & 0 \\ Re\, \mathcal{C}_p & \mathcal{S} \end{bmatrix} \begin{bmatrix} \psi(\tau) \\ u(\tau) \end{bmatrix}$$

Inner product:

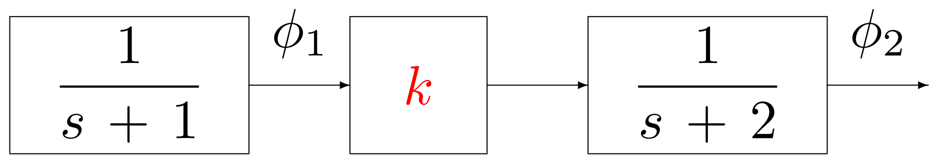
$$\begin{aligned} \langle \phi_1, \phi_2 \rangle_e &= \left\langle \begin{bmatrix} \psi_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} \psi_2 \\ u_2 \end{bmatrix} \right\rangle_e \\ &= \left\langle \begin{bmatrix} \psi_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} -\Delta & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \psi_2 \\ u_2 \end{bmatrix} \right\rangle \\ &= \langle \psi_1, -\Delta\, \psi_2 \rangle + \langle u_1, u_2 \rangle \end{aligned}$$

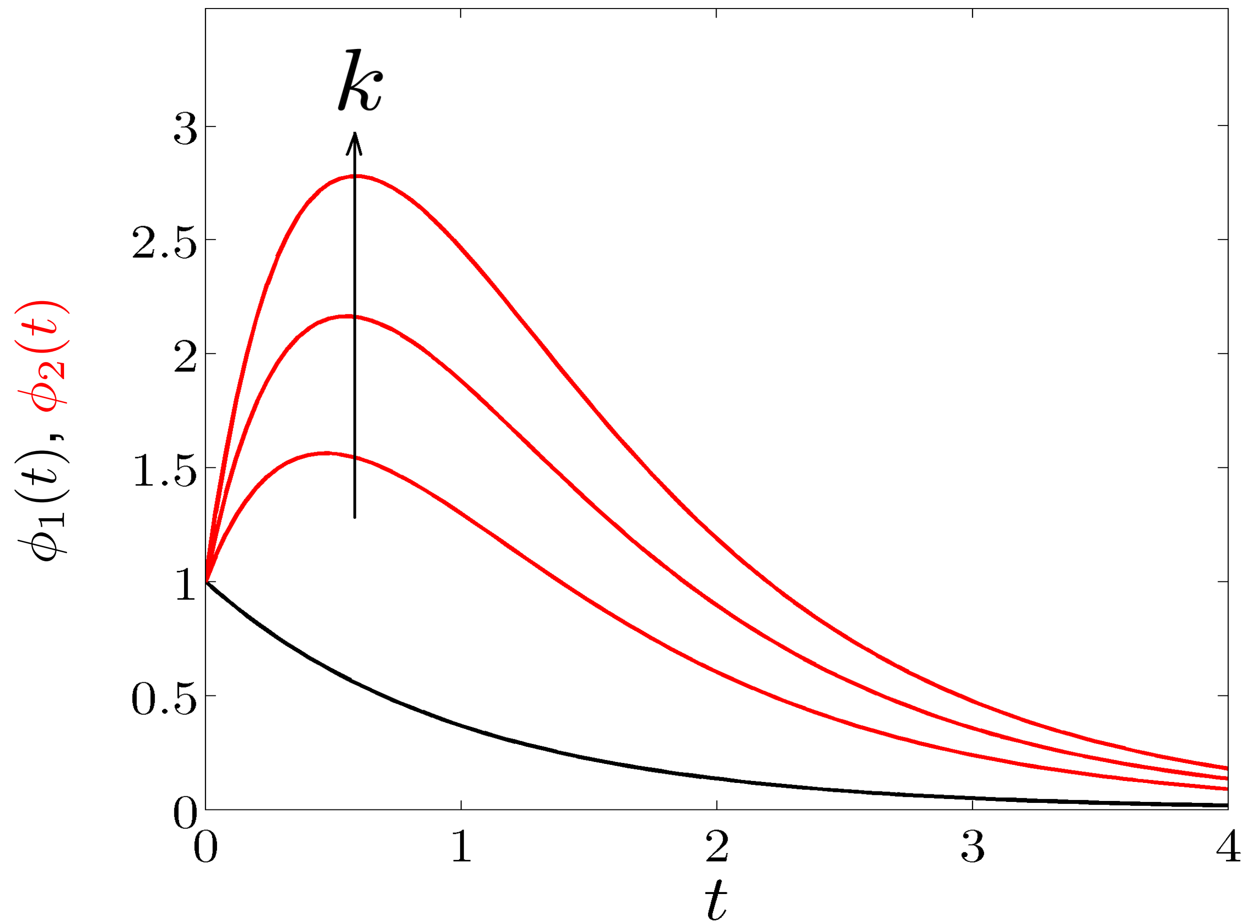
Energy:

