Lectures 6, 7, 8: Kernel representation of linear operators / Hilbert space adjoint of a linear operator

- Kernel representation of an integral operator
  - ⋆ Generalization of matrix/vector multiplication
  - ⋆ Represents action of integral operators and linear dynamical systems

- Adjoint of an operator
  - ⋆ Generalizes notion of complex-conjugate-transpose to operators
  - ⋆ Useful in linear algebra and functional analysis
    (solutions of linear equations, optimization, . . .)

- Self-adjoint operators
  - ⋆ Can be used to characterize complete orthonormal basis of a Hilbert space

Kernel representation

- Recall: Solution of diffusion equation on $L_2\,[-1, 1]$ with Dirichlet BCs

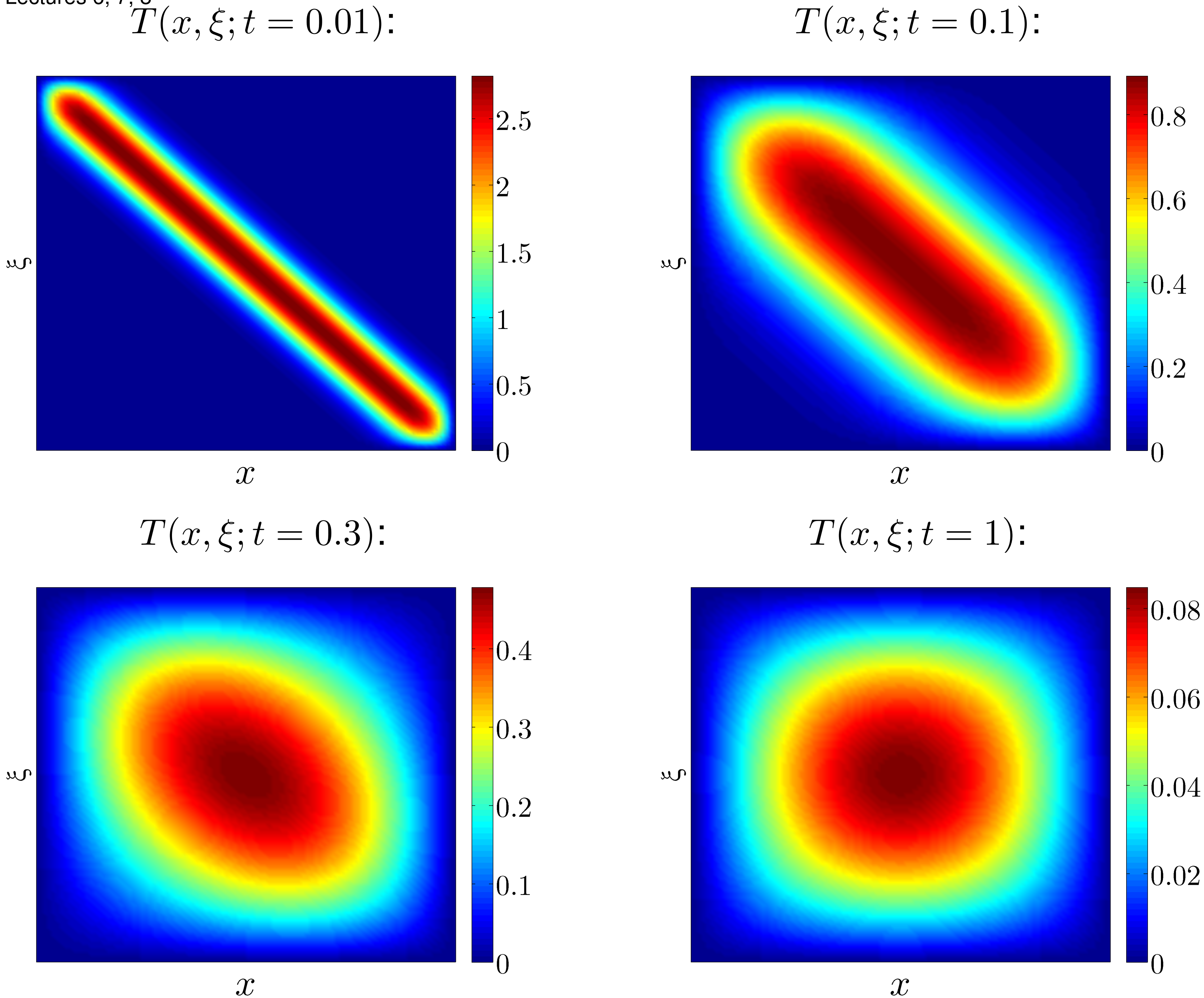
$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) \\ \phi(x,0) &= f(x) \\ \phi(\pm 1, t) &= 0 \end{aligned}$$

given by

$$\phi(x,t) \;=\; [\mathcal{T}(t)\, f]\,(x) \;=\; \int_{-1}^{1} T(x,\xi;t)\, f(\xi)\, \mathrm{d}\xi$$

