$$\mathcal{L} : \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}$$

$$\boxed{\begin{aligned} \mathcal{L}(Q) &= \int_0^\infty e^{At} B Q B^* e^{A^* t}\, dt = P \\ AP + PA^* &= -BQB^* \end{aligned}}$$

$$\begin{tikzcd} \mathbb{C}^{n\times n} \arrow[r, bend left, "\mathcal{L}"] & \mathbb{C}^{n\times n} \arrow[l, bend left, "\mathcal{L}^\dagger"] \end{tikzcd}$$

Appropriate inner product on the space of symetric matrices:

$$\langle R, Q \rangle = \operatorname{trace}(R^* Q)$$

This inner product induces Ferobinuis norm.

$$\langle R, \mathcal{L} Q \rangle = \langle \mathcal{L}^\dagger R, Q \rangle$$

$$\langle R, \mathcal{L} Q \rangle = \operatorname{trace}\left(R^* \int_0^\infty e^{At} B Q B^* e^{A^* t}\, dt\right) =$$

[use Linearity of integral and trace operators]

$$= \int_0^\infty \operatorname{trace}\left(R^* e^{At} B Q B^* e^{A^* t}\right) dt =$$

[use trace (MN) = trace (NM)]

$$= \int_0^\infty \operatorname{trace}\left(B^* e^{A^* t} R^* e^{At} B Q\right) dt =$$

$$= \operatorname{trace}\left(B^* \int_0^\infty e^{A^* t} R^* e^{At}\, dt\, B\, Q\right)$$

$$= \left\langle B^* \int_0^\infty e^{A^* t} R e^{At}\, dt\, B,\ Q \right\rangle$$

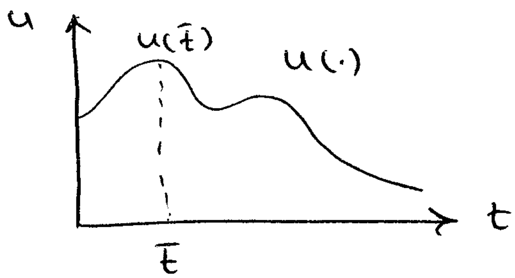
Thus,

$$\boxed{\begin{aligned} \mathcal{L}^\dagger R &= B^* V(R) B \\ A^* V + VA &= -R \end{aligned}}$$

Ex. $\dot{x} = Ax + Bu \quad , \quad x(0) = 0 \quad , \quad x(t) \in \mathbb{R}^n$

$$x(T) = \int_0^T e^{A(T-\tau)} B \cdot u(\tau)\, d\tau$$

$$= [\mathcal{C} u](T)$$



$$\mathcal{C} : L_2[0,T] \longrightarrow \mathbb{R}^n$$

<u>Controllability</u> given a state, can we find input to bring the state to given state at a given time.

$x(T) \in \mathbb{R}^n \quad ; \quad u \in L_2[0,T]$

$$\langle x, \mathcal{C} u \rangle_{\mathbb{R}^n} = \langle \mathcal{C}^{+} x, u \rangle_{L^2[0,T]}$$

$$\langle x, \mathcal{C} u \rangle_{\mathbb{R}^n} = x^* \int_0^T e^{A(T-\tau)} B u(\tau)\, d\tau =$$

$$= \int_0^T x^* e^{A(T-\tau)} B\, u(\tau)\, d\tau$$

$$= \int_0^T \left(B^* e^{A^*(T-\tau)} x \right)^* u(\tau)\, d\tau$$

$$= \langle \underbrace{B^* e^{A^*(T-\tau)}}_{\mathcal{C}^{+}} x , u \rangle_{L^2[0,T]}$$

$$\boxed{\mathcal{C}^{+} = B^* e^{A^*(T-t)}}$$

$$\begin{tikzcd} L_2[0,T] \arrow[r, bend left, "\mathcal{C}"] & \mathbb{R}^n \arrow[l, bend left, "\mathcal{C}^{+}"] \end{tikzcd}$$

$$\boxed{\mathcal{A}\mathcal{A}^{\dagger}} = \int_0^T e^{A(T-\tau)} BB^{*} e^{A^{*}(T-\tau)} \, d\tau$$

$$= \boxed{\int_0^T e^{At} BB^{*} e^{A^{*}t} \, dt}$$

Constant Matrix

Controllability Gramian

$$\begin{tikzcd}
\mathbb{R}^n & L_2[0,T] \arrow[l, "\mathcal{A}"'] & \mathbb{R}^n \arrow[l, "\mathcal{A}^{\dagger}"'] \arrow[ll, bend left, "\mathcal{A}\mathcal{A}^{\dagger}"]
\end{tikzcd}$$

