Last time

Examples of C_0-semigroups

Hille-Yosida and Lumer-Phillips Theorems

Compare implicit Euler with explicit Euler

$$\frac{\partial \psi}{\partial t} = c\mathcal{A}\psi \xrightarrow[\text{Euler}]{\text{Implicit}} \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t} = c\mathcal{A}\psi(t+\Delta t)$$

Evaluate right-hand-side one step ahead

$$\Downarrow$$

$$\boxed{\psi(t+\Delta t) = (I - \Delta t\, c\mathcal{A})^{-1}\psi(t)}$$

$$\frac{\partial \psi}{\partial t} = c\mathcal{A}\psi \xrightarrow[\text{Euler}]{\text{Explicit}} \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t} = c\mathcal{A}\psi(t)$$

Evaluate right-hand-side at current time

$$\Downarrow$$

$$\boxed{\psi(t+\Delta t) = (I + \Delta t\, c\mathcal{A})\psi(t)}$$

Note

$(I - \Delta t\, c\mathcal{A})$ unbounded .... differential operators

$(I - \Delta t\, c\mathcal{A})^{-1}$ bounded .... inverse of differential operators

Implicit Euler involves composition with bounded operators for propagating the state ψ forward in time.

Euler - Bernoulli beam

$$\phi_{tt}(x,t) = -\phi_{xxxx}(x,t)$$

$$\phi(x,0) = f(x) \; ; \quad \phi_t(x,0) = g(x)$$

$$\phi(\pm 1, t) = 0$$

$$\phi_{xx}(\pm 1, t) = 0$$

Abstract evolution model :

$$\begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} = \begin{bmatrix} \phi(\cdot,t) \\ \phi_t(\cdot,t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\frac{d^4}{dx^4} & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}$$

Dynamical generator :

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & 0 \end{bmatrix} \; ; \quad \mathcal{A}_0 = \frac{d^4}{dx^4}$$

$$\mathcal{D}(\mathcal{A}_0) = \left\{ f \in L_2[-1,1], \; \frac{d^4 f}{dx^4} \in L_2[-1,1], \; f(\pm 1) = f''(\pm 1) = 0 \right\}$$

Positive operator:

self-adjoint operator $\mathcal{A} : \mathcal{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}$ is **Positive**

$$\langle \psi, \mathcal{A}\psi \rangle > 0 \quad \text{for all non-zero } \psi \in \mathcal{D}(\mathcal{A})$$

[matrices: $P = P^*$ is positive if

$x^* P x > 0, \quad \forall x \neq 0$ --- positive definite

$x^* P x \geq 0, \quad \forall x$ --- positive semi-definite

$P = P^{1/2} P^{1/2}$

$P^{1/2} = (P^{1/2})^* > 0$]

operator $\mathcal{A}$ is **coercive** if

$\exists \ \varepsilon > 0$

$$\langle \psi, \mathcal{A}\psi \rangle \geq \varepsilon \|\psi\|^2 \quad \forall \ \psi \in \mathcal{D}(\mathcal{A})$$

[In matrices, coercivity is always satisfied

$P = P^*$

$x^* P x \geq \lambda_{min} \|x\|^2$]

λ_{min} ↓ minimum eigenvalue of P

Square-root $\mathcal{A}^{1/2}$ of self-adjoint $\mathcal{A}$

$$\begin{cases} \mathcal{D}(\mathcal{A}^{1/2}) \supset \mathcal{D}(\mathcal{A}) \\ \mathcal{A}^{1/2}\psi \in \mathcal{D}(\mathcal{A}^{1/2}) \\ \mathcal{A}^{1/2}\mathcal{A}^{1/2}\psi = \mathcal{A}\psi \end{cases}$$

reference [Kato]

Examples of positive, self-adjoint operators

$$\mathcal{A}_0 = \frac{-d^2}{dx^2} \; ; \quad \mathcal{D}(\mathcal{A}_0) = \Big\{ f \in L_2[-1,1] ; \; \frac{d^2 f}{dx^2} \in L_2[-1,1], \; f(\pm 1) = 0 \Big\}$$

$$\mathcal{A}_0 = -\frac{d^4}{dx^4} \; ; \quad \mathcal{D}(\mathcal{A}_0) = \Big\{ f \in L_2[-1,1] ; \; \frac{d^4 f}{dx^4} \in L_2[-1,1], \; f(\pm 1) = f''(\pm 1) = 0 \Big\}$$

$\mathcal{D}(\mathcal{A}_0^{1/2})$: determined from the following requirement

$$\langle \mathcal{A}_0^{1/2} f , \mathcal{A}_0^{1/2} g \rangle = \langle f, \mathcal{A}_0 g \rangle , \quad \forall g \in \mathcal{D}(\mathcal{A}_0)$$

