Lectures 10 & 11: Semigroup Theory

- Want to generalize matrix exponential to infinite dimensional setting

- Strongly continuous (C_0) semigroup
  - ⋆ Extension of matrix exponential

- Infinitesimal generator of a C_0-semigroup

- Examples and conditions

Solution to abstract evolution equation

- Abstract evolution equation on a Hilbert space $\mathbb{H}$

$$\frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} \;=\; \mathcal{A}\,\psi(t), \quad \psi(0) \,\in\, \mathbb{H}$$

Dilemma: how to define "$\mathrm{e}^{\mathcal{A}\,t}$"?

Finite dimensional case:

$$M \,\in\, \mathbb{C}^{n \times n} \quad \Rightarrow \quad \mathrm{e}^{M\,t} \;=\; \sum_{k\,=\,1}^{\infty} \frac{(M\,t)^k}{k!}$$

$$\frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} \;=\; \mathcal{A}\,\psi(t), \quad \psi(0) \,\in\, \mathbb{H}$$

- Assume:
  - ⋆ For each $\psi(0) \in \mathbb{H}$, there is a unique solution $\psi(t)$
  - ⋆ There is a well defined mapping $T(t)$: $\mathbb{H} \longrightarrow \mathbb{H}$

$$\psi(t) \;=\; T(t)\,\psi(0)$$

$T(t)$ - time-parameterized family of linear operators on $\mathbb{H}$

  - ⋆ Solution varies continuously with initial state

$T(t)$: a bounded operator (on $\mathbb{H}$)

$$\|T(t)\| \;=\; \sup_{f\,\in\,\mathbb{H}} \frac{\|T(t)\,f\|}{\|f\|} \;<\; \infty$$

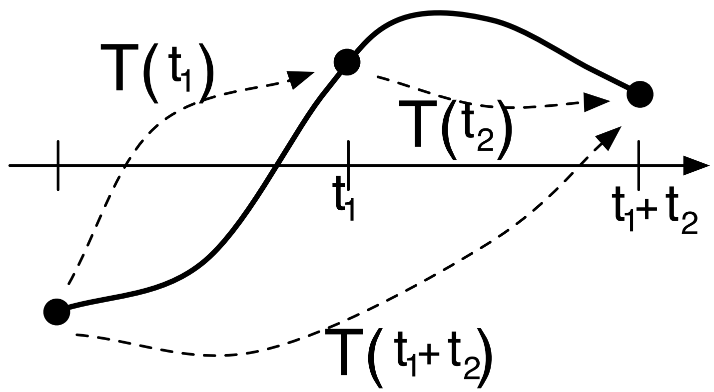
Strongly continuous semigroups

- Properties of $T(t)$: $\psi(t) \;=\; T(t)\,\psi(0)$

- Initial condition: $T(0) \;=\; I$

- Semigroup property:

$$T(t_1 + t_2) \;=\; T(t_2)\,T(t_1) \;=\; T(t_1)\,T(t_2), \quad \text{for all } t_1, t_2 \geq 0$$



- Strong continuity:

$$\lim_{t \to 0^+} \|T(t)\,\psi(0) \;-\; \psi(0)\| \;=\; 0, \;\; \text{for all } \;\psi(0) \;\in\; \mathbb{H}$$

a weaker condition than:

$$\lim_{t \to 0^+} \|T(t) \;-\; I\| \;=\; \lim_{t \to 0^+} \sup_{f \,\in\, \mathbb{H}} \frac{\|\,(T(t) \;-\; I)\,f\|}{\|f\|} \;=\; 0$$

Examples

- Linear transport equation

$$\left.\begin{array}{rcl} \phi_t(x,t) & = & \pm c\,\phi_x(x,t) \\ \phi(x,0) & = & f(x), \quad x \in \mathbb{R} \end{array}\right\} \;\Rightarrow\; \left\{\begin{array}{rcl} \dfrac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} & = & \pm c\,\dfrac{\mathrm{d}}{\mathrm{d}\,x}\,\psi(t) \\ \psi(0) & = & f \,\in\, L_2(-\infty,\infty) \end{array}\right.$$

- Consider:

$$\phi(x,t) \;=\; [T(t)\,f]\,(x) \;=\; f(x \,\pm\, ct)$$

In class: $T(t)$ defines a C_0-semigroup on $L_2(-\infty,\infty)$ ▮

- The infinitesimal generator of a C_0-semigroup $T(t)$ on $\mathbb{H}$

$$\mathcal{A}\,f \;=\; \lim_{t\,\to\,0^+} \frac{T(t)\,f \,-\, f}{t}$$

$$\mathcal{D}(\mathcal{A}) \;=\; \left\{ f \,\in\, \mathbb{H};\; \lim_{t\,\to\,0^+} \frac{T(t)\,f \,-\, f}{t} \text{ exists} \right\}$$