$$\begin{aligned} E &= \frac{1}{2} \left(\langle u, u \rangle + \langle v, v \rangle + \langle w, w \rangle \right) \\ &= \frac{1}{2} \left(\langle u, u \rangle + \langle \psi, -\Delta\, \psi \rangle \right) \end{aligned}$$

A finite dimensional example

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$





$$\dot{\phi}(t) \;=\; A\,\phi(t), \qquad A\,A^* \;\neq\; A^*\,A$$

Let A have a full set of linearly independent e-vectors

$$A\,v_i \;=\; \lambda_i\,v_i \quad\Leftrightarrow\quad A\underbrace{\left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]}_{V} \;=\; \underbrace{\left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]}_{V}\underbrace{\left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array}\right]}_{\Lambda}$$

$$A^*\,w_i \;=\; \bar{\lambda}_i\,w_i \quad\Leftrightarrow\quad A^*\underbrace{\left[\begin{array}{ccc} w_1 & \cdots & w_n \end{array}\right]}_{W} \;=\; \underbrace{\left[\begin{array}{ccc} w_1 & \cdots & w_n \end{array}\right]}_{W}\underbrace{\left[\begin{array}{ccc} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{array}\right]}_{\bar{\Lambda}}$$

choose w_i such that $w_i^*\,v_j \;=\; \delta_{ij}$

- A – diagonalizable: $$A \;=\; \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]\left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array}\right]\left[\begin{array}{c} w_1^* \\ \vdots \\ w_n^* \end{array}\right]$$

- Action of A on $f \in \mathbb{C}^n$

$$
\begin{aligned}
A\,f \;&=\; \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix} f \\
&=\; \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1\, w_1^*\, f \\ \vdots \\ \lambda_n\, w_n^*\, f \end{bmatrix} \\
&=\; \lambda_1\, v_1\, w_1^*\, f \;+\; \cdots \;+\; \lambda_n\, v_n\, w_n^*\, f \\
&=\; \sum_{i\,=\,1}^{n} \lambda_i\, v_i\, \langle w_i, f \rangle
\end{aligned}
$$

- Solution to $\dot{\phi}(t) \;=\; A\,\phi(t)$

$$
\phi(t) \;=\; \mathrm{e}^{A\,t}\,\phi(0) \;=\; \sum_{i\,=\,1}^{n} \mathrm{e}^{\lambda_i\, t}\, v_i\, \langle w_i, \phi(0) \rangle
$$

- E-value decomposition of $A = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}$

