Lectures 4 & 5: Solutions to simple infinite dimensional systems

- Notion of a Hilbert space
  - ⋆ Complete linear vector space with an inner product

- Examples of solutions to infinite dimensional systems
  - ⋆ Infinite number of decoupled scalar states
  - ⋆ Continuum of decoupled states
  - ⋆ 1D heat equation
  - ⋆ 1D wave equation

- Informal discussion
  - ⋆ Serves as a motivation for formal developments (later in the course)

Hilbert space

- Hilbert space $\mathbb{H}$: a linear vector space
  - ⋆ complete (i.e., Cauchy sequences in $\mathbb{H}$ converge to an element in $\mathbb{H}$)
  - ⋆ has an inner product

- Inner product $\langle \cdot, \cdot \rangle$: $\mathbb{H} \times \mathbb{H} \to \mathbb{C}$
  - ⋆ $\langle u, v \rangle \;=\; \overline{\langle v, u \rangle}$
  - ⋆ $\langle u, v + w \rangle \;=\; \langle u, v \rangle \,+\, \langle u, w \rangle$
  - ⋆ $\langle u, \alpha\, v \rangle \;=\; \alpha\, \langle u, v \rangle; \quad \langle \alpha\, u, v \rangle \;=\; \overline{\alpha}\, \langle u, v \rangle$

- $\langle \cdot, \cdot \rangle$: induces a norm on $\mathbb{H}$: for $v \in \mathbb{H}$, $\|v\|^2 \;=\; \langle v, v \rangle$
  - ⋆ $\|v\| \;\geq\; 0$, for all $v \in \mathbb{H}$
  - ⋆ $\|v\| \;=\; 0 \quad \Leftrightarrow \quad v \;=\; 0$
  - ⋆ $\|\alpha\, v\| \;=\; |\alpha|\, \|v\|$
  - ⋆ $\|u \,+\, v\| \;\leq\; \|u\| \,+\, \|v\|$

Examples of Hilbert spaces

- $\mathbb{R}^n$, $\mathbb{C}^n$

- $\ell_2(\mathbb{Z})$, $\ell_2(\mathbb{N})$, $\ell_2(\mathbb{N}_0)$

$$\ell_2(\mathbb{Z}) \;=\; \left\{ \{f_n\}_{n \in \mathbb{Z}}, \;\; \sum_{n=-\infty}^{\infty} f_n^* f_n \;<\; \infty \right\}$$

- $L_2(-\infty, \infty)$, $L_2(0, \infty)$, $L_2[a, b]$

$$L_2(-\infty, \infty) \;=\; \left\{ f, \;\; \int_{-\infty}^{\infty} f^*(x)\, f(x)\, \mathrm{d}x \;<\; \infty \right\}$$

- The geometries of ℓ_2 and L_2 are similar to the geometry of $\mathbb{C}^n$

$\mathbb{C}^n$ vs. $L_2(-\infty, \infty)$

	$\mathbb{C}^n$	$L_2(-\infty, \infty)$
addition	$w = u + v$ $\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$	$w = u + v$ $\begin{bmatrix} w_1(x) \\ \vdots \\ w_n(x) \end{bmatrix} = \begin{bmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{bmatrix} + \begin{bmatrix} v_1(x) \\ \vdots \\ v_n(x) \end{bmatrix}$
inner product	$\langle u, v \rangle = u^* v = \sum_{i=1}^{n} \overline{u}_i v_i$	$\langle u, v \rangle = \int_{-\infty}^{\infty} u^*(x) v(x) \, \mathrm{d}x$ $= \int_{-\infty}^{\infty} \sum_{i=1}^{n} \overline{u}_i(x) v_i(x) \, \mathrm{d}x$
norm	$\|v\|^2 = \langle v, v \rangle = v^* v$	$\|v\|^2 = \langle v, v \rangle = \int_{-\infty}^{\infty} v^*(x) v(x) \, \mathrm{d}x$

Infinite number of decoupled scalar states

$$\dot{\psi}_n(t) \;=\; a_n \, \psi_n(t), \;\; n \,\in\, \mathbb{N}$$

- Abstract evolution equation on $\ell_2\,(\mathbb{N})$

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \vdots \end{bmatrix} \;=\; \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \vdots \end{bmatrix} \;\;\Leftrightarrow\;\; \frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} \;=\; \mathcal{A}\,\psi(t)$$

