Self-adjoint operator $\mathcal{L}$

$\boxed{09-29-11}$

$$\langle f, \mathcal{L}g \rangle = \langle \mathcal{L}f, g \rangle$$

for all $f, g \in \mathcal{D}(\mathcal{L})$; $\mathcal{D}(\mathcal{L}^+) = \mathcal{D}(\mathcal{L})$

(Boundary Conditions matter)

Ex.

$$\begin{cases} \mathcal{L} = \dfrac{d}{dx} & \text{with } g(-1) = 0 \\ \mathcal{L}^+ = -\dfrac{d}{dx} & \text{with } f(1) = 0 \end{cases}$$

Not self-adjoint

Ex.

$$\boxed{\mathcal{L} = j\frac{d}{dx} \quad \text{with } g(-1) = 0}$$

; $j = \sqrt{-1}$

$$\langle f, j\tfrac{d}{dx} g \rangle = j\, f(x) g(x) \Big|_{-1}^{1} + \langle j \tfrac{d}{dx} f, g \rangle$$

$$= j\, f(1) g(1) - j f(-1) \underbrace{g(-1)}_{0} + \langle j \tfrac{d}{dx} f, g \rangle$$

$\downarrow$

Let $f(1) = 0$

$$\boxed{\mathcal{L}^+ = j\frac{d}{dx} \quad \text{with } f(1) = 0}$$

! $\Bigg\{$ $\mathcal{L}$ and $\mathcal{L}^+$ have the same symbol $j\frac{d}{dx}$.
But their domains are different.
So $j\frac{d}{dx}$ is not self-adjoint with the above domains.

We showed that eigenvalues of a self-adjoint operator are <u>real</u>.

Now, we show that eigen-vectors corresponding to two different eigenvalues are orthogonal.

$\mathcal{L}$ ... self-adjoint

$\mathcal{L} v = \lambda v$

1. $\lambda \in \mathbb{R}$
2. $\lambda_n, \lambda_m, \quad \lambda_n \neq \lambda_m \Rightarrow \langle v_n, v_m \rangle = 0$

$$\lambda_m \langle v_n, v_m \rangle = \langle v_n, \lambda_m v_m \rangle = \underline{\langle v_n, \mathcal{L} v_m \rangle = \langle \mathcal{L} v_n, v_m \rangle} = \langle \lambda_n v_n, v_m \rangle$$
$$= \lambda_n \langle v_n, v_m \rangle$$

$$\left.\begin{array}{l} \Rightarrow \langle v_n, v_m \rangle (\lambda_m - \lambda_n) = 0 \\ \lambda_m \neq \lambda_n \end{array}\right\} \Rightarrow \boxed{\langle v_n, v_m \rangle = 0}$$

<u>Ex</u> $\boxed{\mathcal{L} = \frac{d^2}{dx^2} \oplus g(\pm 1) = 0}$

$$\langle f, \mathcal{L} g \rangle = \langle \mathcal{L}^+ f, g \rangle$$

$$\langle f, g'' \rangle = f(x) g'(x) \Big|_{-1}^{1} - \langle f', g' \rangle$$
$$= f(x) g'(x) \Big|_{-1}^{1} - f'(x) g(x) \Big|_{-1}^{1} + \langle f'', g \rangle$$
$$= f(1) g'(1) - f(-1) g'(-1) - f'(1) \underbrace{g(1)}_{0} + f'(-1) \underbrace{g(-1)}_{0}$$
$$+ \langle f'', g \rangle$$

Since $g'(\pm 1)$ is arbitrary, we need $\underline{f(\pm 1) = 0}$.

$\boxed{\mathcal{L}^+ = \frac{d^2}{dx^2} \oplus f(\pm 1) = 0}$ $\qquad \mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L}^+)$

So, $\mathcal{L}$ is self-adjoint.

Ex.
$$\left.\begin{aligned} f'(x) &= g(x) \\ f(-1) &= 0 \end{aligned}\right\} \Rightarrow f(x) - \cancelto{0}{f(-1)} = \int_{-1}^{x} g(\xi)\, d\xi$$

$$[\mathcal{L} f](x) = g(x) \quad \Rightarrow \quad f(x) = [\mathcal{L}^{-1} g](x) = \int_{-1}^{x} g(\xi)\, d\xi = \int_{-1}^{1} \mathbb{1}(x-\xi)\, g(\xi)\, d\xi$$

$$f(x) = \int_{-1}^{x} \underbrace{K(x,\xi)}_{1}\, g(\xi)\, d\xi$$

$\mathcal{L}^{-1}$ bounded operator (because its kernel is bounded)

$(\mathcal{L}^{-1})^{+}$ .... $\quad h(x) = \int_{x}^{1} q(\xi)\, d(\xi)$

$$\mathcal{L}^{-1} : L_2[-1,1] \longrightarrow L_2[-1,1]$$

$$\begin{tikzcd} L_2 \arrow[r, bend left, "\mathcal{L}^{-1}"] & L_2 \arrow[l, bend left, "(\mathcal{L}^{-1})^{+}"] \end{tikzcd} \qquad \begin{aligned} \mathcal{L}^{-1} &: g \longrightarrow f \\ (\mathcal{L}^{-1})^{+} &: q \longrightarrow h \end{aligned}$$

$$h(x) = \int_{x}^{1} q(\xi)\, d\xi = [(\mathcal{L}^{-1})^{+} q](x)$$
$$= [\mathcal{B} q](x)$$

$$h'(x) = -q(x)$$

b.c. $\quad h(1) = 0 = \int_{1}^{1} q(\xi)\, d\xi$

Eigenvalue decomposition of $\left.\dfrac{d^2}{dx^2}\right|_{v(\pm 1)=0}$

$$\underbrace{\frac{d^2 v}{dx^2}}_{\mathcal{A} v} = \lambda v \;;\; (v(\pm 1) = 0)$$