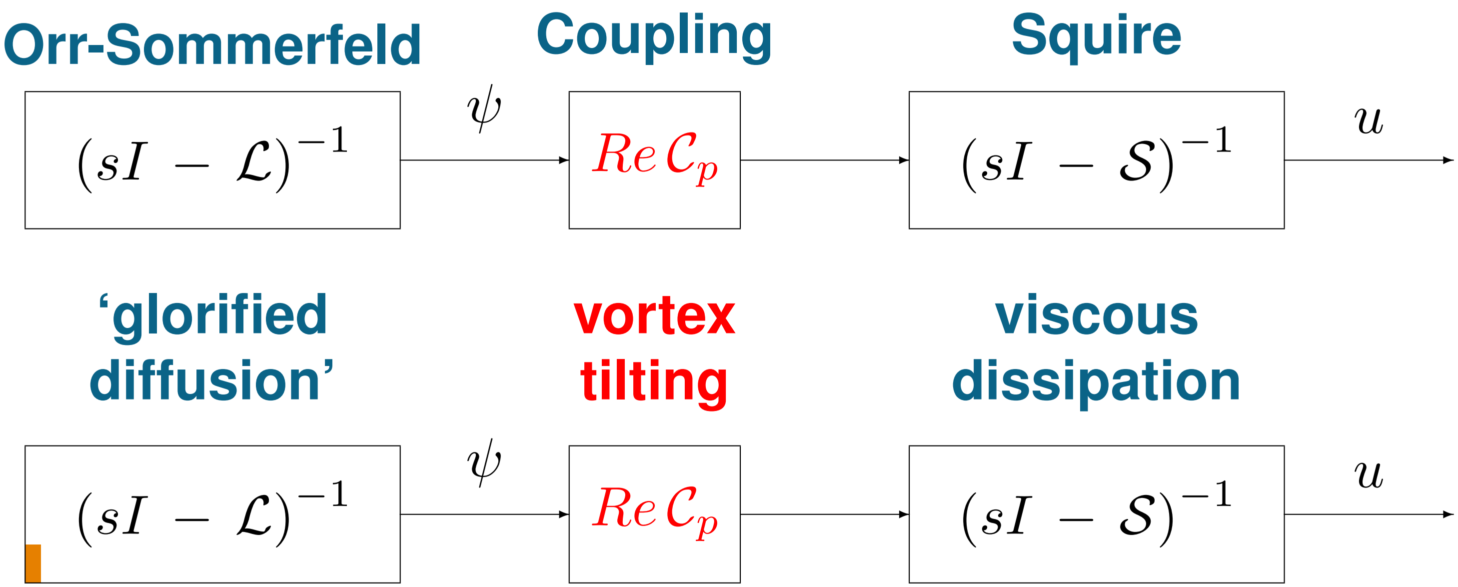
$$\{\lambda_1 = -1,\ \lambda_2 = -2\} \qquad \begin{array}{l} \left\{ v_1 = \dfrac{1}{\sqrt{1 + k^2}} \begin{bmatrix} 1 \\ k \end{bmatrix},\ v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \\ \left\{ w_1 = \begin{bmatrix} \sqrt{1 + k^2} \\ 0 \end{bmatrix},\ w_2 = \begin{bmatrix} -k \\ 1 \end{bmatrix} \right\} \end{array}$$

- Solution to $\dot{\phi}(t) = A\,\phi(t)$

$$\begin{aligned} \phi(t) &= \left(\mathrm{e}^{-t}\, v_1\, w_1^* + \mathrm{e}^{-2t}\, v_2\, w_2^* \right) \phi(0) \\ &= \begin{bmatrix} \mathrm{e}^{-t} & 0 \\ k \left(\mathrm{e}^{-t} - \mathrm{e}^{-2t} \right) & \mathrm{e}^{-2t} \end{bmatrix} \begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix} \end{aligned}$$

Back to fluids

$$\begin{bmatrix} \psi_\tau(\tau) \\ u_\tau(\tau) \end{bmatrix} = \begin{bmatrix} \mathcal{L} & 0 \\ Re\,\mathcal{C}_p & \mathcal{S} \end{bmatrix} \begin{bmatrix} \psi(\tau) \\ u(\tau) \end{bmatrix}$$