Solution

$$\begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \vdots \end{bmatrix} \;=\; \begin{bmatrix} \mathrm{e}^{a_1\,t} & & \\ & \mathrm{e}^{a_2\,t} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \psi_2(0) \\ \vdots \end{bmatrix}$$

looks like $\psi(t) \;=\; \mathrm{e}^{\mathcal{A}\,t}\,\psi(0)$

- Later: conditions for well-posedness on $\ell_2\,(\mathbb{N})$

Continuum of decoupled scalar states

$$\dot{\psi}(\kappa, t) \;=\; a(\kappa)\,\psi(\kappa, t), \quad \kappa \in \mathbb{R}$$

- Generator of the dynamics

multiplication operator: $\left[M_a\,\psi(\cdot, t)\right](\kappa) \;=\; a(\kappa)\,\psi(\kappa, t)$

Solution

$\psi(\kappa, t) \;=\; \mathrm{e}^{a(\kappa)\,t}\,\psi(\kappa, 0)$ looks like $\psi(\kappa, t) \;=\; \left[\mathrm{e}^{M_a\,t}\,\psi(\cdot, 0)\right](\kappa)$

- Later: conditions for well-posedness on $L_2\,(-\infty, \infty)$

Diffusion equation on $L_2(-\infty, \infty)$

$$
\begin{aligned}
\phi_t(x,t) &= \phi_{xx}(x,t) + u(x,t) \\
\phi(x,0) &= f(x), \;\; x \in \mathbb{R}
\end{aligned}
$$

Spatial Fourier transform:

$$
\left.
\begin{aligned}
\dot{\hat{\phi}}(\kappa,t) &= -\kappa^2\,\hat{\phi}(\kappa,t) + \hat{u}(\kappa,t) \\
\hat{\phi}(\kappa,0) &= \hat{f}(\kappa), \;\; \kappa \in \mathbb{R}
\end{aligned}
\right\} \Rightarrow \hat{\phi}(\kappa,t) = \mathrm{e}^{-\kappa^2 t}\,\hat{f}(\kappa) + \int_0^t \mathrm{e}^{-\kappa^2 (t-\tau)}\,\hat{u}(\kappa,\tau)\,\mathrm{d}\tau
$$

- Abstractly

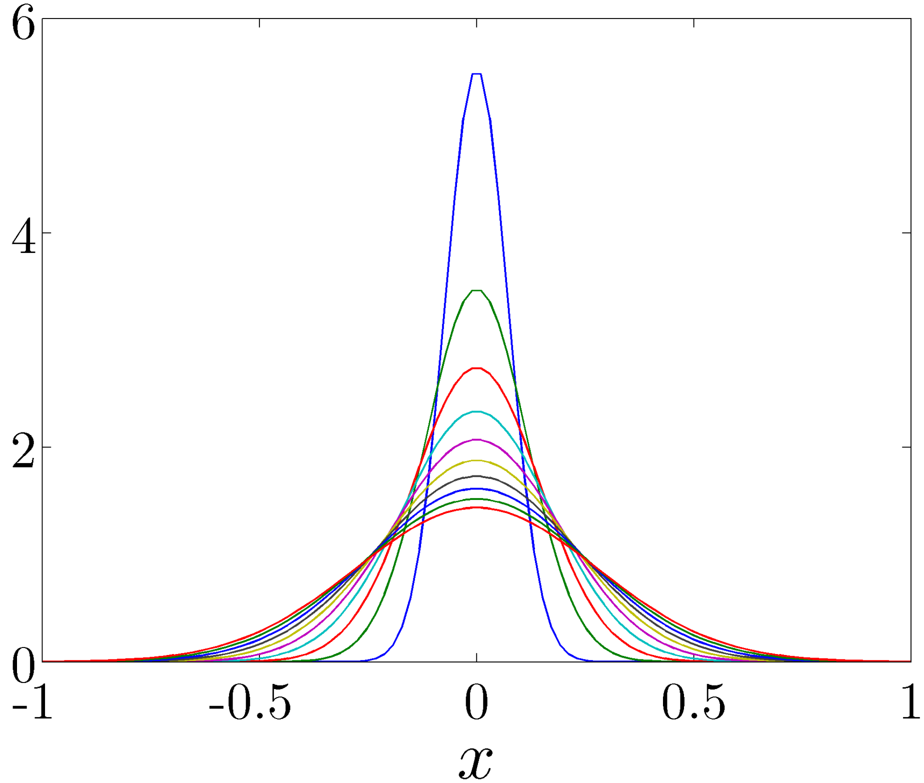
$$
\hat{\phi}(\kappa,t) = \hat{T}(\kappa,t)\,\hat{f}(\kappa) + \int_0^t \hat{T}(\kappa,t-\tau)\,\hat{u}(\kappa,\tau)\,\mathrm{d}\tau
$$

$$\Updownarrow$$

$$
\phi(x,t) = \int_{-\infty}^{\infty} T(x-\xi,t)\,f(\xi)\,\mathrm{d}\xi + \int_0^t \int_{-\infty}^{\infty} T(x-\xi,t-\tau)\,u(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau
$$

