Finite dimensional

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

given $\left.\begin{matrix} f \in \mathbb{R}^n \\ g \in \mathbb{R}^m \end{matrix}\right\} \quad g = Af$

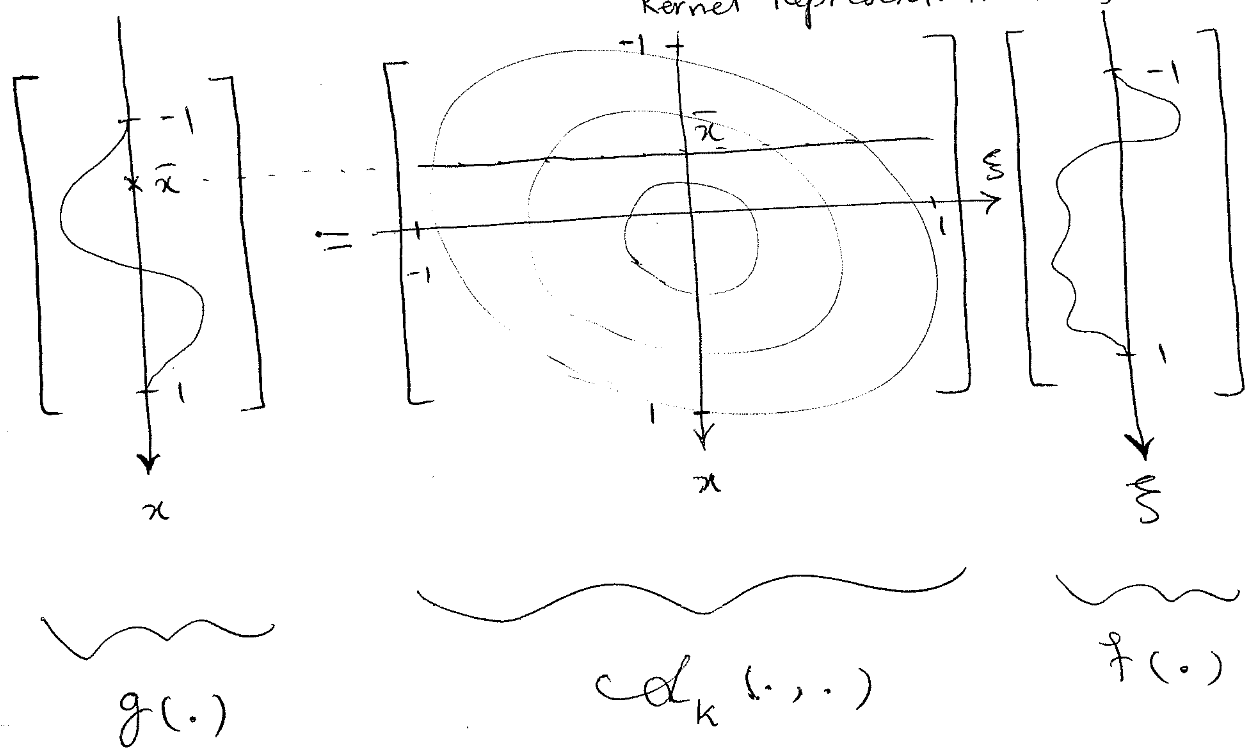
$g \in \mathbb{R}^2 \; ; \; f \in \mathbb{R}^3$

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Infinite dimensional

$\mathcal{A} : L_2[-1,1] \longrightarrow L_2[-1,1]$

$$g(x) = [\mathcal{A} f](x) = \int_{-1}^{1} \underbrace{\mathcal{A}_K(x, \xi)}_{\text{Kernel representation of } \mathcal{A}} f(\xi)\, d\xi$$



$I : L_2[-1,1] \longrightarrow L_2[-1,1]$

$g(x) = [If](x) = f(x) =$

$$g(x) = \int_{-1}^{1} \delta(x-\xi) f(\xi)\, d\xi = f(x)$$

Multiplication operator : $M_a : L_2[-1,1] \longrightarrow L_2[-1,1]$

$g(x) = [M_a f](x) = a(x) \cdot f(x)$

$$g(x) = \int_{-1}^{1} a(x)\, \delta(x-\xi) f(\xi)\, d\xi = a(x) \cdot f(x)$$

Note kernel representation:

Kernel is a distribution, it doesn't have to be a function.

