Lecture 9: Spectral theory for compact normal operators

- Resolvent and spectrum of an operator

- Compact operators
  - ⋆ Direct extension of matrices

- Normal operators
  - ⋆ Commute with its adjoint

- Compact normal operators
  - ⋆ Unitarily diagonalizable
  - ⋆ E-functions provide a complete orthonormal basis of $\mathbb{H}$

- Riesz-spectral operators

Resolvent

- Want to study equations of the form

$$(\lambda I - \mathcal{A})\,\psi = u, \quad \{\mathcal{A} : \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}; \;\; \lambda \in \mathbb{C}; \;\; \psi, u \in \mathbb{H}\}$$

Determine conditions under which $\mathcal{A}_\lambda = (\lambda I - \mathcal{A})$ is boundedly invertible

Relevant conditions:

$$\begin{cases} (1) & \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ exists} \\ (2) & \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ is bounded} \\ (3) & \text{The domain of } \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ is dense in } \mathbb{H} \end{cases}$$

- The resolvent set of $\mathcal{A}$:

$$\rho(\mathcal{A}) := \{\lambda \in \mathbb{C}; \text{ (1), (2), (3) hold}\}$$

- The spectrum of $\mathcal{A}$:

$$\sigma(\mathcal{A}) := \mathbb{C} \setminus \rho(\mathcal{A})$$

Spectrum

$$(1) \quad \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ exists}$$

$$(2) \quad \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ is bounded}$$

$$(3) \quad \text{The domain of } \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ is dense in } \mathbb{H}$$

- $\sigma(\mathcal{A})$ can be decomposed into

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_c(\mathcal{A}) \cup \sigma_r(\mathcal{A})$$

⋆ Point spectrum

$$\sigma_p(\mathcal{A}) := \{\lambda \in \mathbb{C};\ (\lambda I - \mathcal{A}) \text{ is not one-to-one}\}$$

⋆ Continuous spectrum

$$\sigma_c(\mathcal{A}) := \{\lambda \in \mathbb{C};\ \text{(1) and (3) hold, but (2) doesn't}\}$$

⋆ Residual spectrum

$$\sigma_r(\mathcal{A}) := \{\lambda \in \mathbb{C};\ \text{(1) holds but (3) doesn't}\}$$

Examples

- Point spectrum

$$\{\lambda \in \sigma_p(\mathcal{A}): \text{ e-values; } \; v \in \mathcal{N}(\lambda I - \mathcal{A}): \text{ e-functions}\}$$

- Continuous spectrum

$$\text{multiplication operator on } L_2[a,b]: \;\; [M_a \, f(\cdot)]\,(x) \;=\; a(x)\,f(x)$$

- Residual spectrum

$$\text{right-shift operator on } \ell_2(\mathbb{N}): \;\; [S_r \, f(\cdot)]\,(n) \;=\; f_{n-1}$$

Spectral decomposition of compact normal operators

- compact, normal operator $\mathcal{A}$ on $\mathbb{H}$ admits a dyadic decomposition

$$\left.\begin{array}{rcl} [\mathcal{A}\, v_n]\,(x) & = & \lambda_n\, v_n(x) \\ \langle v_n, v_m \rangle & = & \delta_{nm} \end{array}\right\} \Rightarrow \; [\mathcal{A}\, f]\,(x) \; = \; \sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\, \langle v_n, f \rangle \quad \text{for all } f \, \in \, \mathbb{H}$$

$$\mathcal{A}\colon \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \; \longrightarrow \; \mathbb{H}, \quad \text{with compact and normal } \mathcal{A}^{-1}$$

$$\downarrow$$

$$\left.\begin{array}{rcl} \left[\mathcal{A}^{-1}\, v_n\right](x) & = & \lambda_n^{-1}\, v_n(x) \\ \langle v_n, v_m \rangle & = & \delta_{nm} \end{array}\right\} \Rightarrow \; \left[\mathcal{A}^{-1} f\right](x) \; = \; \sum_{n\,=\,1}^{\infty} \lambda_n^{-1}\, v_n(x)\, \langle v_n, f \rangle\,, \quad f \, \in \, \mathbb{H}$$

$$[\mathcal{A}\, f]\,(x) \; = \; \sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\, \langle v_n, f \rangle\,, \quad f \, \in \, \mathcal{D}(\mathcal{A})$$

$$\mathcal{D}(\mathcal{A}) \; = \; \left\{ f \, \in \, \mathbb{H}; \;\; \sum_{n\,=\,1}^{\infty} |\lambda_n|^2\, |\langle v_n, f \rangle|^2 \, < \, \infty \right\}$$