- Kernel representation of operator $\mathcal{T}(t)$: $L_2\,[-1,1] \longrightarrow L_2\,[-1,1]$

$$T(x,\xi;t) \;=\; \sum_{n\,=\,1}^{\infty} \mathrm{e}^{-\left(\frac{n\pi}{2}\right)^2 t} \sin\left(\frac{n\pi}{2}\,(x\,+\,1)\right) \sin\left(\frac{n\pi}{2}\,(\xi\,+\,1)\right)$$


$T(x, \xi; t = 0.01)$:
$T(x, \xi; t = 0.1)$:
$T(x, \xi; t = 0.3)$:
$T(x, \xi; t = 1)$:
x
ξ
0
0.5
1
1.5
2
2.5
0.2
0.4
0.6
0.8
0.1
0.3
0.02
0.04
0.06
0.08

- For operator $\mathcal{T}$: $f \longrightarrow g$ given by

$$g(x) \;=\; [\mathcal{T} f]\,(x) \;=\; \int_a^b T(x,\xi)\, f(\xi)\, \mathrm{d}\xi$$

- Vector-valued f and g $\;\Rightarrow\;$ matrix-valued $T(\cdot,\cdot)$

$$g(x) \;=\; \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix},\; f(x) \;=\; \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} \;\Rightarrow\; T(\cdot,\cdot) \;=\; \begin{bmatrix} T_{11}(\cdot,\cdot) & T_{12}(\cdot,\cdot) & T_{13}(\cdot,\cdot) \\ T_{21}(\cdot,\cdot) & T_{22}(\cdot,\cdot) & T_{23}(\cdot,\cdot) \end{bmatrix}$$

- Kernels of identity and multiplication operators are distributions

$$g(x) \;=\; [I\, f]\,(x) \;=\; f(x) \;=\; \int_a^b \delta(x-\xi)\, f(\xi)\, \mathrm{d}\xi$$

$$g(x) \;=\; [M_a\, f]\,(x) \;=\; a(x)\, f(x) \;=\; \int_a^b a(x)\, \delta(x-\xi)\, f(\xi)\, \mathrm{d}\xi$$

- Kernel of M_a: $\begin{cases} \text{impulse sheet supported along the line } x = \xi \text{ in } [a,\, b] \times [a,\, b] \\ \text{strength "modulated" by the function } a(\cdot) \end{cases}$

Generalizations

- Can be generalized to $\mathcal{T}$: $L_2(\Omega) \longrightarrow L_2(\Omega)$, $\Omega \subset \mathbb{R}^n$

$$g(x) \;=\; [\mathcal{T} f]\,(x) \;=\; \int_{\Omega} T(x,\xi)\, f(\xi)\, \mathrm{d}\xi$$

- Examples of bounded $\mathcal{T}$: $L_2(\Omega) \longrightarrow L_2(\Omega)$
  - ⋆ Ω compact; $T(\cdot,\cdot)$ has no distributions; $T(\cdot,\cdot)$ bounded
  - ⋆ Ω compact; $\sup\limits_{x \,\in\, \Omega} \int_{\Omega} |T(x,\xi)|\, \mathrm{d}\xi \;<\; \infty$; $\sup\limits_{\xi \,\in\, \Omega} \int_{\Omega} |T(x,\xi)|\, \mathrm{d}x \;<\; \infty$
  - ⋆ $\mathcal{T}$ Hilbert-Schmidt, i.e., $\int_{\Omega}\int_{\Omega} |T(x,\xi)|^2\, \mathrm{d}x\, \mathrm{d}\xi \;<\; \infty$

- $\mathcal{T}$: discrete spectrum and complete set of orthonormal e-functions

$$[\mathcal{T} f]\,(x) \;=\; \sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\, \langle v_n, f \rangle \;=\; \int_{\Omega} \underbrace{\left(\sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\, v_n^*(\xi) \right)}_{T(x,\xi)} f(\xi)\, \mathrm{d}\xi$$

