10 - 06 - 11

Eigenvalue - Decomposition

<u>Ex</u> $\quad v'' = a v, \quad v(\pm 1) = 0$

$$\begin{cases} \psi_1 = v \\ \psi_2 = v' \end{cases}$$

$$\begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}}^{A} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{N_1} \begin{bmatrix} \psi_1(-1) \\ \psi_2(-1) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{N_2} \begin{bmatrix} \psi_1(1) \\ \psi_2(1) \end{bmatrix}$$

$$\boxed{\begin{aligned} \psi(x) &= e^{A(x+1)} \psi(-1) \\ 0 &= N_1 \psi(-1) + N_2 \psi(1) \\ &= (N_1 + N_2 e^{2A}) \psi(-1) \end{aligned}}$$

$$\boxed{\text{Non-trivial solutions: } \det(N_1 + N_2 e^{2A}) = 0}$$

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① Continuity with respect to initial condition:

i.c. $\quad f, g \in \mathbb{H}$

$$\| f - g \| \text{ small} \implies \| T(t)(f - g) \| \text{ small}$$

(small changes in initial condition, don't change response by much.)

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③ <u>Strong Continuity</u>:

For fixed initial condition, small change in time shouldn't change the responce by much.

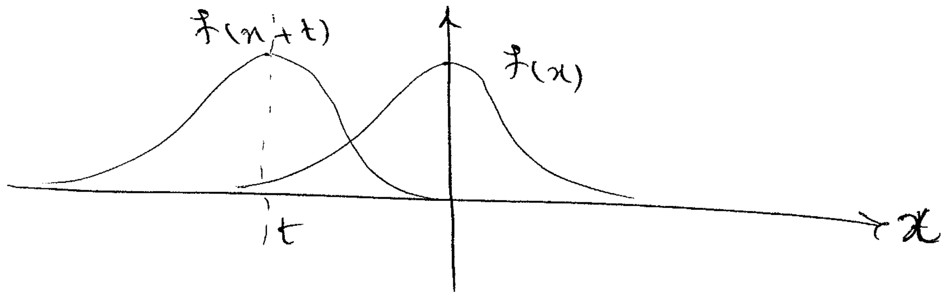
$$\lim_{\Delta t \to 0} \| T(t + \Delta t) \psi(0) - \psi(t) \| = \quad \text{(by semi-group property)}$$

$$= \lim_{\Delta t \to 0} \| T(t) (T(\Delta t) - I) \psi(0) \| \leq \quad \text{(by boundedness of } T(t) \text{ that follows from ①)}$$

$$\leq M \underbrace{\lim_{\Delta t \to 0} \| T(\Delta t) \psi(0) - \psi(0) \|}_{=0 \quad \text{strong Continuity}} = 0$$

Ex Transport operator

$$\phi(x,t) = [T(t)f](x) = f(x+t)$$



Properties of $T(t)$:

① $T(0) \Rightarrow \phi(x,0) = f(x) = [T(0)f](x)$

$\Rightarrow T(0) = I$ ✓

② $T(t_1+t_2) = T(t_2)T(t_1)$

③ Is $T(t)$ bounded?

if $f \in L_2 \Rightarrow \phi(\cdot,t) \in L_2$

Yes! why? $\int_{-\infty}^{\infty} (f(x+t))^2 dx = \int_{-\infty}^{\infty} (f(x))^2 dx$

$T(t)$ bounded with norm:

$$\|T(t)\| = \sup_{\substack{f \in L_2 \\ f \neq 0}} \frac{\|T(t)f\|}{\|f\|} = 1 < \infty$$

④ Strong Continuity:

$$\lim_{t \to 0^+} \|T(t)f - f\|^2 = 0 \qquad \forall f \in L_2(-\infty,\infty)$$

$$\int_{-\infty}^{\infty} (f(x+t) - f(x))^2 dx$$

<u>Fact</u> any function $f \in L_2$ can be approximated by (in L_2 sense) a continuous function h with compact support.

$$\int_{-\infty}^{\infty} (f(x+t) - f(x))^2 \, dx$$

$$= \int_{-\infty}^{\infty} (T(t) f(x) - f(x))^2 \, dx$$

$$= \| T(t) f - f \|^2 = \| T(t)(f - h + h) - (f - h + h) \|^2$$

$$= \| T(t)(f-h) + (T(t)h - h) - (f-h) \|^2 \leq$$

$$\underbrace{\| T(t)(f-h) \|^2}_{< \varepsilon/3} + \underbrace{\| T(t)h - h \|^2}_{< \varepsilon/3} + \underbrace{\| f - h \|^2}_{< \varepsilon/3}$$

As $t \to 0$

Can be made smaller than ε.

boundedness of $T(t)$

the fact above

$$\lim_{t \to 0} \int_{-\infty}^{\infty} (h(x+t) - h(x))^2 \, dx = 0$$

<u>Ex</u> Infinite number of decoupled scalar states

$\mathcal{A} : l_2 \longrightarrow l_2$

$$\frac{d}{dt} \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \end{bmatrix}$$

① if $\mathcal{A}$ is bounded ?

$g = \mathcal{A} f \Rightarrow g_n = a_n f_n$

$\{f_n\} \in l_2 \overset{?}{\Longrightarrow} \{g_n\} \in l_2$

$$\sum_{n=1}^{\infty} |g_n|^2 = \sum_{n=1}^{\infty} |a_n f_n|^2 = \sum_{n=1}^{\infty} |a_n|^2 |f_n|^2 \leq \sup_n |a_n|^2 \sum_{n=1}^{\infty} |f_n|^2$$

$$= \sup_n |a_n|^2 \, \|f\|^2_{l_2}$$