- Back to physical space

$$T(x,t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{T}(\kappa,t)\,\mathrm{e}^{\mathrm{j}\kappa\,x}\,\mathrm{d}\kappa \;=\; \frac{1}{2\sqrt{\pi\,t}}\,\mathrm{e}^{-x^2/(4t)}$$



Solution can be represented as:

$$\phi(x,t) \;=\; \left[\mathcal{T}(t)\,f(\cdot)\right](x) \;+\; \left[\int_0^t \mathcal{T}(t-\tau)\,u(\cdot,\tau)\,\mathrm{d}\tau\right](x)$$

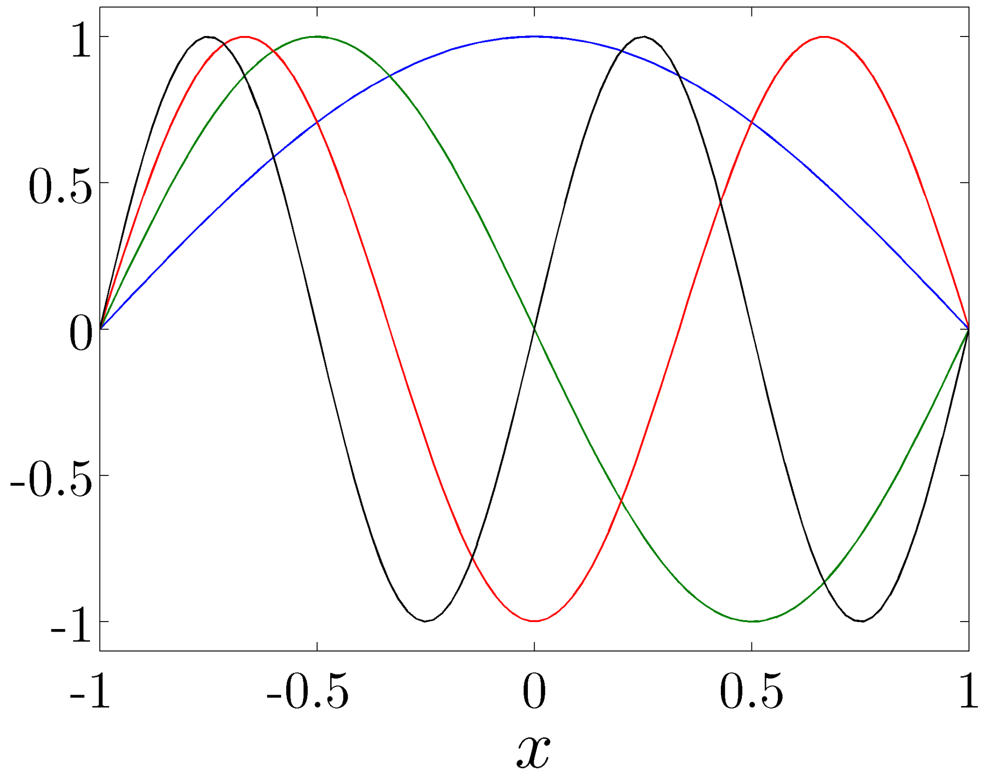
$$\left[\mathcal{T}(t)\,f(\cdot)\right](x) \;=\; \int_{-\infty}^{\infty} T(x-\xi,t)\,f(\xi)\,\mathrm{d}\xi$$

Diffusion equation on $L_2[-1, 1]$ with Dirichlet BCs

$$
\begin{aligned}
\phi_t(x,t) &= \phi_{xx}(x,t) + u(x,t) \\
\phi(x,0) &= f(x) \\
\phi(\pm 1,t) &= 0
\end{aligned}
$$

- Consider

$$
\left\{ v_n(x) = \sin\left(\frac{n\pi}{2}(x + 1)\right) \right\}_{n \in \mathbb{N}}
$$



- Properties of $\left\{ v_n(x) \;=\; \sin\left(\frac{n\pi}{2}\,(x\,+\,1)\right) \right\}_{n\,\in\,\mathbb{N}}$