Hilbert space adjoint

- The adjoint of a bounded operator $\mathcal{A}\colon \mathbb{H}_1 \longrightarrow \mathbb{H}_2$
  - ⋆ the operator $\mathcal{A}^\dagger\colon \mathbb{H}_2 \longrightarrow \mathbb{H}_1$ defined by

$$\langle \psi_2, \mathcal{A}\,\psi_1 \rangle_2 \;=\; \left\langle \mathcal{A}^\dagger\,\psi_2, \psi_1 \right\rangle_1, \quad \text{for all } \psi_1 \in \mathbb{H}_1 \text{ and } \psi_2 \in \mathbb{H}_2$$

- Examples
  - ⋆ $\mathcal{A}\colon \mathbb{C}^n \longrightarrow \mathbb{C}^m$ with standard inner product $\qquad \mathcal{A}^\dagger = \mathcal{A}^*$
  - ⋆ $\mathcal{A}\colon \mathbb{C}^n \longrightarrow \mathbb{C}^m$ with $\{\langle f_i, g_i \rangle_i = f_i^*\,Q_i\,g_i;\, Q_i = Q_i^* > 0\}$ $\qquad \mathcal{A}^\dagger = Q_1^{-1}\,\mathcal{A}^*\,Q_2$
  - ⋆ $\mathcal{A}\colon \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}$, $\mathcal{A}(Q) = \int_0^\infty \mathrm{e}^{At}\,Q\,\mathrm{e}^{A^*t}\,\mathrm{d}t$ with $\langle R, Q \rangle = \mathrm{trace}\,(R^*\,Q)$

$$\mathcal{A}^\dagger(R) = \int_0^\infty \mathrm{e}^{A^*t}\,R\,\mathrm{e}^{At}\,\mathrm{d}t$$

  - ⋆ $\mathcal{A}\colon L_2[0,t] \longrightarrow \mathbb{C}^n$, $[\mathcal{A}\,u]\,(t) = \int_0^t \mathrm{e}^{A(t-\tau)}\,B\,u(\tau)\,\mathrm{d}\tau$ with standard inner products on $L_2[0,t]$ and $\mathbb{C}^n$ $\qquad \left[\mathcal{A}^\dagger\,x(t)\right](\tau) = B^*\,\mathrm{e}^{A^*(t-\tau)}\,x(t)$

⋆ $\mathcal{A}$: $L_2[a,b] \longrightarrow L_2[a,b]$, $[\mathcal{A}\, f]\,(x) \,=\, \int_a^b A(x,\xi)\, f(\xi)\,\mathrm{d}\xi$ with standard inner product on $L_2[0,t]$ ▮

$$\left[\mathcal{A}^\dagger\, g\right](x) \;=\; \int_a^b A^*(\xi,x)\, g(\xi)\,\mathrm{d}\xi$$

⋆ $\mathcal{A}$: $L_2[a,b] \longrightarrow L_2[a,b]$, $[\mathcal{A}\, f]\,(x) \,=\, \int_a^x A(x,\xi)\, f(\xi)\,\mathrm{d}\xi$ with standard inner product on $L_2[0,t]$ ▮

$$\left[\mathcal{A}^\dagger\, g\right](x) \;=\; \int_x^b A^*(\xi,x)\, g(\xi)\,\mathrm{d}\xi$$

▮

- For bounded $\mathcal{A}$: $\mathbb{H}_1 \longrightarrow \mathbb{H}_2$, $\mathcal{B}$: $\mathbb{H}_2 \longrightarrow \mathbb{H}_3$, $\alpha \in \mathbb{C}$

$$I^\dagger \;=\; I, \quad (\alpha\,\mathcal{A})^\dagger \;=\; \overline{\alpha}\,\mathcal{A}^\dagger, \quad \|\mathcal{A}^\dagger\| \;=\; \|\mathcal{A}\|$$

$$(\mathcal{A}_1 \,+\, \mathcal{A}_2)^\dagger \;=\; \mathcal{A}_1^\dagger \,+\, \mathcal{A}_2^\dagger, \quad (\mathcal{B}\,\mathcal{A})^\dagger \;=\; \mathcal{A}^\dagger\,\mathcal{B}^\dagger, \quad \|\mathcal{A}^\dagger\,\mathcal{A}\| \;=\; \|\mathcal{A}\|^2$$