- A couple of additional notes

  - Change of coordinates:

$$\left.\begin{array}{rcl}\phi_t(x,t) & = & \pm c\,\phi_x(x,t) \\ \phi(x,0) & = & f(x), \quad x \in \mathbb{R}\end{array}\right\} \xrightarrow{z \,=\, x \,\pm\, ct} \left\{\begin{array}{rcl}\phi_t(z,t) & = & 0 \\ \phi(z,0) & = & f(z), \quad z \in \mathbb{R}\end{array}\right.$$

  - Reaction-convection equation:

$$\left.\begin{array}{rcl}\phi_t(x,t) & = & \pm c\,\phi_x(x,t) \;+\; a\,\phi(x,t) \\ \phi(x,0) & = & f(x), \quad x \in \mathbb{R}\end{array}\right\}$$

C_0-semigroup:

$$\phi(x,t) \;=\; [T(t)\,f]\,(x) \;=\; \mathrm{e}^{a\,t} f(x \pm ct)$$

$a > 0$ exponentially growing traveling wave

$a < 0$ exponentially decaying traveling wave

Infinite number of decoupled scalar states

- Abstract evolution equation on $\ell_2(\mathbb{N})$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\begin{bmatrix}\psi_1(t)\\ \psi_2(t)\\ \vdots\end{bmatrix} = \begin{bmatrix}a_1 & & \\ & a_2 & \\ & & \ddots\end{bmatrix}\begin{bmatrix}\psi_1(t)\\ \psi_2(t)\\ \vdots\end{bmatrix} \;\Leftrightarrow\; \frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} = \mathcal{A}\,\psi(t)$$