Side note Bounded operator

$\mathcal{A} : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$

$\mathcal{A}$ bounded means

$\exists$ a constant $C < +\infty$ s.t. $\| \mathcal{A} f \|_2 \leq c \| f \|_1$ for all $f \in \mathcal{H}_1$

$\| f \|_1^2 = \langle f, f \rangle_{\mathcal{H}_1}$

$\| \mathcal{A} f \|_2^2 = \langle \mathcal{A} f, \mathcal{A} f \rangle_{\mathcal{H}_2}$

Example of unbounded operator

$$\mathcal{A} = \frac{d}{dx} \quad ; \quad x \in [-1,1]$$

$$f_n(x) = \sin(nx) \quad ; \quad n = 1, 2, \ldots$$

$$f_n'(x) = n \cos(nx)$$

$$\| \mathcal{A} f_n \|^2 = \| f_n' \|^2 = \underbrace{n^2 \| f \|^2}$$

Cannot be written as or bounded by $\quad C \| f \|^2$

$$\| \mathcal{A} f_n \|^2 = n^2 \| f \|^2 \not\leq C \| f \|^2$$

An operator norm (an induced norm)

$$\| \mathcal{A} \| = \sup_{f \neq 0} \frac{\| \mathcal{A} f \|_2}{\| f \|_2}$$

Recall $\quad A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$

$$\| A \|_F^2 = \text{trace}(A^* A) = \text{trace}(A A^*)$$

where $\quad \text{trace}(M) = \sum_{i=1}^{n} m_{ii}$

$\| \cdot \|_F$ .... not an induced norm !

$$\mathcal{A} : L_2[-1,1] \longrightarrow L_2[-1,1]$$

$$\|\mathcal{A}\|_{HS}^2 = \int_{-1}^{1}\int_{-1}^{1} \underbrace{\text{trace}\left(\mathcal{A}_K^*(x,\xi)\,\mathcal{A}_K(x,\xi)\right)}_{\|\mathcal{A}_K(x,\xi)\|_F^2} \, dx \, d\xi$$