- Adjoint of $\mathcal{A}$ (w.r.t. $\langle \cdot, \cdot \rangle_e$):

$$\mathcal{A} = \begin{bmatrix} \mathcal{L} & 0 \\ Re\,\mathcal{C}_p & \mathcal{S} \end{bmatrix} \;\Rightarrow\; \left\{ \mathcal{A}^\dagger = \begin{bmatrix} \mathcal{L} & Re\,\mathcal{C}_p^\dagger \\ 0 & \mathcal{S} \end{bmatrix},\; \mathcal{C}_p^\dagger = -\mathrm{j} k_z \Delta^{-1} U'(y) \right\}$$

☞ $\mathcal{A}$: not normal $\Leftrightarrow$ not diagonalizable by a unitary coordinate transformation

Spectral decomposition of $\mathcal{A}$ and $\mathcal{A}^\dagger$

$$\begin{bmatrix} \mathcal{L} & 0 \\ Re\,\mathcal{C}_p & \mathcal{S} \end{bmatrix} \begin{bmatrix} \psi \\ u \end{bmatrix} = \lambda \begin{bmatrix} \psi \\ u \end{bmatrix} \;\Rightarrow\; \left\{ \begin{array}{rcl} \mathcal{L}\,\psi & = & \lambda\,\psi \\ \mathcal{S}\,u & = & \lambda\,u \;-\; Re\,\mathcal{C}_p\,\psi \end{array} \right.$$

- Two sets of eigenvalues

$$(\lambda I \,-\, \mathcal{L}) \text{ not one-to-one} \;\Rightarrow\; \left\{ \lambda_{os},\; \begin{bmatrix} \psi_{os} \\ u_{os} \end{bmatrix} \right\}$$

$$(\lambda I \,-\, \mathcal{S}) \text{ not one-to-one} \;\Rightarrow\; \left\{ \lambda_{sq},\; \begin{bmatrix} 0 \\ u_{sq} \end{bmatrix} \right\}$$

- Homework:
  - ⋆ fill in details for the e-value decomposition of $\mathcal{A}$ and $\mathcal{A}^\dagger$

Orr-Sommerfeld: $$\left\{ \begin{array}{rcl} \mathcal{L}\,\psi_{os} & = & \lambda_{os}\,\psi_{os}, \qquad \psi_{os}(\pm 1) \;=\; \psi_{os}'(\pm 1) \;=\; 0 \\ \mathcal{S}\,u_{os} & = & \lambda_{os}\,u_{os} \;-\; Re\,\mathcal{C}_p\,\psi_{os}, \qquad u_{os}(\pm 1) \;=\; 0 \end{array} \right.$$

Squire: $$\left\{ \lambda_{sq} = -\left(\left(\frac{n\pi}{2} \right)^2 + k_z^2 \right),\; \begin{bmatrix} 0 \\ u_{sq} \end{bmatrix} = \begin{bmatrix} 0 \\ \sin\left(\frac{n\pi}{2}(y \,+\, 1) \right) \end{bmatrix} \right\}$$

  - ⋆ show that $\mathcal{A}$ is a Riesz-spectral operator

Riesz-spectral operator

- Action of $\mathcal{A}$ on $f \in \mathbb{H}$

$$[\mathcal{A} f](y) = \sum_{n=1}^{\infty} \lambda_{os,n} v_{os,n}(y) \langle w_{os,n}, f \rangle_e + \sum_{n=1}^{\infty} \lambda_{sq,n} v_{sq,n}(y) \langle w_{sq,n}, f \rangle_e$$

- Solution to $\phi_\tau(\tau) = \mathcal{A}\phi(\tau), \quad \phi(0) = f$

$$\phi(y,\tau) = \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_{os,n} \tau} v_{os,n}(y) \langle w_{os,n}, f \rangle_e + \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_{sq,n} \tau} v_{sq,n}(y) \langle w_{sq,n}, f \rangle_e$$

