Resolvent

10 - 04 - 11

$$\frac{\partial \psi}{\partial t} = \mathcal{A}\psi + u$$

Apply Lapplas transform : $\quad s\psi(s) - \psi(0) = \mathcal{A}\psi + u$

$$\Rightarrow \quad \psi(s) = (sI - \mathcal{A})^{-1}(\psi(0) + u)$$

In finite dimensions : $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

$$y(s) = \underbrace{\left[C(sI - A)^{-1}B\right]}_{\text{Transfer function}} u(s) + C(sI - A)^{-1}x(0)$$

$(sI - A)^{-1} = R_s(A)$ ---- resolvent of A

<u>Ex</u> $\quad A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

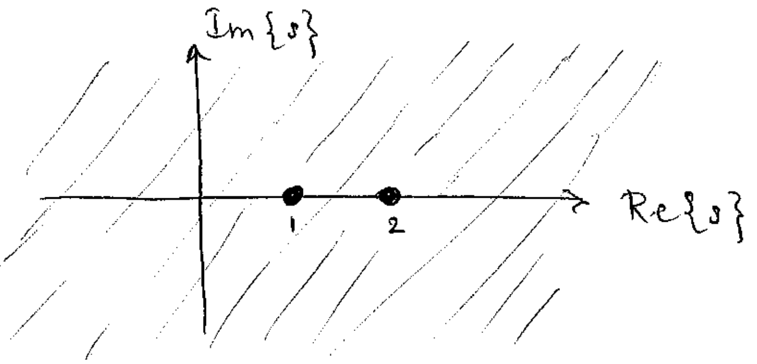
$$sI - A = \begin{bmatrix} s-1 & 0 \\ -3 & s-2 \end{bmatrix}$$

$$\det(sI - A) = (s-1)(s-2)$$

$sI - A$ invertible $\iff s \neq 1,\ s \neq 2$

resolvent set of A : $\rho(A) = \{ s \in \mathbb{C} ;\ s \neq 1,\ s \neq 2 \}$

Spectrum of A : $\sigma(A) = \mathbb{C} \setminus \rho(A) = \{ s = 1, s = 2 \}$



Shaded region denotes $\rho(A)$.

$\sigma(A)$ is only the two points $s=1, s=2$.

So, in finite dimensions, spectrum of A is only a point spectrum:

$$\sigma(A) = \sigma_p(A)$$

In infinite dimensions:

$$\sigma(A) = \underbrace{\sigma_p(A)}_{\text{Point spectrum}} \cup \underbrace{\sigma_c(A)}_{\text{Continuous spectrum}} \cup \underbrace{\sigma_r(A)}_{\text{residual spectrum}}$$

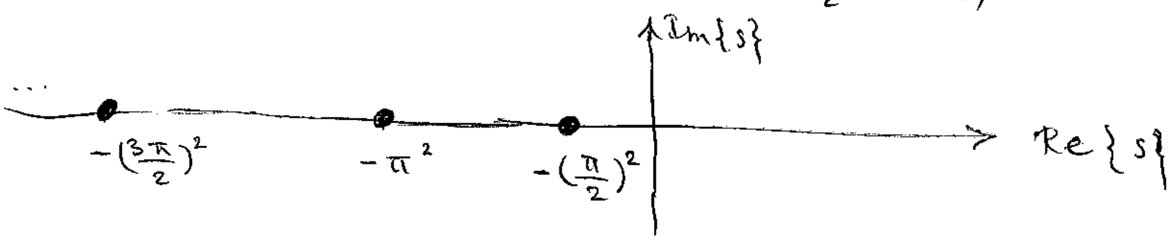
$$\sigma_p(A) = \{ s \in \mathbb{C},\ \text{s.t. } (sI - \mathcal{A}) \text{ is not } \underbrace{1 \text{ to } 1}_{\text{(invertable)}} \}$$

These points are called eigenvalues with eigen-vectors $v \in \mathcal{N}(sI - \mathcal{A})$

Example: $\mathcal{A} = \dfrac{d^2}{dx^2}$; $v(\pm 1) = 0$

$$\sigma_p(\mathcal{A}) = \left\{ -\left(\frac{n\pi}{2}\right)^2 ;\ n = 1, 2, \ldots \right\}$$

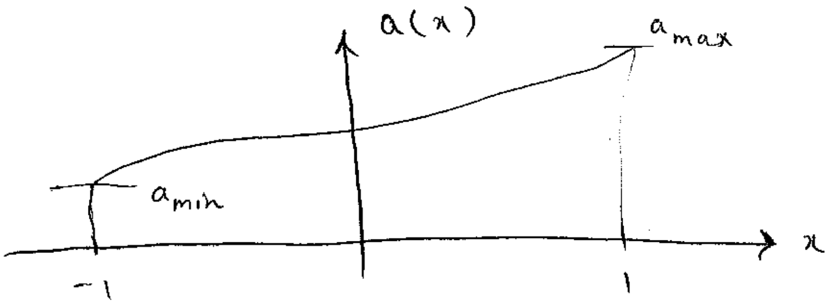
$$v_n(x) = \sin\left(\frac{n\pi}{2}(x+1)\right)$$