<u>Ex.</u> $\mathcal{A} : L_2[a,b] \longrightarrow L_2[a,b]$

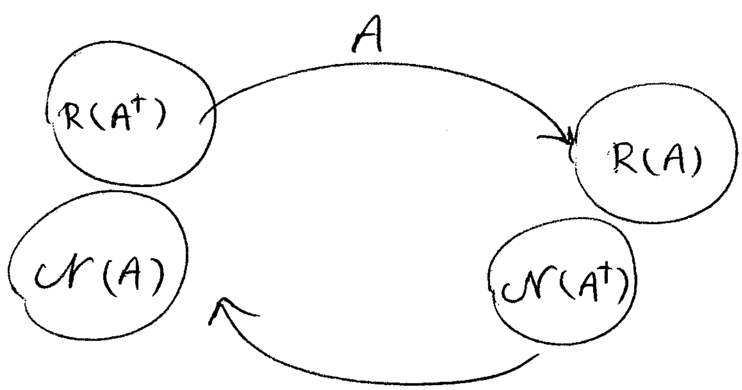
↳ bounded operator with kernel representation

$$[\mathcal{A}f](x) = \int_a^b A_k(x,\xi) f(\xi) \, d\xi$$

$$[\mathcal{A}^{\dagger}g](x) = \int_a^b A_k^{*}(\xi,x) f(\xi) \, d\xi$$

<u>Finite Dimensions</u>

$A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$



$$\mathcal{N}(A^{\dagger}) = [R(A)]^{\perp}$$

$x(T) \leftarrow \boxed{\mathcal{A}} \leftarrow u$

$x(T) = [\mathcal{A}u](T) \iff [\ \ ]_{n\times 1} = \int_0^T \left[\xrightarrow{\quad A_k \quad} \right] \left[\ \ \right]$

System Controllable

$\Updownarrow$

$R(\mathcal{A}) = \mathbb{R}^n$

$R(\mathcal{A}) = R(\mathcal{A}\mathcal{A}^\dagger) \longleftarrow$ use

if Controllable, then interested in finding u with smallest energy.

$$\min \|u\|^2_{L_2[0,T]}$$

u is given by psuedo-inverse of $\mathcal{A}$

$$\boxed{u = \mathcal{A}^\dagger (\mathcal{A}\mathcal{A}^\dagger)^{-1} x(T)}$$

Adjoint of unbounded operators:

<u>Ex</u> $\mathcal{A} : L_2[-1,1] \supset \mathcal{D}(\mathcal{A}) \longrightarrow L_2[-1,1]$

$\mathcal{A} = \frac{d}{dx}$; $\mathcal{D}(\mathcal{A}) = \{ f \in L_2 \,;\, f' \in L_2 ,\, f(-1) = 0 \}$

$$\langle g, \mathcal{A} f \rangle_{L_2} = \langle \mathcal{A}^\dagger g, f \rangle_{L^2}$$

$f \in \mathcal{D}(\mathcal{A})$ $\qquad$ $g \in \mathcal{D}(\mathcal{A}^\dagger)$

$$\langle g, cd f\rangle_{L_2} = \langle g, \frac{df}{dx}\rangle_{L_2^0} = \int_{-1}^{1} g^* \frac{df}{dx}(x)\,dx$$

[use integration by parts]

$$= g(x) f(x)\Big]_{-1}^{1} - \int_{-1}^{1} \frac{dg^*}{dx}(x)\, f(x)\,dx$$

$$= g(1)f(1) - g(-1)\underbrace{f(-1)}_{\substack{0 \\ f \in \mathcal{D}(cd)}} + \int_{-1}^{1} \left(\frac{-dg^*}{dx}(x)\right) f(x)\,dx$$

$$= \underbrace{g(1)f(1) + \langle \frac{-d}{dx} g, f\rangle_{L_2[-1,1]}}_{\langle cd^\dagger g, f\rangle_{L_2[-1,1]}}$$

? want

Candidate for adjoint :

$$\boxed{\begin{gathered} cd^\dagger = \frac{-d}{dx} \quad \text{with} \\ \mathcal{D}(cd^\dagger) = \left\{ g \in L_2[-1,1] ; \frac{dg}{dx} \in L_2[-1,1],\ g(1) = 0 \right\} \end{gathered}}$$

Note : $\frac{d}{dx}$ is not invertable unless we specify that

e.g. $\mathcal{D}(\frac{d}{dx})$ is functions with b.c. $f(-1) = 0$.

This is to restrict the Null space of $\frac{d}{dx}$ to $f = 0$, (instead of $f = \text{Const.}$ without b.c.) so that $\frac{d}{dx}$ is invertible.

Eigen-values of <u>Self-adjoint</u> operators are <u>real</u>.

$$\lambda \|\psi\|^2 = \langle \psi, \lambda\psi \rangle = \langle \psi, \mathcal{A}\psi \rangle = \langle \mathcal{A}\psi, \psi \rangle$$

$$= \bar{\lambda} \|\psi\|^2$$

$$(\lambda - \bar{\lambda}) \|\psi\|^2 = 0 \implies \lambda = \bar{\lambda}.$$