Undamped Wave Equation $(a_1 = 0)$ $\boxed{10-18-11}$

$$\phi_{tt} = \phi_{xx} \quad , \quad \phi(\pm 1) = 0$$

$$\begin{bmatrix} \psi_{1t} \\ \psi_{2t} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & -a_1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

Abstractly :

$$\begin{bmatrix} \psi_{1t} \\ \psi_{2t} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & -a_1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

<u>Lumer-Phillips</u> : sufficient conditions for <u>well-posedness</u>

$$\left\{ \begin{aligned} &\mathrm{Re}\{\langle \psi, \mathcal{A}\psi \rangle_e\} \leq w \|\psi\|_e^2 \quad , \forall \psi \in \mathcal{D}(\mathcal{A}) \\ &\mathrm{Re}\{\langle \psi, \mathcal{A}^+\psi \rangle_e\} \leq w \|\psi\|_e^2 \quad , \forall \psi \in \mathcal{D}(\mathcal{A}^+) \end{aligned} \right.$$

$$\langle \psi, \mathcal{A}\psi \rangle_e = \left\langle \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} \psi_2 \\ -\mathcal{A}_0\psi_1 - a_1\psi_2 \end{bmatrix} \right\rangle_e$$

$$= \langle \mathcal{A}_0^{1/2}\psi_1 , \mathcal{A}_0^{1/2}\psi_2 \rangle_2 + \langle \psi_2 , -\mathcal{A}_0\psi_1 \rangle_2 + \langle \psi_2 , -a_1\psi_2 \rangle_2$$

$$= \langle \psi_1 , \mathcal{A}_0\psi_2 \rangle_2 - \langle \psi_2 , \mathcal{A}_0\psi_1 \rangle_2 - a_1 \langle \psi_2 , \psi_2 \rangle_2$$

$$\mathrm{Re}\{\langle \psi, \mathcal{A}\psi \rangle_e\} = -a_1 \langle \psi_2 , \psi_2 \rangle_2 \leq -a_1 \|\psi\|_e^2$$

$$\leq 0 \, \|\psi\|_e^2$$

same holds for $\mathrm{Re}\{\langle \psi, \mathcal{A}^+\psi \rangle_e\}$

<u>Note</u> : Worst case happens for undamped equation $(a_1 = 0)$

<u>Thus</u> :

well-posed for $\psi \in \mathcal{H} = \begin{bmatrix} \mathcal{D}(\mathcal{A}_0^{1/2}) \\ L_2[-1,1] \end{bmatrix}$

<u>Check orthonormality</u> of the eigen-functions of the undamped wave equation:

$$v_n = \begin{bmatrix} v_{1n} \\ v_{2n} \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_n} \phi_n(x) \\ \phi_n(x) \end{bmatrix}$$

$$\phi_n(x) = \sin\left(\frac{n\pi}{2}(x+1)\right), \quad \lambda_n = j\,\frac{n\pi}{2}, \quad n \in \mathbb{N}$$

$$\langle v_m, v_n \rangle_e = \langle v_{1m}, \underset{\substack{\downarrow \\ -\frac{d^2}{dx^2}}}{\mathcal{A}_0}\, v_{1n} \rangle_2 + \langle v_{2m}, v_{2n} \rangle_2$$

$$v_{1n}'' = \frac{1}{\lambda_n}\phi_n'' = \frac{1}{\lambda_n}\underset{\substack{\downarrow \\ \lambda_n \bar{\lambda}_n}}{(\lambda_n^2)}\,\phi_n = \bar{\lambda}_n \phi_n(x)$$

So,

$$\langle v_m, v_n \rangle_e = \frac{\bar{\lambda}_n}{\bar{\lambda}_m}\langle \phi_m, \phi_n \rangle_2 + \underbrace{\langle \phi_m, \phi_n \rangle_2}_{\delta_{m,n}}$$

$$= \left(\frac{\bar{\lambda}_n}{\bar{\lambda}_m} + 1\right)\delta_{m,n}$$

$$= \begin{cases} 2 & ; \quad m = n \\ 0 & ; \quad m \neq n \end{cases}$$

$\frac{1}{2}\langle v_m, v_n \rangle_e$ gives you energy.

<u>Check Completeness</u> Need to show that the orthogonal complement of span $\{v_n\}_{n \in \mathbb{Z}\setminus\{0\}}$ is zero.

i.e. $\boxed{\langle \theta, v_n \rangle_e = 0 \Rightarrow \theta = 0}$

<u>Note</u> Since $\{v_n\}$ is orthonormal, it generates a Reisz basis. (on $\begin{bmatrix} L_2[-1,1] \\ L_2[-1,1] \end{bmatrix}$, it does not !)