- Dependence of $u(y, k_z, \tau)$ on $\psi(y, k_z, 0) = \sum_{n=1}^{\infty} \alpha_n(k_z)\, \psi_{os,n}(y, k_z)$

$$u(y, k_z, \tau) \;=\; Re \sum_{n\,=\,1}^{\infty} \Bigg(\alpha_n\, \mathrm{e}^{\lambda_{os,n}\,\tau}\, u_{os,n}(y, k_z, \tau) \;-\; \sum_{m\,=\,1}^{\infty} \frac{\alpha_m}{\lambda_{os,m} \,-\, \lambda_{sq,n}}\, \mathrm{e}^{\lambda_{sq,n}\,\tau}\, u_{sq,n}(y, k_z, \tau)\, \langle u_{sq,n}, \mathcal{C}_p\, \psi_{os,m} \rangle \Bigg)$$

Orr-Sommerfeld: $\left\{ \begin{array}{rclrcl} \mathcal{L}\,\psi_{os} & = & \lambda_{os}\,\psi_{os}, & \psi_{os}(\pm 1) & = & \psi'_{os}(\pm 1) \;=\; 0 \\ \mathcal{S}\,u_{os} & = & \lambda_{os}\,u_{os} \,-\, \mathcal{C}_p\,\psi_{os}, & u_{os}(\pm 1) & = & 0 \end{array} \right.$

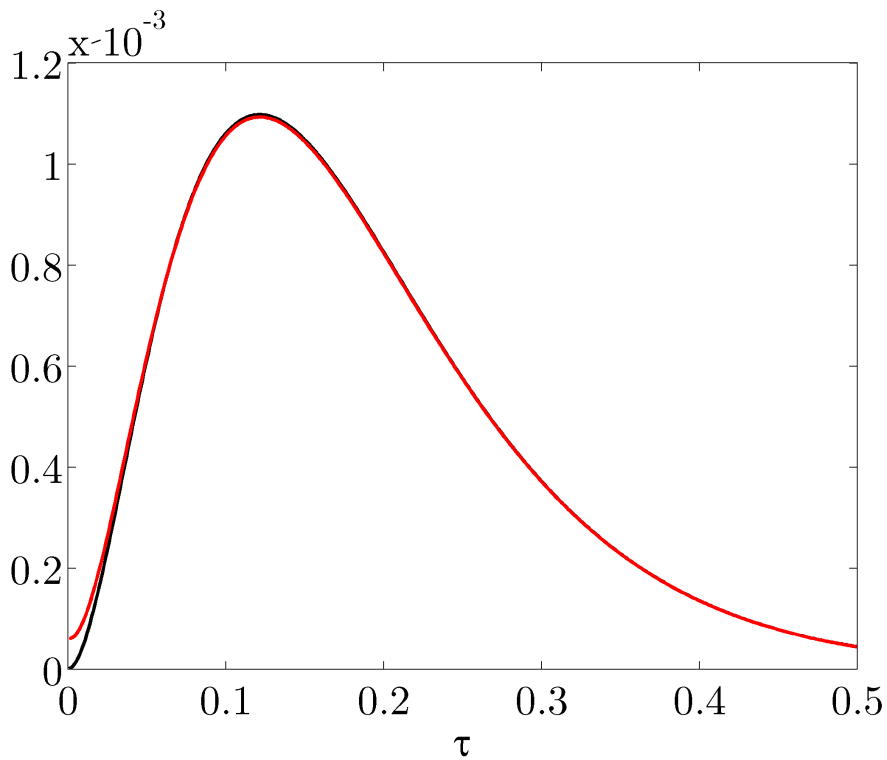
Squire: $\left\{ \lambda_{sq} \;=\; -\left(\left(\frac{n\pi}{2}\right)^2 \,+\, k_z^2 \right),\; \left[\begin{array}{c} 0 \\ u_{sq} \end{array} \right] \;=\; \left[\begin{array}{c} 0 \\ \sin\left(\frac{n\pi}{2}(y \,+\, 1)\right) \end{array} \right] \right.$

Energy growth

- Worst case energy of u caused by the initial condition in ψ
  - ⋆ $Re = 1$, $k_z = 2$

shear-driven flow

(one OS and one Squire mode)



pressure-driven flow

(one OS and two Squire modes)