$$= \text{trace}\left(\mathcal{A}^*\mathcal{A}\right)$$

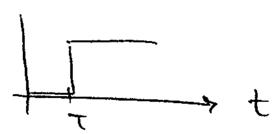
infinite dimensions

Linear System

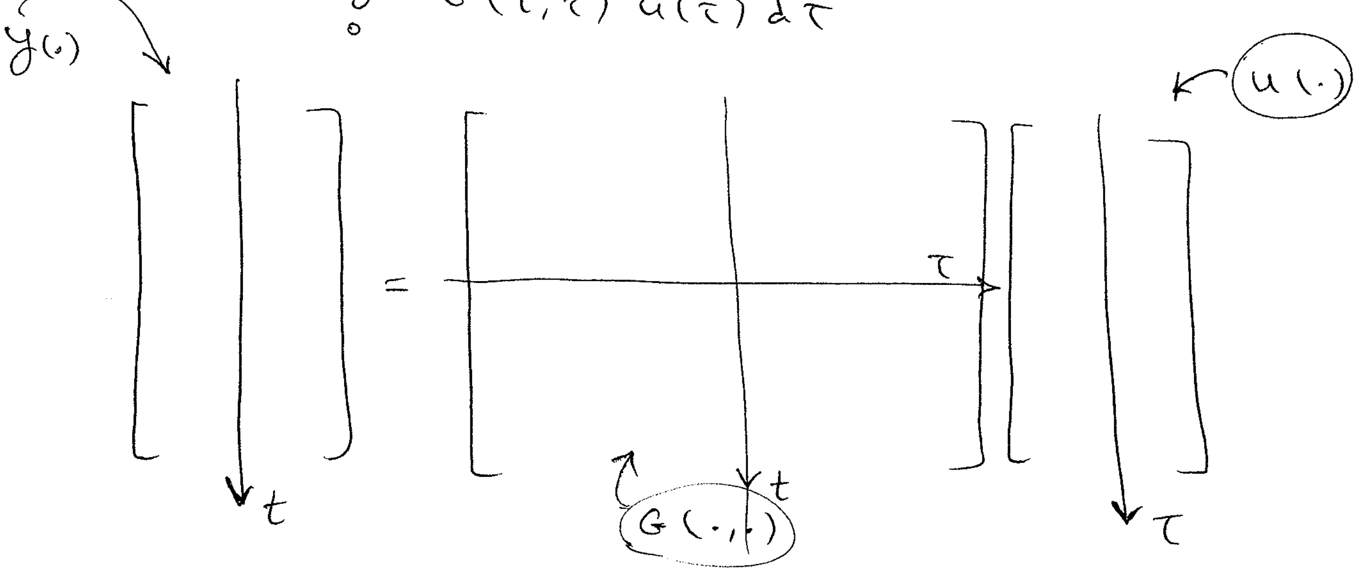
$$\left.\begin{aligned} \dot{x} &= A(t)\,x + B(t)\,u \\ y &= C(t)\,x \end{aligned}\right\} \quad \begin{aligned} & t \in [0,T] \\ & x(0) = 0 \end{aligned}$$

$$y(t) = C(t)\,x(t) = C(t)\int_0^t \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau$$