<u>Example</u> $\quad \mathcal{A}_0 = \frac{d^4}{dx^4} \; ; \quad g(\pm 1) = g''(\pm 1) = 0$

$$\langle f, \mathcal{A}_0 g \rangle = \langle \mathcal{A}_0^{1/2} f , \mathcal{A}_0^{1/2} g \rangle \quad \text{for all } g \in \mathcal{D}(\mathcal{A}_0)$$

$$\langle f, \frac{d^4}{dx^4} g \rangle = \langle f, g^{(4)} \rangle = f(x) g^{(3)}(x) \Big|_{-1}^{1} - \langle f', g^{(3)} \rangle =$$

$$= \underbrace{f(x) \underbrace{g^{(3)}(x)}_{\text{arbitrary}} \Big|_{-1}^{1}}_{\text{Need } f(\pm 1) = 0} - \underbrace{f'(x) \cancel{g''(x)}^{\,0} \Big|_{-1}^{1}}_{g''(\pm 1) = 0} + \langle f'', g'' \rangle$$

$$= \langle f'', g'' \rangle \quad \text{if} \quad f(\pm 1) = 0$$

Thus,

$$\underbrace{\mathcal{A}_0^{1/2} = -\frac{d^2}{dx^2}}_{\text{want } \mathcal{A}_0^{1/2} \text{ to be positive operator.}} \; ; \quad \mathcal{D}(\mathcal{A}_0^{1/2}) = \Big\{ f \in L_2[-1,1], \; f'' \in L_2[-1,1], \; f(\pm 1) = 0 \Big\}$$

$\boxed{\text{E-values of } \frac{d^2}{dx^2}\Big|_{f(\pm 1) = 0} \text{ are } -\left(\frac{n\pi}{2}\right)^2}$

Adjoint of $\mathcal{A}$ with respect to the energy inner product $\langle \cdot , \cdot \rangle_e$

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & -a_1 I \end{bmatrix}$$

$$\langle \phi_1 , \phi_2 \rangle_e = \left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} , \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_e =$$

$$= \langle \mathcal{A}_0^{1/2} f_1 , \mathcal{A}_0^{1/2} f_2 \rangle_2 + \langle g_1 , g_2 \rangle_2$$

Definition : $\langle \phi_1 , \mathcal{A}\phi_2 \rangle_e = \langle \mathcal{A}^{\dagger} \phi_1 , \phi_2 \rangle_e$

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} , \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & -a_1 I \end{bmatrix} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_e = \left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} , \begin{bmatrix} g_2 \\ -\mathcal{A}_0 f_2 - a_1 g_2 \end{bmatrix} \right\rangle_e$$

$\mathcal{A}_0^{1/2}$

$$= \langle \mathcal{A}_0^{1/2} f_1 , \mathcal{A}_0^{1/2} g_2 \rangle_2 + \langle g_1 , -\mathcal{A}_0 f_2 - a_1 g_2 \rangle_2$$

[using the slides : guess for $\mathcal{A}^{\dagger}$]

$$= \langle \mathcal{A}_0 f_1 , g_2 \rangle_2 + \langle -\mathcal{A}_0^{1/2} g_1 , \mathcal{A}_0^{1/2} f_2 \rangle_2$$

$$- \langle a_1 g_1 , g_2 \rangle_2$$

$$\Rightarrow \mathcal{A}^{\dagger} = \begin{bmatrix} 0 & -I \\ \mathcal{A}_0 & -a_1 I \end{bmatrix}$$

Spectral decomposition for wave equation

$$\phi_{tt} = \phi_{xx} \quad \text{w/} \quad \phi(\pm 1) = 0$$

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix}$$

$$\mathcal{A} v = \lambda v$$

$$\left\{ \begin{array}{l} v_2 = \lambda v_1 \\ v_1'' = \lambda v_2 \\ v_1(\pm 1) = 0 \end{array} \right. \Rightarrow \left. \begin{array}{l} v_1'' = \lambda^2 v_1 \\ v_1(\pm 1) = 0 \end{array} \right\}$$

Compare with

$$\left\{ \begin{array}{l} v'' = s v \\ v(\pm 1) = 0 \end{array} \right.$$

Heat equation

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Know

$$s_n = -\left(\frac{n\pi}{2}\right)^2 ; \quad n = 1, 2, \ldots$$

$$v_n = \sin\left(\frac{n\pi}{2}(x+1)\right)$$

So, $\lambda_n^2 = -\left(\frac{n\pi}{2}\right)^2$

$$\Rightarrow \boxed{\lambda_n = \pm j \left(\frac{n\pi}{2}\right) ; \quad n = 1, 2, \ldots}$$

- There are two sets of eigen-vectors.

<u>Summary</u>

$$\lambda_n = j\left(\frac{n\pi}{2}\right) ; \quad n = \pm 1, \pm 2, \ldots$$

$$\lambda_n = -\lambda_n \quad , \text{ use } \sin(-x) = -\sin(x)$$

$$v_n(x) = \begin{bmatrix} \frac{1}{\lambda_n} \sin\left(\frac{n\pi}{2}(x+1)\right) \\ \sin\left(\frac{n\pi}{2}(x+1)\right) \end{bmatrix} \begin{array}{l} \longrightarrow \text{Same for } \pm n \\ \longrightarrow \text{Changes sign when } n \to -n \end{array}$$

Normalization is done such that $\langle v_n, v_m \rangle_e = \delta_{n,m}$