Fundamental subspaces

- The range space of $\mathcal{A} : \mathbb{H}_1 \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}_2$

$$\mathcal{R}(\mathcal{A}) \;=\; \{g \in \mathbb{H}_2;\;\; g \,=\, \mathcal{A}f,\; f \in \mathcal{D}(\mathcal{A})\}$$

- The null space of $\mathcal{A} : \mathbb{H}_1 \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}_2$

$$\mathcal{N}(\mathcal{A}) \;=\; \{f \in \mathbb{H}_1;\;\; \mathcal{A}f \,=\, 0\}$$

- For a bounded $\mathcal{A}$: $\mathbb{H}_1 \longrightarrow \mathbb{H}_2$
  - ⋆ $[\mathcal{R}(\mathcal{A})]^\perp = \mathcal{N}(\mathcal{A}^\dagger)$; $\quad \overline{[\mathcal{R}(\mathcal{A})]} = [\mathcal{N}(\mathcal{A}^\dagger)]^\perp$
  - ⋆ $[\mathcal{R}(\mathcal{A}^\dagger)]^\perp = \mathcal{N}(\mathcal{A})$; $\quad \overline{[\mathcal{R}(\mathcal{A}^\dagger)]} = [\mathcal{N}(\mathcal{A})]^\perp$

- For bounded $\mathcal{A}$: $\mathbb{H}_1 \longrightarrow \mathbb{H}_2$, $\mathcal{B}$: $\mathbb{H}_2 \longrightarrow \mathbb{H}_3$
  - ⋆ $\mathcal{N}(\mathcal{B}\,\mathcal{A}) \supseteq \mathcal{N}(\mathcal{A})$ but $\mathcal{N}(\mathcal{A}) = \mathcal{N}(\mathcal{A}^\dagger\,\mathcal{A})$
  - ⋆ $\mathcal{R}(\mathcal{B}\,\mathcal{A}) \subseteq \mathcal{R}(\mathcal{B})$ but $\overline{\mathcal{R}(\mathcal{A})} = \overline{\mathcal{R}(\mathcal{A}\,\mathcal{A}^\dagger)}$

Adjoint of an unbounded operator

- The adjoint of an unbounded operator $\left\{ \begin{array}{l} \mathcal{A} : \mathbb{H}_1 \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}_2 \\ \mathcal{D}(\mathcal{A}) \text{ dense in } \mathbb{H}_1 \end{array} \right.$

  ⋆ the operator $\mathcal{A}^\dagger$: $\mathbb{H}_2 \supset \mathcal{D}(\mathcal{A}^\dagger) \longrightarrow \mathbb{H}_1$ defined by

$$\left\{ \begin{array}{rcl} \mathcal{D}(\mathcal{A}^\dagger) & = & \{\psi_2 \in \mathbb{H}_2;\ \exists\, \phi_1 \in \mathbb{H}_1 \text{ s.t. } \langle \psi_2, \mathcal{A}\,\psi_1 \rangle_2 = \langle \phi_1, \psi_1 \rangle_1 \text{ for all } \psi_1 \in \mathcal{D}(\mathcal{A})\} \\ \mathcal{A}^\dagger\,\psi_2 & = & \phi_1 \end{array} \right.$$

- Informally

$$\langle \psi_2, \mathcal{A}\,\psi_1 \rangle_2 \;=\; \left\langle \mathcal{A}^\dagger\,\psi_2, \psi_1 \right\rangle_1 \left\{ \begin{array}{l} \text{for all } \psi_1 \in \mathcal{D}(\mathcal{A}) \text{ and } \psi_2 \text{ for which the RHS is finite} \\ \text{such } \psi_2 \in \mathbb{H}_2 \text{ determine } \mathcal{D}(\mathcal{A}^\dagger) \end{array} \right.$$

Examples (to be solved in class)

- $$\left\{\begin{array}{rcl} [\mathcal{A}\, f]\,(x) & = & \left[\dfrac{\mathrm{d}f}{\mathrm{d}x}\right](x) \\[2ex] \mathcal{D}(\mathcal{A}) & = & \left\{f \,\in\, L_2\,[-1,\,1],\, \dfrac{\mathrm{d}f}{\mathrm{d}x} \in\, L_2\,[-1,\,1],\; f(-1) \,=\, 0\right\} \end{array}\right.$$

