Lecture 12: Waves, beams, . . .

- Objective: study dynamics of waves and beams

- Approach: identify commonalities between the two equations
  - ⋆ Inner product that induces energy of wave/beam
  - ⋆ Square-root of a positive self-adjoint operator

Wave equation

$$\begin{aligned} \phi_{tt}(x,t) &= \phi_{xx}(x,t) \\ \phi(x,0) &= f(x), \quad \phi_t(x,0) \;=\; g(x) \\ \phi(\pm 1,t) &= 0 \end{aligned}$$

Define $\psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} = \begin{bmatrix} \phi(\cdot,t) \\ \phi_t(\cdot,t) \end{bmatrix}$ and write an abstract evolution equation:

$$\begin{aligned} \begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ \mathrm{d}^2/\mathrm{d}x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \\ \phi(t) &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \end{aligned}$$

- Dynamical generator

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & 0 \end{bmatrix}, \quad \mathcal{A}_0 \;=\; -\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} \\ \mathcal{D}(\mathcal{A}_0) &= \left\{ f \in L_2\,[-1,\,1],\, \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \in L_2\,[-1,\,1],\; f(\pm 1) \;=\; 0 \right\} \end{aligned}$$

Euler-Bernoulli beam

$$\begin{aligned} \phi_{tt}(x,t) &= -\phi_{xxxx}(x,t) \\ \phi(x,0) &= f(x), \quad \phi_t(x,0) \;=\; g(x) \\ \phi(\pm 1,t) &= 0 \\ \phi_{xx}(\pm 1,t) &= 0 \end{aligned}$$

Define $\psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} = \begin{bmatrix} \phi(\cdot,t) \\ \phi_t(\cdot,t) \end{bmatrix}$ and write an abstract evolution equation:

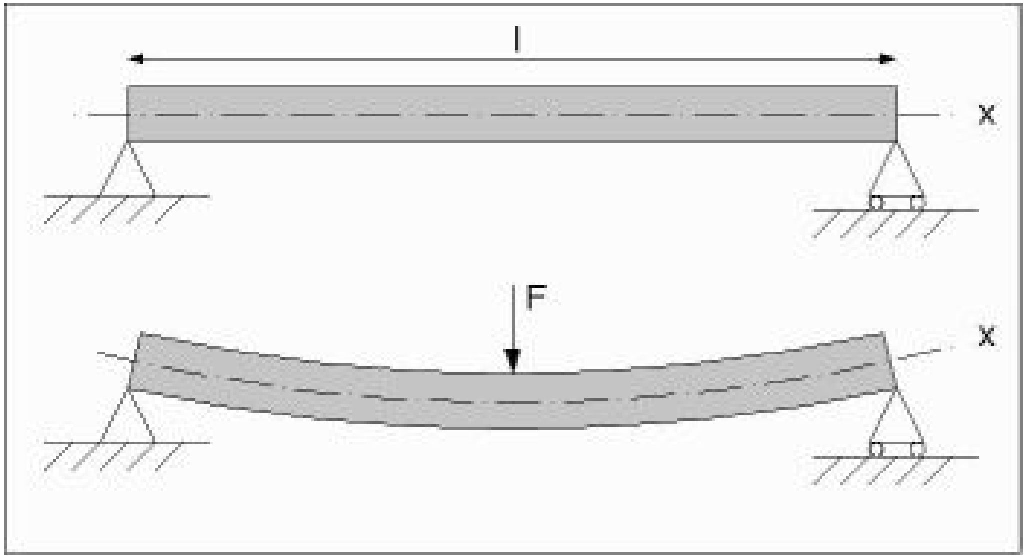
$$\begin{aligned} \begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -\mathrm{d}^4/\mathrm{d}x^4 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \\ \phi(t) &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \end{aligned}$$