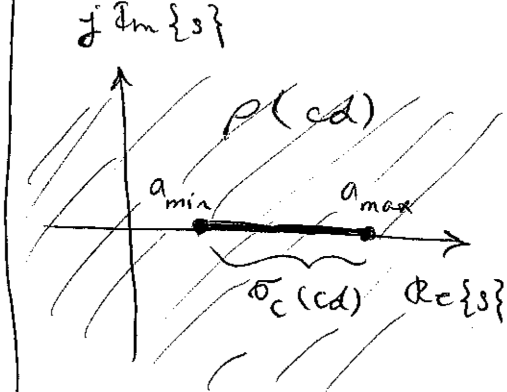


Example $\boxed{\text{Continuous spectrum}}$

$$[\mathcal{A} f](x) = a(x) f(x)$$

Multiplication operator on $L_2[-1,1]$





$$g(x) = a(x) \cdot f(x)$$

Want to study: $(\lambda I - \mathcal{A}) f = g$

$$(\lambda - a(x)) f(x) = g(x) \longrightarrow f(x) = \frac{1}{\lambda - a(x)} g(x)$$

$$= [(\lambda I - \mathcal{A})^{-1} g](x)$$

$$\mathcal{D}(R_\lambda(\mathcal{A})) = \{ g, \text{ s.t. } (\lambda I - \mathcal{A})^{-1} g \in L_2 \}$$

Easy to see that $\mathcal{D}(\mathcal{A})$ is dense in L_2

<u>Q</u>: under what conditions $R_\lambda(\mathcal{A})$ is bounded ?

$$\|f\|^2_{L_2} = \int_{-1}^{1} \frac{1}{(\lambda - a(x))^2} g^2(x)\, dx \leq \sup_x \frac{1}{(\lambda - a(x))^2} \|g\|^2_{L_2}$$

Note: $\left(\sup_x \frac{1}{(\lambda - a(x))^2} < \infty \right) \Leftrightarrow \left(\lambda \notin (a_{min}, a_{max}) \right)$

<u>Then</u>

$$\boxed{\begin{aligned} \rho(\mathcal{A}) &= \{ \lambda \in \mathbb{C}, \text{ s.t. } \lambda \notin (a_{min}, a_{max}) \} \\ \sigma(\mathcal{A}) &= \sigma_c(\mathcal{A}) = \{ \lambda \in \mathbb{C}, \lambda \in (a_{min}, a_{max}) \} \end{aligned}}$$

Ex $\boxed{\text{Residual spectrum}}$

$$\mathcal{A} : \ell_2 \longrightarrow \ell_2$$

$$g = [\mathcal{A} f] = [S_r f] = \{ f_{n-1} \}$$

Right shift

$$\begin{bmatrix} g_1 \\ g_2 \\ \vdots \end{bmatrix} = S_r \begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}$$

$$(\lambda I - \mathcal{A}) f = (\lambda I - S_r) f = \lambda \begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix} - \begin{bmatrix} 0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \lambda f_1 \\ \lambda f_2 - f_1 \\ \lambda f_3 - f_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

if $\lambda \neq 0 \Rightarrow f = 0$ $(\lambda I - \mathcal{A})$ invertable.

if $\lambda = 0 \Rightarrow -S_r f = g = \begin{bmatrix} 0 \\ -f_1 \\ -f_2 \\ \vdots \end{bmatrix}$

$$\Rightarrow f = (\lambda I - \mathcal{A})^{-1} g \Big|_{\lambda = 0}$$

$$= -\lambda_\ell g$$

So inverse of $(\lambda I - S_r)\Big|_{\lambda = 0}$ exists, it is $(-S_\ell)$,

But its domain in not dense in ℓ_2.

$$R_0(\mathcal{A}) = (\lambda I - \mathcal{A})^{-1}\Big|_{\lambda = 0} = -S_\ell$$

$$\mathcal{D}(R_0(\mathcal{A})) = \left\{ g \in \ell_2 ; \; g = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}^{\perp} \right\} \text{ or } g = \begin{bmatrix} 0 \\ g_1 \\ g_2 \\ \vdots \end{bmatrix}$$

$$g = \begin{bmatrix} 0 \\ g_1 \\ g_2 \\ \vdots \end{bmatrix}$$

Consider sequences of the form $\begin{bmatrix} 0 \\ g_{1n} \\ g_{2n} \\ \vdots \end{bmatrix}$

So, this can never recover an element of l_2 in the direction of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$.

<u>Thus</u> $\mathcal{D}(R_0(cd))$ is not dense in l_2.

$$\boxed{\sigma_r(cd) = \{\lambda = 0\}}$$

$cd : H \supset \mathcal{D}(cd) \longrightarrow H$

cd^{-1} : Compact, normal
$\downarrow$
finite HS norm :

$$\int_a^b \int_a^b \text{trace}\left(A_K(x,\xi) A_K^*(x,\xi)\right) dx\, d\xi < +\infty$$

$$\begin{cases} cd^{-1} v_n = \dfrac{1}{\lambda_n} v_n \\ cd\, v_n = \lambda_n v_n \end{cases}$$

Matrix M:

$$M = U\Sigma V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

$$UU^* = U^*U = I$$

$$VV^* = V^*V = I$$

$$MM^* = U\Sigma \underbrace{V^*V}_{I} \Sigma^* U^* = U\Sigma\Sigma^* U^*$$

$$MM^* u_i = \sigma_i^2 u_i \quad ; \quad U = [u_1, \dots, u_m]$$

$$M^*M v_i = \sigma_i^2 v_i$$

$$Mf = \sum_{i=1}^{r} \sigma_i u_i \langle v_i, f \rangle$$

$$\|M\|_{2i} = \underbrace{\sigma_1}_{\text{maximum singular value of } M} \quad \dots \quad \text{induced 2-norm}$$

$$\|M\|_F^2 = \sum_{i=1}^{r} \sigma_i^2$$

↑ overview of SVD in finite dimensions.