Navier - Stokes Equations (NSE)

$$(1) \qquad \underline{v}_t + (\underline{v} \cdot \nabla)\underline{v} = -\nabla p + \frac{1}{Re}\Delta \underline{v}$$

$$(2) \qquad \underbrace{\underline{v}}_{\text{total velocity}} = \underbrace{\underline{V}}_{\substack{\text{mean velocity} \\ \text{(equilibrium point)}}} + \underbrace{\tilde{\underline{v}}}_{\text{fluctuations}}$$

Linearized NSE $\left\{ \begin{array}{l} \text{Put (2) into (1)} \\ \text{neglect } (\tilde{\underline{v}} \cdot \nabla)\tilde{\underline{v}} \end{array} \right.$

$$(3) \qquad \tilde{\underline{v}}_t + (\tilde{\underline{v}} \cdot \nabla)\underline{V} + (\underline{V} \cdot \nabla)\tilde{\underline{v}} = -\nabla p + \frac{1}{Re}\Delta \tilde{\underline{v}}$$

Continuity: $\nabla \cdot \tilde{\underline{v}} = 0$

$$\left\{ \begin{array}{lll} \text{Pressure driven flow:} & \underline{V} = \left[U(y) = 1 - y^2 \quad 0 \quad 0 \right]^T \\ \text{Shear driven flow:} & \underline{V} = \left[U(y) = y \quad 0 \quad 0 \right]^T \end{array} \right.$$

Linearized NSE $\qquad \tilde{\underline{v}} = [u \ \ v \ \ w]^T$

$$\left\{ \begin{array}{lll} u_t + U(y)\, u_x + U'(y)\, v & = & -p_x + \frac{1}{Re}\Delta u \\ v_t + U(y)\, v_x & = & -p_y + \frac{1}{Re}\Delta v \\ w_t + U(y)\, w_x & = & -p_z + \frac{1}{Re}\Delta w \\ u_x + v_y + w_z = 0 & & \text{(Constraint)} \end{array} \right.$$

Note:

$$\begin{bmatrix} u_t \\ v_t \\ w_t \\ 0\, p_t \end{bmatrix} = \bar{\mathcal{A}} \begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix}$$

!

Not in standard evolution form.

set $\partial_x = 0$.

$$\begin{cases} u_t + U'(y)v = \frac{1}{Re}\Delta u \\ v_t = -p_y + \frac{1}{Re}\Delta v \\ w_t = -p_z + \frac{1}{Re}\Delta w \\ v_y + w_z = 0 \end{cases}$$

Let $v = -\psi_z$, $w = \psi_y$;

then $v_y + w_z = 0$ Automatically !

Obtain

$$\begin{cases} \Delta\psi_t = \frac{1}{Re}\Delta^2\psi \\ u_t = \frac{1}{Re}\Delta u - U'\psi_z \\ \Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{cases}$$

$$\begin{bmatrix} \psi_t \\ u_t \end{bmatrix} = \begin{bmatrix} \frac{1}{Re}\mathcal{L} & 0 \\ C_p & \frac{1}{Re}S \end{bmatrix} \begin{bmatrix} \psi \\ u \end{bmatrix}$$

$$\left.\begin{array}{lll} \text{Orr-Sommerfeld:} & \mathcal{L} = & \Delta^{-1}\Delta^2 \\ \text{Squire:} & S = & \Delta \\ \text{Coupling:} & C_p = & -U'(y)\partial_z \end{array}\right\}$$

- Can we cancel $\Delta^{-1}\Delta^2$ to obtain Δ ?!

Q : what is $\Delta^{-1}\Delta^2$?

A : an operator that maps $\Delta^{-1}\Delta^2 : f \longrightarrow g$

$$\Delta^{-1}\Delta^2 f = g \Rightarrow \Delta^2 f = \Delta g$$

F.T in z $\Rightarrow$ $\begin{cases} \Delta = \frac{\partial^2}{\partial y^2} - k_z^2 I \\ \Delta^2 = \frac{\partial^4}{\partial y^4} - 2k_z^2\frac{\partial^2}{\partial y^2} + k_z^4 I \end{cases}$

So $\Delta^{-1}\Delta^2$ is an integro-differential operator.

F.T in z:

$$\begin{cases} \Delta = \partial_{yy} - k_z^2 I \\ \Delta^2 = \partial_{yyyy} - 2k_z^2 \partial_{yy} + k_z^4 I \\ C_p = -j k_z U' \end{cases}$$

B.C.s $\quad \Delta$: Dirichlet

Δ^2 : Dirichlet + Neumann