- Dynamical generator

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & 0 \end{bmatrix}, \quad \mathcal{A}_0 \;=\; \frac{\mathrm{d}^4}{\mathrm{d}\,x^4} \\ \mathcal{D}(\mathcal{A}_0) &= \left\{ f \in L_2[-1,\,1],\; \frac{\mathrm{d}^4 f}{\mathrm{d}x^4} \in L_2[-1,\,1],\; f(\pm 1) \;=\; f''(\pm 1) \;=\; 0 \right\} \end{aligned}$$

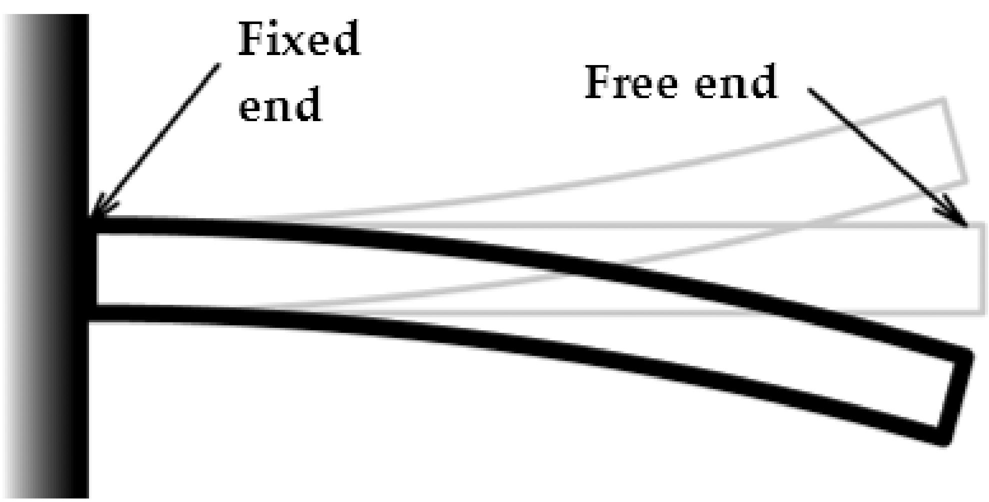
Simply supported and cantilever beams

- Simply supported beams



$$\begin{aligned} \phi(0,t) &= \phi(L,t) = 0 \\ \phi_{xx}(0,t) &= \phi_{xx}(L,t) = 0 \end{aligned}$$

- Cantilever beams



$$\begin{aligned} \phi(0,t) &= 0, \;\; \phi_x(0,t) = 0 \\ \phi_{xx}(L,t) &= 0, \;\; \phi_{xxx}(L,t) = 0 \end{aligned}$$

Square-root of a positive operator

- Self-adjoint operator $\mathcal{A}$: $\mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}$ is

  ⋆ positive

$$\langle \psi, \mathcal{A}\psi \rangle \; > \; 0 \;\; \text{for all non-zero } \psi \in \mathcal{D}(\mathcal{A})$$

  ⋆ coercive: if there is $\epsilon > 0$ such that

$$\langle \psi, \mathcal{A}\psi \rangle \; > \; \epsilon \, \|\psi\|^2 \;\; \text{for all } \psi \in \mathcal{D}(\mathcal{A})$$

- Self-adjoint, non-negative $\mathcal{A}$ has a unique non-negative square-root $\mathcal{A}^{\frac{1}{2}}$

$$\left\{ \begin{array}{ll} \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \supset \mathcal{D}(\mathcal{A}) & \\ \mathcal{A}^{\frac{1}{2}} \psi \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) & \text{for all } \psi \in \mathcal{D}(\mathcal{A}) \\ \mathcal{A}^{\frac{1}{2}} \mathcal{A}^{\frac{1}{2}} \psi = \mathcal{A}\psi & \text{for all } \psi \in \mathcal{D}(\mathcal{A}) \end{array} \right.$$

positive $\mathcal{A}$ $\Rightarrow$ positive $\mathcal{A}^{\frac{1}{2}}$

- Examples of positive, self-adjoint operators:

$$\mathcal{A}_0 \;=\; -\,\frac{\mathrm{d}^2}{\mathrm{d}\,x^2},\;\; \mathcal{D}(\mathcal{A}_0) \;=\; \left\{f \,\in\, L_2\,[-1,\,1],\; \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \in\, L_2\,[-1,\,1],\; f(\pm 1) \;=\; 0\right\}$$

$$\mathcal{A}_0 \;=\; \frac{\mathrm{d}^4}{\mathrm{d}\,x^4},\;\; \mathcal{D}(\mathcal{A}_0) \;=\; \left\{f \,\in\, L_2\,[-1,\,1],\; \frac{\mathrm{d}^4 f}{\mathrm{d}x^4} \in\, L_2\,[-1,\,1],\; f(\pm 1) \;=\; f''(\pm 1) \;=\; 0\right\}$$

$\mathcal{D}(\mathcal{A}_0^{\frac{1}{2}})$ – determined from the following requirement:

$$\left\langle \mathcal{A}_0^{\frac{1}{2}}\, f, \mathcal{A}_0^{\frac{1}{2}}\, g \right\rangle \;=\; \langle f, \mathcal{A}_0\, g\rangle\,, \;\; \text{for all } g \,\in\, \mathcal{D}(\mathcal{A}_0)$$

- For beam (wave left for homework):

$$\mathcal{A}_0^{\frac{1}{2}} \;=\; -\,\frac{\mathrm{d}^2}{\mathrm{d}\,x^2},\;\; \mathcal{D}(\mathcal{A}_0^{\frac{1}{2}}) \;=\; \left\{f \,\in\, L_2\,[-1,\,1],\; \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \in\, L_2\,[-1,\,1],\; f(\pm 1) \;=\; 0\right\}$$

Abstract evolution equation

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & -a_1 I \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}$$

Hilbert space:

$$\mathbb{H} = \begin{bmatrix} \mathcal{D}(\mathcal{A}_0^{\frac{1}{2}}) \\ L_2[-1, 1] \end{bmatrix}$$

Inner product:

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle_e &= \left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_e \\ &= \left\langle \mathcal{A}_0^{\frac{1}{2}} f_1, \mathcal{A}_0^{\frac{1}{2}} f_2 \right\rangle + \langle g_1, g_2 \rangle \end{aligned}$$

Energy:

$$E(t) = \begin{cases} \dfrac{1}{2} \langle \psi_{1x}, \psi_{1x} \rangle + \dfrac{1}{2} \langle \psi_2, \psi_2 \rangle & \text{wave} \\ \dfrac{1}{2} \langle \psi_{1xx}, \psi_{1xx} \rangle + \dfrac{1}{2} \langle \psi_2, \psi_2 \rangle & \text{beam} \end{cases}$$

- Adjoint of $\mathcal{A}$ (w.r.t. $\langle \cdot, \cdot \rangle_e$):

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & -a_1 I \end{bmatrix} \Rightarrow \mathcal{A}^\dagger = \begin{bmatrix} 0 & -I \\ \mathcal{A}_0 & -a_1 I \end{bmatrix}, \; \mathcal{D}(\mathcal{A}^\dagger) = \mathcal{D}(\mathcal{A}) = \begin{bmatrix} \mathcal{D}(\mathcal{A}_0) \\ \mathcal{D}(\mathcal{A}_0^{\frac{1}{2}}) \end{bmatrix}$$

- In class:
  - ⋆ well-posedness on $\mathbb{H} = \begin{bmatrix} \mathcal{D}(\mathcal{A}_0^{\frac{1}{2}}) \\ L_2[-1, 1] \end{bmatrix}$ using Lumer-Phillips
  - ⋆ spectral decomposition of $\mathcal{A}$ for the undamped wave equation
  - ⋆ solution to the undamped wave equation
  - ⋆ mention different forms of internal damping in beams