1. Satisfy BCs

$$v_n(\pm 1) \;=\; 0$$

2. Of unit length and mutually orthogonal (i.e., orthonormal)

$$\langle v_n, v_m \rangle \;=\; \delta_{nm} \;=\; \begin{cases} 1, & n \,=\, m \\ 0, & n \,\neq\, m \end{cases}$$

3. Complete basis of $L_2\,[-1, 1]$

$$\overline{\mathrm{span}\,\{v_n\}_{n\,\in\,\mathbb{N}}} \;=\; L_2\,[-1, 1]$$

4. Eigenfunctions of $\dfrac{\mathrm{d}^2}{\mathrm{d}\,x^2}$ with Dirichlet BCs

$$\frac{\mathrm{d}^2\, v_n(x)}{\mathrm{d}\,x^2} \;=\; \lambda_n\, v_n(x), \;\; \lambda_n \;=\; -\left(\frac{n\pi}{2}\right)^2$$

Solution technique

1. Represent the solution as

$$\phi(x,t) \;=\; \sum_{n=1}^{\infty} \alpha_n(t)\, v_n(x)$$

$$\alpha_n(t) \;=\; \langle v_n, \phi \rangle$$

2. Substitute into the PDE and use $v_n''(x) \;=\; \lambda_n\, v_n(x)$

$$\sum_{n=1}^{\infty} \dot{\alpha}_n(t)\, v_n(x) \;=\; \sum_{n=1}^{\infty} \lambda_n\, \alpha_n(t)\, v_n(x) \;+\; u(x,t)$$

3. Take an inner product with v_m

$$\left\langle v_m, \sum_{n\,=\,1}^{\infty} \dot{\alpha}_n(t)\, v_n \right\rangle \;=\; \left\langle v_m, \sum_{n\,=\,1}^{\infty} \lambda_n\, \alpha_n(t)\, v_n \right\rangle \;+\; \langle v_m, u \rangle$$

4. Use orthonormality of $\{v_n(x)\}_{n \,\in\, \mathbb{N}}$

$$\dot{\alpha}_m(t) \;=\; \lambda_m\, \alpha_m(t) \;+\; u_m(t)$$

$$\Downarrow$$

$$\alpha_m(t) \;=\; \mathrm{e}^{\lambda_m\, t}\, \underbrace{\alpha_m(0)}_{\langle v_m, f\rangle} \;+\; \int_0^t \mathrm{e}^{\lambda_m\,(t - \tau)}\, \underbrace{u_m(\tau)}_{\langle v_m, u\rangle}\, \mathrm{d}\tau$$

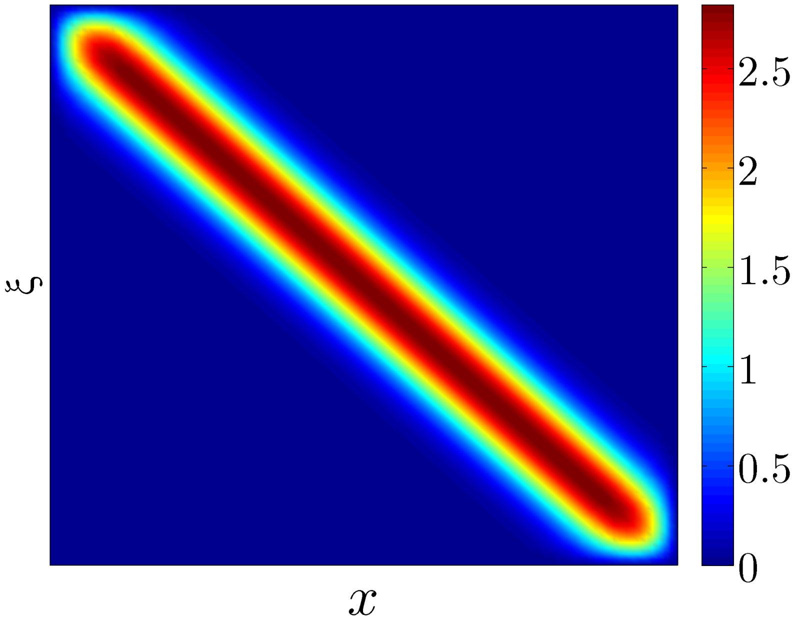
5. Express solution as

$$\begin{aligned}\phi(x,t) \;&=\; \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n t}\, v_n(x)\, \langle v_n, f\rangle \;+\; \int_0^t \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n (t-\tau)}\, v_n(x)\, \langle v_n, u(\cdot,\tau)\rangle \; \mathrm{d}\tau \\ &=\; \int_{-1}^{1} \underbrace{\sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n t}\, v_n(x)\, v_n^*(\xi)}_{T(x,\xi;t)}\, f(\xi)\, \mathrm{d}\xi \;+\; \int_0^t \int_{-1}^{1} \underbrace{\sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n (t-\tau)}\, v_n(x)\, v_n^*(\xi)}_{T(x,\xi;t-\tau)}\, u(\xi,\tau)\, \mathrm{d}\xi\, \mathrm{d}\tau\end{aligned}$$