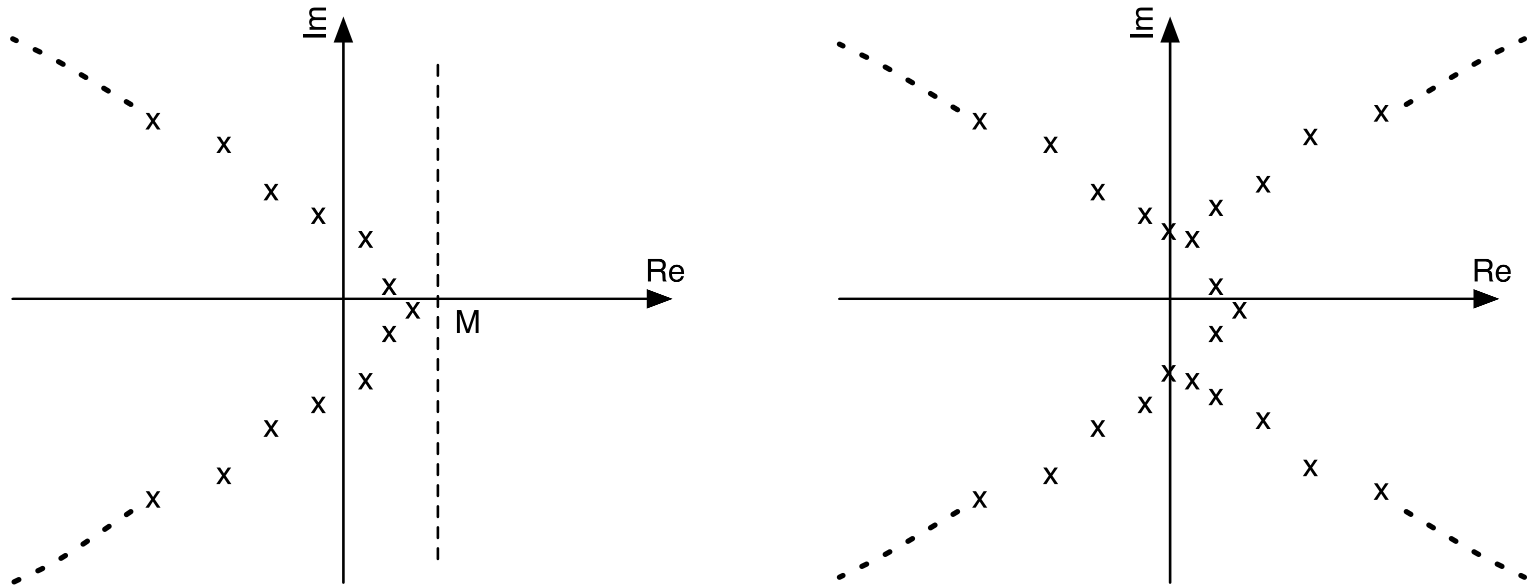
Solution

$$\psi(t) = \begin{bmatrix}\psi_1(t)\\ \psi_2(t)\\ \vdots\end{bmatrix} = \begin{bmatrix}\mathrm{e}^{a_1\,t} & & \\ & \mathrm{e}^{a_2\,t} & \\ & & \ddots\end{bmatrix}\begin{bmatrix}\psi_1(0)\\ \psi_2(0)\\ \vdots\end{bmatrix} = T(t)\,\psi(0)$$

- In class: conditions for well-posedness on $\ell_2(\mathbb{N})$

- Half-plane condition:

$$\sup_{n} Re\,(a_n) \;<\; M \;<\; \infty$$



(a) (b)

Same condition for:

$$T(t)\, f \;=\; \sum_{n\,=\,1}^{\infty} \mathrm{e}^{a_n\, t}\, v_n\, \langle v_n, f \rangle$$

Continuum of decoupled scalar states

$$\dot{\psi}(\kappa, t) \;=\; a(\kappa)\,\psi(\kappa, t), \quad \kappa \,\in\, \mathbb{R}$$

Solution

$$\psi(\kappa, t) \;=\; [T(t)\,\psi(\cdot, 0)]\,(\kappa) \;=\; \mathrm{e}^{a(\kappa)\,t}\,\psi(\kappa, 0)$$

- Homework: conditions for well-posedness on $L_2\,(-\infty, \infty)$ ▮

Half-plane condition:

$$\sup_{\kappa \,\in\, \mathbb{R}} \; Re\,(a(\kappa)) \;<\; M \;<\; \infty$$

Hille-Yosida Theorem

closed, densely defined operator $\mathcal{A}$ on $\mathbb{H}$:

$\mathcal{A}$ - infinitesimal generator of a C_0-semigroup with $\|T(t)\| \leq M\,\mathrm{e}^{\omega\,t}$

$$\Updownarrow$$

every real $\lambda > \omega$ is in $\rho(\mathcal{A})$ and $\|\,(\lambda I - \mathcal{A})^{-n}\,\| \leq \dfrac{M}{(\lambda - \omega)^n}$ for all $n \geq 1$

- Difficult to check
- Important consequence: a method for computing $T(t)$

$$T(t) = \lim_{N \to \infty} \left(I - \frac{t}{N}\mathcal{A} \right)^{-N}$$

Implicit Euler:

$$\frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} = \mathcal{A}\,\psi(t) \quad \Rightarrow \quad \frac{\psi(t + \Delta t) - \psi(t)}{\Delta t} = \mathcal{A}\,\psi(t + \Delta t)$$

Lumer-Phillips Theorem

closed, densely defined operator $\mathcal{A}$ on $\mathbb{H}$:

$$Re\,(\langle \psi, \mathcal{A}\,\psi \rangle) \;\leq\; \omega\,\|\psi\|^2 \quad \text{for all } \psi \in \mathcal{D}(\mathcal{A})$$

$$Re\,\left(\left\langle \psi, \mathcal{A}^\dagger\,\psi \right\rangle\right) \;\leq\; \omega\,\|\psi\|^2 \quad \text{for all } \psi \in \mathcal{D}(\mathcal{A}^\dagger)$$

$$\Downarrow$$

$\mathcal{A}$ - infinitesimal generator of a C_0-semigroup with $\|T(t)\| \leq \mathrm{e}^{\omega\,t}$

- Examples:

$$\left\{ \begin{array}{rcl} [\mathcal{A}\,f]\,(x) & = & \left[\dfrac{\mathrm{d}f}{\mathrm{d}x}\right](x) \\ \mathcal{D}(\mathcal{A}) & = & \left\{ f \in L_2\,[-1,\,1],\; \dfrac{\mathrm{d}f}{\mathrm{d}x} \in L_2\,[-1,\,1],\; f(1) = 0 \right\} \end{array} \right.$$

$$\left\{ \begin{array}{rcl} [\mathcal{A}\,f]\,(x) & = & \left[\dfrac{\mathrm{d}^2f}{\mathrm{d}x^2}\right](x) \\ \mathcal{D}(\mathcal{A}) & = & \left\{ f \in L_2\,[-1,\,1],\; \dfrac{\mathrm{d}^2f}{\mathrm{d}x^2} \in L_2\,[-1,\,1],\; f(\pm 1) = 0 \right\} \end{array} \right.$$