$$= \int_0^T C(t)\,\Phi(t,\tau)\,B(\tau)\,\mathbb{1}(t-\tau)\,u(\tau)\,d\tau$$



$$= \int_0^T G(t,\tau)\,u(\tau)\,d\tau$$

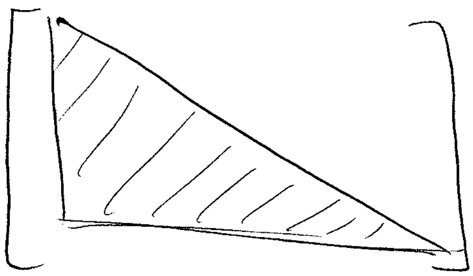


① Causality : G is lower triangular

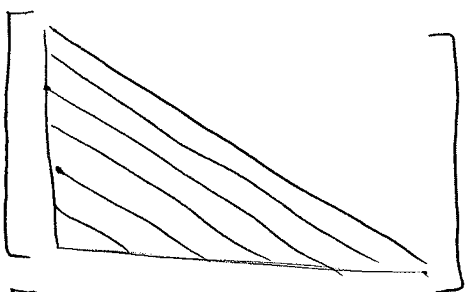
② Time-invariance : G is Toeplitz

③ Time-periodic : $G(t+T, \tau+T) = G(t,\tau)$

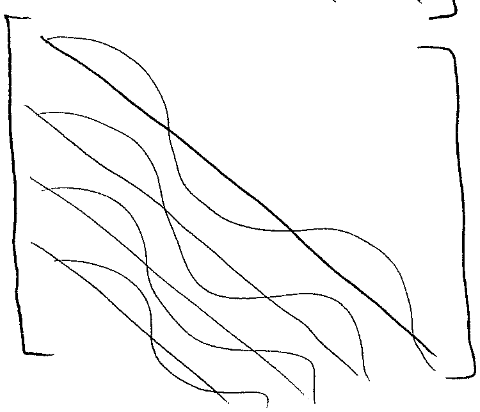
①



②

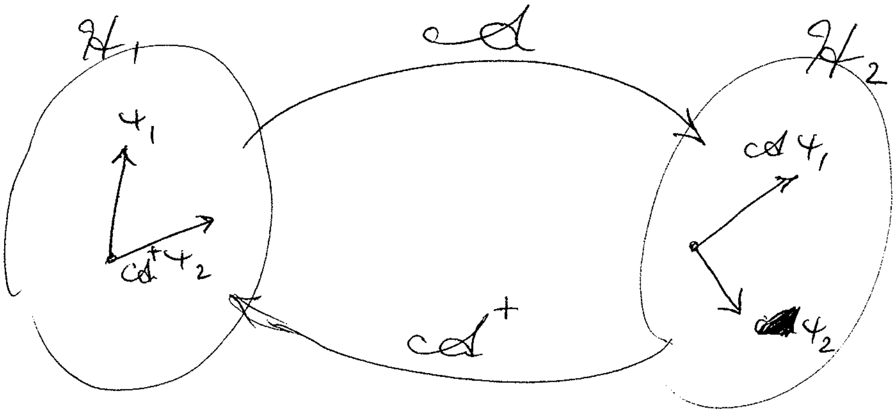


③



The kernel $G(\cdot,\cdot)$ for the linear system can have a distribution part if there is a D term in the system dynamics ; $y = Cx + Du$

$$G(t,\tau) = C(t)\,\Phi(t,\tau)\,B(\tau) + D(\tau)\,\delta(t-\tau)$$



Adjoints

$$\langle \psi_2 , \mathcal{A}\psi_1 \rangle_{\mathcal{H}_2} = \langle \mathcal{A}^+ \psi_2 , \psi_1 \rangle_{\mathcal{H}_1}$$

for all $\psi_1 \in \mathcal{H}_1$, $\psi_2 \in \mathcal{H}_2$

$\exists \ \phi_1 \in \mathcal{H}_1$

$$\langle \psi_2 , \mathcal{A}\psi_1 \rangle_{\mathcal{H}_2} = \langle \phi_1 , \psi_1 \rangle_{\mathcal{H}_1}$$ for all $\psi_1 \in \mathcal{H}_1$, $\psi_2 \in \mathcal{H}_2$

$\mathcal{A}^+ \psi_2 = \phi_1$

Adjoint is unique if inner product is fixed.

Example $A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ $\langle f, g \rangle = f^* \cdot g$

$$\langle \psi_2 , A\psi_1 \rangle_{\mathbb{C}^m} = \langle A^+ \psi_2 , \psi_1 \rangle_{\mathbb{C}^n}$$

$$\psi_2^* A \psi_1 = (A^+ \psi_2)^* \psi_1 = \langle A^+ \psi_2 , \psi_1 \rangle_{\mathbb{C}^n}$$

$$\Rightarrow \boxed{A^+ = A^*}$$

in finite dimensions,

adjoint is equal to complex conjugate transpose.

$$\boxed{\mathcal{L}(Q) = \int_0^\infty e^{At} B Q B^* e^{A^*t} \, dt}$$

$$AP + PA^* = -BQB^*$$

$$\langle R, Q \rangle = \operatorname{trace}(R^* Q)$$

$$\langle R, \mathcal{L}(Q) \rangle = \langle \mathcal{L}^{+}(R), Q \rangle$$

Let $B = I$,

$$\left\langle R, \int_0^\infty e^{At} Q e^{A^*t} \, dt \right\rangle = \operatorname{trace}\left(R^* \int_0^\infty e^{At} Q e^{A^*t} \, dt\right)$$

$$= \operatorname{trace}\left(\int_0^\infty e^{A^*t} R^* e^{At} \, dt \cdot Q \right)$$

$$= \left\langle \int_0^\infty e^{A^*t} R e^{At} \, dt, Q \right.$$

$\downarrow$

R is Hermitian

$$\boxed{\mathcal{L}^{+}(R) = \int_0^\infty e^{A^*t} R e^{At} \, dt}$$