- compact, normal operator $\mathcal{A}$ on $\mathbb{H}$

$$\left.\begin{array}{rcl} [\mathcal{A} v_n](x) & = & \lambda_n v_n(x), \quad \lambda_n \neq 0 \\ \langle v_n, v_m \rangle & = & \delta_{nm} \end{array}\right\} \quad \left\{\begin{array}{rcl} u & = & u_{\mathcal{R}(\mathcal{A})} + u_{\mathcal{N}(\mathcal{A})} \\ & = & \displaystyle\sum_{n=1}^{\infty} v_n \langle v_n, u \rangle + u_{\mathcal{N}(\mathcal{A})} \end{array}\right.$$

- Solutions to

$$(\lambda I - \mathcal{A}) \psi = u, \quad \lambda \neq 0$$

1. λ – not an eigenvalue of $\mathcal{A}$ $\Rightarrow$ unique solution

$$\psi = \sum_{n=1}^{\infty} \frac{\langle v_n, u \rangle}{\lambda - \lambda_n} v_n + \frac{1}{\lambda} u_{\mathcal{N}(\mathcal{A})}$$

2. $\left.\begin{array}{l} \lambda \ - \ \text{eigenvalue of } \mathcal{A} \\ J \ - \ \text{index set s.t. } \lambda_j = \lambda \end{array}\right\} \Rightarrow$ there is a solution iff $\langle v_j, u \rangle = 0$ for all $j \in J$

$$\psi = \sum_{j \in J} c_j v_j + \sum_{j \in \mathbb{N} \setminus J} \frac{\langle v_j, u \rangle}{\lambda - \lambda_j} v_j + \frac{1}{\lambda} u_{\mathcal{N}(\mathcal{A})}$$

Singular Value Decomposition of compact operators

- compact operator $\mathcal{A}$: $\mathbb{H}_1 \longrightarrow \mathbb{H}_2$ admits a Schmidt Decomposition (i.e., an SVD)

$$[\mathcal{A}\, f]\,(x) \;=\; \sum_{n\,=\,1}^{\infty} \sigma_n\, u_n(x)\, \langle v_n, f \rangle$$

$$\left[\mathcal{A}\,\mathcal{A}^\dagger\, u_n\right](x) \;=\; \sigma_n^2\, u_n(x) \;\Rightarrow\; \{u_n\}_{n\,\in\,\mathbb{N}} \text{ orthonormal basis of } \mathbb{H}_2$$

$$\left[\mathcal{A}^\dagger\,\mathcal{A}\, v_n\right](x) \;=\; \sigma_n^2\, v_n(x) \;\Rightarrow\; \{v_n\}_{n\,\in\,\mathbb{N}} \text{ orthonormal basis of } \mathbb{H}_1$$

- matrix M: $\mathbb{C}^n \longrightarrow \mathbb{C}^m$

$$M \;=\; U\,\Sigma\,V^* \;=\; \sum_{i\,=\,1}^{r} \sigma_i\, u_i\, v_i^* \;\Rightarrow\; M\,f \;=\; \sum_{i\,=\,1}^{r} \sigma_i\, u_i\, \langle v_i, f \rangle$$

$$M\,M^*\,u_i \;=\; \sigma_i^2\, u_i$$

$$M^*\,M\,v_i \;=\; \sigma_i^2\, v_i$$

Riesz basis

- $\{v_n\}_{n \in \mathbb{N}}$: Riesz basis of $\mathbb{H}$ if
  - $\overline{\text{span}\,\{v_n\}_{n \in \mathbb{N}}} \;=\; \mathbb{H}$
  - there are $m, M > 0$ such that for any $N \in \mathbb{N}$ and any $\{\alpha_n\}$, $n = 1, \ldots, N$

$$m \sum_{n=1}^{N} |\alpha_n|^2 \;\leq\; \| \sum_{n=1}^{N} \alpha_n \, v_n \|^2 \;\leq\; M \sum_{n=1}^{N} |\alpha_n|^2$$