Spectral decomposition of the undamped wave equation

$$\begin{bmatrix} 0 & I \\ \partial_{xx} & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \lambda \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \;\Rightarrow\; \left\{ \begin{array}{rcl} \psi_2 & = & \lambda\,\psi_1 \\ \psi_1'' & = & \lambda\,\psi_2 \\ 0 & = & \psi_1(\pm 1) \end{array} \right.$$

- Showed:

$$\left. \begin{array}{rcl} \psi_1'' & = & \lambda^2\,\psi_1 \\ 0 & = & \psi_1(\pm 1) \end{array} \right\} \xrightarrow{n \,\in\, \mathbb{N}} \left\{ \begin{array}{rclrcl} \lambda_n & = & +\,\mathrm{j}\,\dfrac{n\pi}{2}, & v_n(x) & = & \begin{bmatrix} (1/\lambda_n)\,\phi_n(x) \\ \phi_n(x) \end{bmatrix} \\ \lambda_{-n} & = & -\,\mathrm{j}\,\dfrac{n\pi}{2}, & v_{-n}(x) & = & \begin{bmatrix} (1/\lambda_n)\,\phi_n(x) \\ -\phi_n(x) \end{bmatrix} \\ & & & \phi_n(x) & = & \sin\left(\dfrac{n\pi}{2}\,(x\,+\,1)\right) \end{array} \right.$$

☞ $\{v_n\}_{n \,\in\, \mathbb{Z}\backslash 0}$ – complete orthonormal basis (w.r.t. $\langle\cdot,\cdot\rangle_e$)

Solution of the undamped wave equation

- Represent the solution as

$$\begin{aligned}\psi(x,t) &= \sum_{n=1}^{\infty} \alpha_n(t)\, v_n(x) \;+\; \sum_{n=1}^{\infty} \alpha_{-n}(t)\, v_{-n}(x) \\ &= \sum_{n=1}^{\infty} \left[\begin{array}{c} (\alpha_n(t) + \alpha_{-n}(t))\, \frac{1}{\lambda_n}\, \phi_n(x) \\ (\alpha_n(t) - \alpha_{-n}(t))\, \phi_n(x) \end{array}\right] \\ &= \sum_{n=1}^{\infty} \left[\begin{array}{c} a_n(t)\, \frac{1}{\lambda_n}\, \phi_n(x) \\ b_n(t)\, \phi_n(x) \end{array}\right] \;\Rightarrow\; \{a_n(t) \in \mathrm{j}\,\mathbb{R},\; b_n(t) \in \mathbb{R}\}\end{aligned}$$

- Substitute into the evolution model

$$\left.\begin{aligned}\dot{\alpha}_n(t) &= +\mathrm{j}\,\frac{n\pi}{2}\,\alpha_n(t) \\ \dot{\alpha}_{-n}(t) &= -\mathrm{j}\,\frac{n\pi}{2}\,\alpha_{-n}(t)\end{aligned}\right\} \Rightarrow \begin{array}{l} \left[\begin{array}{c} \dot{a}_n(t) \\ \dot{b}_n(t) \end{array}\right] = \left[\begin{array}{cc} 0 & \mathrm{j}n\pi/2 \\ \mathrm{j}n\pi/2 & 0 \end{array}\right] \left[\begin{array}{c} a_n(t) \\ b_n(t) \end{array}\right] \\ \left[\begin{array}{c} a_n(t) \\ b_n(t) \end{array}\right] = \left[\begin{array}{cc} \cos\left(\frac{n\pi}{2}\,t\right) & \mathrm{j}\sin\left(\frac{n\pi}{2}\,t\right) \\ \mathrm{j}\sin\left(\frac{n\pi}{2}\,t\right) & \cos\left(\frac{n\pi}{2}\,t\right) \end{array}\right] \left[\begin{array}{c} a_n(0) \\ b_n(0) \end{array}\right] \end{array}$$