$v'' - \lambda v = 0$

$s^2 - \lambda = 0 \Rightarrow s = \pm\sqrt{\lambda}$

$\left.\begin{array}{l} \lambda \in \mathbb{R} \\ \text{because} \\ \mathcal{A} \text{ self-adjoint} \end{array}\right\}$

$$v(x) = a e^{\sqrt{\lambda} x} + b e^{-\sqrt{\lambda} x}$$

Constants to be determined such that $v(\pm 1) = 0$.

λ : real $\begin{cases} \lambda > 0 \\ \lambda < 0 \end{cases}$

$\boxed{\text{if } \lambda > 0}$

B.C. $\begin{cases} a e^{\sqrt{\lambda}} + b e^{-\sqrt{\lambda}} = 0 & (x = 1) \\ a e^{-\sqrt{\lambda}} + b e^{\sqrt{\lambda}} = 0 & (x = -1) \end{cases}$

$$\underbrace{\begin{bmatrix} e^{\sqrt{\lambda}} & e^{-\sqrt{\lambda}} \\ e^{-\sqrt{\lambda}} & e^{\sqrt{\lambda}} \end{bmatrix}}_{M} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

For non-trivial solution, we need

$$\det(M) = 0 = e^{2\sqrt{\lambda}} - e^{-2\sqrt{\lambda}} = 0$$

$$\Rightarrow e^{4\sqrt{\lambda}} = 1 \;; \quad \text{only option } \lambda = 0$$

$\Rightarrow v(x) = a + bx$

but, cannot satisfy b.c.

So $\lambda \geq 0$ cannot be an eigenvalue of $\left.\mathcal{A}\right|_{v(\pm 1)=0}$.

Therefore, $\lambda < 0$.

$$\lambda < 0 \Rightarrow s^2 = \lambda = -|\lambda|$$

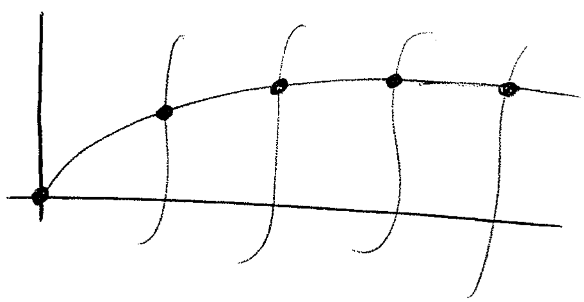
$$\Rightarrow \quad s = \pm j\sqrt{|\lambda|}$$

$$v(x) = a\sin(\sqrt{\lambda}\, x) + b\cos(\sqrt{\lambda}\, x)$$

$$\lambda = -\left(\frac{\pm n\pi}{2}\right)^2 \; ; \quad n \in \{1, 2, \dots\}$$

$$v_n(x) = \sin\left(\frac{n\pi}{2}(x+1)\right)$$

<u>HW.</u> $\mathcal{A} = \dfrac{d^2}{dx^2}$ with $\begin{cases} v(-1) = 0 \\ v(1) = v'(1) \end{cases}$



<u>Alternative.</u>

Bring $\dfrac{d^2 v}{dx^2} - \lambda v = 0$ to state-space form.

$$v'' - \lambda v = 0$$

$$\left.\begin{aligned} \psi_1 &= v \\ \psi_2 &= v' \end{aligned}\right\} \text{states} \qquad \begin{aligned} \psi_1' &= v' = \psi_2 \\ \psi_2' &= v'' = \lambda v = \lambda \psi_1 \end{aligned}$$

$$\begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix}}_{A} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \longrightarrow \begin{aligned} \psi' &= A\psi \\ v &= C\psi \; ; \quad C = [1 \;\; 0] \end{aligned}$$

$$\left.\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{N_1} \begin{bmatrix} \psi_1(-1) \\ \psi_2(-1) \end{bmatrix} + \\ \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{N_2} \begin{bmatrix} \psi_1(1) \\ \psi_2(1) \end{bmatrix} \end{aligned}\right\} \rightarrow \quad 0 = N_1\psi(-1) + N_2\psi(1)$$

$$\begin{cases} \psi' = A\psi \\ 0 = N_1 \psi(-1) + N_2 \psi(1) \end{cases} \quad ; \quad v = C\psi$$

$$\psi(x) = e^{A(x-(-1))}\psi(-1) = e^{A(x+1)}\psi(-1)$$

Problem : donot know $\psi(-1) = \begin{bmatrix} \psi_1(-1) \\ \psi_2(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ ? \end{bmatrix}$

use BCs :

$$N_1\psi(-1) + N_2\psi(1) = N_1\psi(-1) + N_2 e^{2A}\psi(-1)$$

$$= \underbrace{(N_1 + N_2 e^{2A})}_{\det(N_1 + N_2 e^{2A}) = 0}(\psi(-1)) = 0$$

$\downarrow$

gives λ .