- closed $\mathcal{A} : \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}$

$$[\mathcal{A}\, v_n]\,(x) \;=\; \lambda_n\, v_n(x) \quad \begin{cases} \{\lambda_n\}_{n\,\in\,\mathbb{N}} & \text{simple e-values} \\ \{v_n\}_{n\,\in\,\mathbb{N}} & \text{Riesz basis of } \mathbb{H} \end{cases}$$

⋆ $\left[\mathcal{A}^{\dagger}\, w_n\right](x) \;=\; \bar{\lambda}_n\, w_n(x) \;\Rightarrow\; \{w_n\}_{n\,\in\,\mathbb{N}}$ can be scaled s.t. $\langle w_n, v_m \rangle \;=\; \delta_{nm}$

⋆ every $f \in \mathbb{H}$ can be represented uniquely by

$$f(x) \;=\; \sum_{n\,=\,1}^{\infty} v_n(x)\, \langle w_n, f \rangle$$

$$m \sum_{n\,=\,1}^{\infty} |\, \langle w_n, f \rangle\, |^2 \;\leq\; \|f\|^2 \;\leq\; M \sum_{n\,=\,1}^{\infty} |\, \langle w_n, f \rangle\, |^2$$

or by

$$f(x) \;=\; \sum_{n\,=\,1}^{\infty} w_n(x)\, \langle v_n, f \rangle$$

$$\frac{1}{M} \sum_{n\,=\,1}^{\infty} |\, \langle v_n, f \rangle\, |^2 \;\leq\; \|f\|^2 \;\leq\; \frac{1}{m} \sum_{n\,=\,1}^{\infty} |\, \langle v_n, f \rangle\, |^2$$

Riesz-spectral operator

- closed $\mathcal{A} : \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}$ is Riesz-spectral operator if

$$[\mathcal{A}\, v_n]\,(x) \;=\; \lambda_n\, v_n(x) \qquad \begin{cases} \{\lambda_n\}_{n\,\in\,\mathbb{N}} & \text{simple e-values} \\ \{v_n\}_{n\,\in\,\mathbb{N}} & \text{Riesz basis of } \mathbb{H} \\ \overline{\{\lambda_n\}_{n\,\in\,\mathbb{N}}} & \text{totally disconnected} \end{cases}$$

$\mathcal{A}$ – Riesz-spectral operator with e-pair $\{(\lambda_n, v_n)\}_{n\,\in\,\mathbb{N}}$

$\{w_n\}_{n\,\in\,\mathbb{N}}$ – e-functions of $\mathcal{A}^\dagger$ s.t. $\langle w_n, v_m \rangle \;=\; \delta_{nm}$

$$\downarrow$$

$$\begin{cases} \sigma(\mathcal{A}) \;=\; \overline{\{\lambda_n\}_{n\,\in\,\mathbb{N}}}, \quad \rho(\mathcal{A}) \;=\; \{\lambda_n \,\in\, \mathbb{C};\;\; \inf_{n\,\in\,\mathbb{N}} |\lambda \,-\, \lambda_n| \;>\; 0\} \\ \lambda \,\in\, \rho(\mathcal{A}) \;\Rightarrow\; \left[(\lambda I \,-\, \mathcal{A})^{-1} f\right](x) \;=\; \displaystyle\sum_{n\,=\,1}^{\infty} \frac{1}{\lambda \,-\, \lambda_n}\, v_n(x)\, \langle w_n, f \rangle \\ [\mathcal{A}\, f]\,(x) \;=\; \displaystyle\sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\, \langle w_n, f \rangle\,, \quad \mathcal{D}(\mathcal{A}) \;=\; \left\{ f \in \mathbb{H};\; \sum_{n\,=\,1}^{\infty} |\lambda_n|^2\, |\langle w_n, f \rangle|^2 \;<\; \infty \right\} \end{cases}$$