09-20-11

Fourier Transform : $\mathcal{F}$

$$
\begin{tikzcd}
L_2(-\infty,\infty) \supset \mathcal{D}(\mathcal{A}) \arrow[r, "\mathcal{A} = \frac{\partial^2}{\partial x^2}"] \arrow[d, shift right, "\mathcal{F}"'] & L_2(-\infty,\infty) \arrow[d, shift right] \\
L_2(-\infty,\infty) \supset \mathcal{D}(\hat{\mathcal{A}}) \arrow[r, "\hat{\mathcal{A}}(k) = -k^2", "\substack{\text{multiplication}\\ \text{operator}}"'] \arrow[u, shift right] & L_2(-\infty,\infty) \arrow[u, shift right, "\mathcal{F}^{-1}"']
\end{tikzcd}
$$

F.T. brings $\phi_t = \phi_{xx} + u$ to

$$\dot{\hat{\phi}}(k,t) = -k^2 \hat{\phi}(k,t) + \hat{u}(k,t)$$

Continuum of decoupled scalar states (parameterized by $k \in \mathbb{R}$)

k → wave-number

F.T. "diagonalizes" generator of our dynamics

(i.e. operator $\mathcal{A} = \left.\frac{d^2}{dx^2}\right|_{L_2(-\infty,\infty)}$)

Heat equation

$\phi_t = \phi_{xx}$ with B.C. $\phi(x=\pm 1, t) = 0$

I.C. $\phi(x, t=0) = f(x)$

$$\phi(x,t) = \sum_{n=1}^{\infty} \alpha_n(t) v_n(x)$$

$$\langle v_n, v_m \rangle = \int_{-1}^{1} v_n^*(x) v_m(x)\, dx = \delta_{m,n} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Cont'd →

$$\sum_{n=1}^{\infty} \dot{\alpha}_n(t) v_n(x) = \sum_{n=1}^{\infty} \alpha_n(t) \underbrace{v_n''(x)}$$

Use the fact that

$$v_n''(x) = \lambda_n v_n(x)$$

$$\sum_{n=1}^{\infty} \dot{\alpha}_n(t) v_n(x) = \sum_{n=1}^{\infty} \lambda_n \alpha_n(t) v_n(x)$$

$$\left\langle v_m, \sum_{n=1}^{\infty} \dot{\alpha}_n(t) v_n(x) \right\rangle = \left\langle v_m, \sum_{n=1}^{\infty} \lambda_n \alpha_n(t) v_n(x) \right\rangle$$

$$\sum_{n=1}^{\infty} \dot{\alpha}_n \underbrace{\langle v_n, v_m \rangle}_{\delta_{n,m}} = \sum_{n=1}^{\infty} \lambda_n \alpha_n(t) \underbrace{\langle v_n, v_m \rangle}_{\delta_{n,m}}$$

$$\boxed{\dot{\alpha}_m(t) = \lambda_m \alpha_m(t)}$$

Eigen-function expansion brings

$$\begin{cases} \phi_t = \phi_{xx} \\ \phi(\pm 1, t) = 0 \end{cases} \quad \text{into}$$

$$\frac{d}{dt} \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \end{bmatrix}$$

$$\begin{tikzcd}
L_2[-1,1] \supset \mathcal{D}(\mathcal{A}) \arrow[r, "\mathcal{A} = \frac{d^2}{dx^2}"] \arrow[d, "\text{Basis expansion}"'] & L_2[-1,1] \\
\ell_2(\mathbb{N}) \supset \mathcal{D}(\hat{\mathcal{A}}) \arrow[r, "\hat{\mathcal{A}} = \operatorname{diag}\{\lambda_n\}_{n \in \mathbb{N}}", "\text{multiplication operator}"'] & \ell_2(\mathbb{N}) \arrow[u]
\end{tikzcd}$$