- $$\left\{\begin{array}{rcl} [\mathcal{A}\, f]\,(x) & = & \left[\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right](x) \\[2ex] \mathcal{D}(\mathcal{A}) & = & \left\{f \,\in\, L_2\,[-1,\,1],\, \dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} \in\, L_2\,[-1,\,1],\; f(\pm 1) \,=\, 0\right\} \end{array}\right.$$

Useful property

- $\begin{cases} \mathcal{A}: & \text{unbounded densely defined operator with domain } \mathcal{D}(\mathcal{A}) \subset \mathbb{H} \\ \mathcal{B}: & \text{bounded operator defined on the whole } \mathbb{H} \end{cases}$

  ⋆ $(\alpha\,\mathcal{A})^\dagger = \overline{\alpha}\,\mathcal{A}^\dagger; \quad \mathcal{D}\left((\alpha\,\mathcal{A})^\dagger\right) = \begin{cases} \mathcal{D}\left(\mathcal{A}^\dagger\right), & \alpha \neq 0 \\ \mathbb{H}, & \alpha = 0 \end{cases}$

  ⋆ $(\mathcal{A} + \mathcal{B})^\dagger = \mathcal{A}^\dagger + \mathcal{B}^\dagger$, with domain $\mathcal{D}\left((\mathcal{A} + \mathcal{B})^\dagger\right) = \mathcal{D}\left(\mathcal{A}^\dagger\right)$

  ⋆ $\mathcal{A}$ has bounded inverse $\Rightarrow$ $\mathcal{A}^\dagger$ has bounded inverse: $\left(\mathcal{A}^\dagger\right)^{-1} = \left(\mathcal{A}^{-1}\right)^\dagger$

- Examples on $L_2[-1,1]$

$$\left.\begin{array}{rcl} f'(x) & = & g(x) \\ f(-1) & = & 0 \end{array}\right\} \Rightarrow f(x) = \int_{-1}^{x} g(\xi)\,\mathrm{d}\xi = \int_{-1}^{1} \mathbb{1}(x-\xi)\,g(\xi)\,\mathrm{d}\xi$$

$$\left.\begin{array}{rcl} f''(x) & = & g(x) \\ f(\pm 1) & = & 0 \end{array}\right\} \Rightarrow f(x) = \int_{-1}^{1} \left((x-\xi)\,\mathbb{1}(x-\xi) + \frac{(x+1)(\xi-1)}{2}\right) g(\xi)\,\mathrm{d}\xi$$

Self-adjoint operators

$$\begin{cases} \langle \psi_2, \mathcal{A}\,\psi_1 \rangle_2 &= \langle \mathcal{A}\,\psi_2, \psi_1 \rangle_1 \text{ for all } \psi_1,\ \psi_2 \in \mathcal{D}(\mathcal{A}) \\ \mathcal{D}(\mathcal{A}^\dagger) &= \mathcal{D}(\mathcal{A}) \end{cases}$$

$\mathcal{A}$ self-adjoint $\Rightarrow$ $\begin{cases} \text{all e-values of } \mathcal{A} \text{ are real} \\ v_n,\, v_m\text{: e-vectors corresponding to } \lambda_n \neq \lambda_m \;\Rightarrow\; \langle v_n, v_m \rangle = 0 \end{cases}$

$\mathcal{A}$: densely defined self-adjoint operator in $\mathbb{H}$ with discrete spectrum

$\Downarrow$

$\mathcal{A}$ has an orthonormal set of e-functions that span $\mathbb{H}$

Example (to be solved in class)

- E-value decomposition of $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $L_2[-1,1]$ with Dirichlet BCs

$$\left\{ \begin{array}{rcl} [\mathcal{A}\, f]\,(x) & = & \left[\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right](x) \\[2ex] \mathcal{D}(\mathcal{A}) & = & \left\{ f \,\in\, L_2\,[-1,\,1],\ \dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} \,\in\, L_2\,[-1,\,1],\ f(\pm 1) \,=\, 0 \right\} \end{array} \right.$$

- Need to solve

$$\left\{ \begin{array}{rcl} \left[\dfrac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right](x) & = & \lambda\, v(x) \\[2ex] v(\pm 1) & = & 0 \end{array} \right.$$

$$\downarrow$$

$$\left\{ v_n(x) \;=\; \sin\left(\frac{n\pi}{2}\,(x\,+\,1)\right);\ \lambda_n \;=\; -\left(\frac{n\pi}{2}\right)^2 \right\}_{n\,\in\,\mathbb{N}}$$