Last time

10-11-11

$$\dot{\Psi}(t) = \mathcal{A}\Psi(t) \ ; \quad \mathbb{H} = \ell_2(\mathbb{N})$$

$$\mathcal{A} = \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{bmatrix} = \operatorname{diag}\{a_n\}_{n\in\mathbb{N}}$$

We've shown that

$$\text{if} \quad \sup_n |a_n|^2 < +\infty \Rightarrow \mathcal{A} : \text{bounded from } \ell_2 \text{ to } \ell_2$$

$$(\text{i.e. } g = \mathcal{A}f, \ f \in \ell_2 \Rightarrow g \in \ell_2)$$

Ex second derivative $\quad a_n = -\left(\frac{n\pi}{2}\right)^2$

$$\sup_n |a_n| = \sup_n \left|\left(\frac{n\pi}{2}\right)^2\right| \not< +\infty$$

unbounded operator.

$$\text{Let} \quad \mathcal{D}(\mathcal{A}) = \left\{ f \in \ell_2 \ \text{ s.t. } \sum_{n=1}^{\infty} |a_n f_n|^2 < +\infty \right\}$$

Important property to check:

Boundedness of $T(t)$ ... state-transition operator (C_0 semigroup)

$$T(t) = \begin{bmatrix} e^{a_1 t} & & \\ & e^{a_2 t} & \\ & & \ddots \end{bmatrix} ; \quad g = T(t)f = \left\{ e^{a_n t} f_n \right\}_{n\in\mathbb{N}}$$

Q: determine conditions on $\{a_n\}_{n\in\mathbb{N}}$ s.t.

$$f \in \ell_2 \Rightarrow g \in \ell_2$$

<u>Answer</u> : $\|g\|^2_{\ell_2} = \sum_{n=1}^{\infty} |e^{a_n t} \cdot f_n|^2 = \sum_{n=1}^{\infty} |e^{a_n t}|^2 \cdot |f_n|^2$

$$\leq \underbrace{\sup_n |e^{a_n t}|^2}_{\downarrow} \cdot \underbrace{\sum_{n=1}^{\infty} |f_n|^2}_{\|f\|^2_{\ell_2}}$$

What is important for bounding this term is for fixed 't'. What happens as $t \to \infty$ is a question of stability.

<u>Note:</u> Aside

$$a_n = \mathrm{Re}(a_n) + j\,\mathrm{Im}(a_n)$$

$$|e^{a_n t}|^2 = |e^{\mathrm{Re}(a_n)t} e^{j\,\mathrm{Im}(a_n)t}|^2$$

$$= e^{2\mathrm{Re}(a_n)t} \underbrace{|e^{j\,\mathrm{Im}(a_n)t}|^2}_{1}$$

$$\|g\|^2_{\ell_2} \leq e^{2\mathrm{Re}(a_n)t} \|f\|^2_{\ell_2}$$

if $\sup_n \mathrm{Re}(a_n) < M < +\infty$, then

So-called Half-plane Condition

$$\|g\|^2_{\ell_2} \leq M \cdot \|f\|^2_{\ell_2} \Rightarrow g \in \ell_2$$

<u>So</u> for fixed t,

$$T(t) : \ell_2 \longrightarrow \ell_2 \quad \text{is bounded.}$$

Example $\quad a_n = -\left(\frac{n\pi}{2}\right)^2$

$\mathcal{A}$ is unbounded. but,

$$\sup_n Re(a_n) = \sup_n -\left(\frac{n\pi}{2}\right)^2 < \underbrace{M}_{\text{Some positive number}} < +\infty$$

So, $T(t)$ is bounded.

Example $\quad$ backward-in-time heat equation

$$\psi_t = -\psi_{xx} \Rightarrow a_n = \left(\frac{n\pi}{2}\right)^2$$

$$\sup_n Re(a_n) = +\infty \qquad \text{(unbounded)}$$

Finally Determine conditions on $\{a_n\}_{n\in\mathbb{N}}$ s.t.

$$\lim_{t\to 0^+} \| T(t)\underbrace{\psi(\cdot)}_{f} - \underbrace{\psi(\cdot)}_{f}\| = 0 \qquad \forall \psi(\cdot) \in \ell_2$$

$$\|(T(t)-I)f\|^2_{\ell_2} = \sum_{n=1}^{\infty} |(e^{a_n t}-1)f_n|^2$$

$$= \sum_{n=1}^{N} |(e^{a_n t}-1)f_n|^2 + \sum_{n=N+1}^{\infty} |(e^{a_n t}-1)f_n|^2$$

$$\leq \sup_{1\leq n\leq N} |e^{a_n t}-1|^2 \underbrace{\sum_{n=1}^{N}|f_n|^2}_{\leq \|f\|^2_{\ell_2}} + \sup_{n>N} |e^{a_n t}-1|^2 \sum_{n=N+1}^{\infty} |f_n|^2$$

Summary if half-plane condition is satisfied:

$$\sup_n \mathrm{Re}(a_n) < M < +\infty$$

then

$$\lim_{t \to 0^+} \| T(t)\psi(0) - \psi(0) \| = 0$$

Thus, $T(t)$ generates a C_0 - semigroup
(read: equivalent of Matrix exponential on ℓ_2)

In the proof of Hille-Yosida Theorem:

Implicit Euler: $\dfrac{d\psi}{dt} = \mathcal{A}\psi$

$$\Rightarrow \quad \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t} = \mathcal{A}\psi(t+\Delta t)$$

$$\psi(t+\Delta t) = (I - \underbrace{\Delta t}_{\frac{t}{N}}\mathcal{A})^{-1}\psi(t)$$

A method for computing $T(t) = \lim_{N \to \infty} \left(I - \frac{t}{N}\mathcal{A}\right)^{-N}$.

Hille - Yosida ... Necessary & Sufficient Conditions
(difficult to use)

Lumer - Phillips ... Sufficient Conditions
(easy to use)

Example Lumer-Phillips

$$\mathcal{A} = \frac{d}{dx} \quad ; \quad L_2[-1,1] \quad ; \quad f(1) = 0$$

$$\langle \psi, \mathcal{A}\psi \rangle = \langle \psi, \psi' \rangle = \psi(x)\psi(x)\Big|_{-1}^{1} - \langle \psi', \psi \rangle$$

$$= \psi(1)\underbrace{\cancel{\psi(1)}}_{0} - \psi(-1)\psi(-1) - \langle \psi', \psi \rangle$$

$$2\,\mathrm{Re}\{\langle \psi, \mathcal{A}\psi \rangle\} = -\psi^2(-1) \leqslant 0$$

$$\left.\begin{array}{l} \mathrm{Re}\{\langle \psi, \mathcal{A}\psi \rangle\} \leqslant e^{0t} \\ \text{similarly:} \\ \mathrm{Re}\{\langle \psi, \mathcal{A}^{+}\psi \rangle\} \leqslant e^{0t} \end{array}\right\} \Rightarrow T(t) \text{ generates } C_0\text{-semigroup.}$$