Remaining Task

Find dependence of

$\alpha_n(0)$ on $\phi(x,0) = f(x)$ — $\underbrace{\text{initial}}$ condition

So far: $\phi(x,t) = \sum_{n=1}^{\infty} e^{\lambda_n t} \alpha_n(0) v_n(x)$

$$f(x) = \phi(x,0) = \sum_{n=1}^{\infty} \alpha_n(0) v_n(x)$$

$$\langle v_m, f \rangle = \sum_{n=1}^{\infty} \alpha_n(0) \underbrace{\langle v_m, v_n \rangle}_{\delta_{m,n}}$$

$$\boxed{\alpha_m(0) = \langle v_m, f \rangle}$$

Then,

$$\phi(x,t) = \sum_{n=1}^{\infty} e^{\lambda_n t} v_n(x) \langle v_n, f \rangle$$

$$= \sum_{n=1}^{\infty} e^{\lambda_n t} v_n(x) \int_{-1}^{1} v_n^*(\xi) f(\xi)\, d\xi$$

$$= \int_{-1}^{1} \underbrace{\sum_{n=1}^{\infty} e^{\lambda_n t} v_n(x) v_n^*(\xi)}_{T(x,\xi,t)} f(\xi)\, d\xi$$

$$\boxed{\phi(x,t) = \int_{-1}^{1} T(x,\xi,t) f(\xi)\, d\xi}$$

Abstractly

$$\phi(x,t) = [T(t)\cdot f](x)$$

propagator of the dynamics

initial condition

$T(t)$ is an operator with a kernel representation $T(x, \xi, t)$

Finite dimensional

$$\boxed{\frac{d\psi}{dt} = A\psi} \qquad (1)$$

$$A = v\Lambda v^* \iff v^* A v = \Lambda$$

introduce a coordinate transformation:

$$\phi = v^*\psi \longrightarrow \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} = \begin{bmatrix} v_1^* \\ \vdots \\ v_n^* \end{bmatrix} \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_n \end{bmatrix}$$

$$\psi = v^{-*}\phi = v\phi$$

$$v^*\left(v\frac{d\phi}{dt} = Av\phi\right) \Rightarrow$$

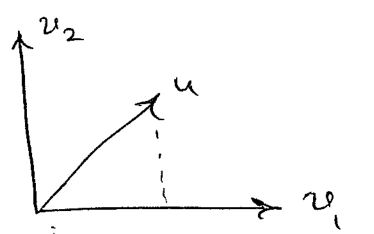
$$v^* v \frac{d\phi}{dt} = v^* A v \phi \Rightarrow \boxed{\frac{d\phi}{dt} = \Lambda\phi}$$

$$\begin{tikzcd} \psi \in \mathbb{R}^n \arrow[r, "A"] \arrow[d, "v^*"'] & \mathbb{R}^n \\ \phi \in \mathbb{R}^n \arrow[r, "\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}"] & \mathbb{R}^n \arrow[u] \end{tikzcd} \qquad \Rightarrow \psi(t) = e^{At}\psi(0)$$

Summary

$$A u = \sum_{i=1}^{n} \lambda_i v_i \underbrace{\langle v_i , u \rangle}_{\text{Projection of } u \text{ on } v_i}$$

for $u \in \mathbb{R}^2$



projection of u on v_1

Then,
$$e^{At} u = \sum_{i=1}^{n} e^{\lambda_i t} v_i \langle v_i , u \rangle \qquad (1)$$

Note : if $u = v_m$

$$e^{At} u = e^{At} v_m = \sum_{i=1}^{n} e^{\lambda_i t} v_i \underbrace{\langle v_i , v_m \rangle}_{\delta_{i,m}}$$

$$= e^{\lambda_m t} \cdot v_m$$

In the case of Heat equation

$$\phi(x,t) = \sum_{n=1}^{\infty} e^{\lambda_n t} \cdot v_n(x) \langle v_n , f \rangle \qquad (2)$$

(1) & (2) are Conceptually similar.

The main difference is the inner - product that is used in (1) or (2).