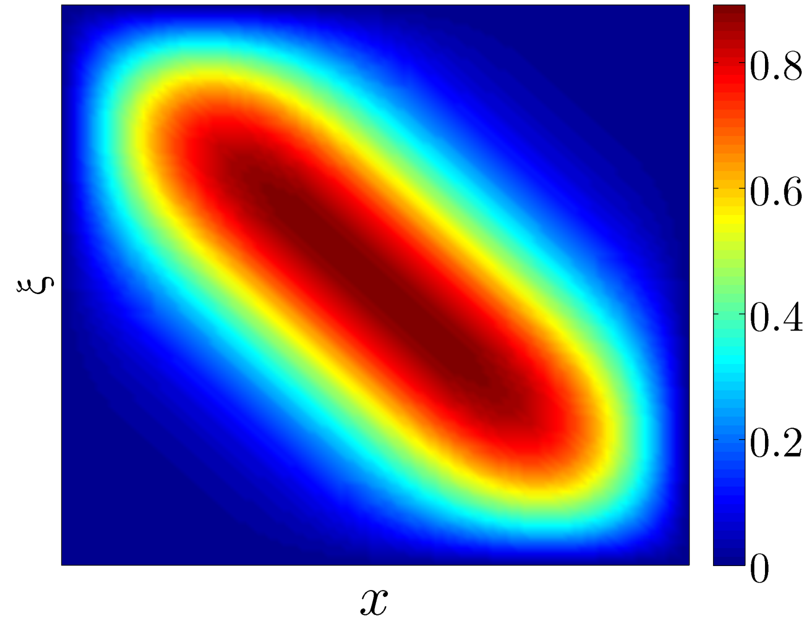
- Green's function for diffusion equation on $L_2\,[-1, 1]$ with Dirichlet BCs

$$\begin{aligned}T(x,\xi;t) \;&=\; \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n t}\, v_n(x)\, v_n^*(\xi) \\ &=\; \sum_{n=1}^{\infty} \mathrm{e}^{-\left(\frac{n\pi}{2}\right)^2 t}\, \sin\left(\frac{n\pi}{2}\,(x+1)\right) \sin\left(\frac{n\pi}{2}\,(\xi+1)\right)\end{aligned}$$

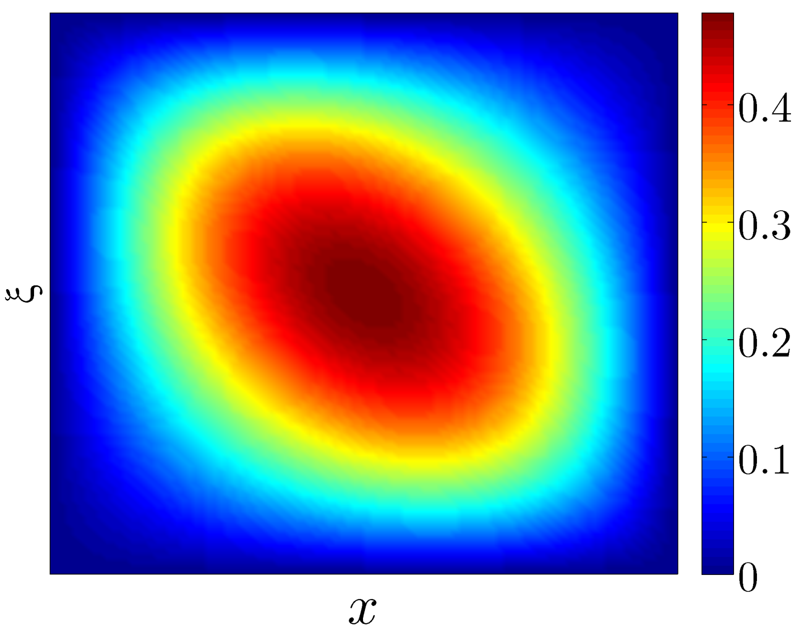
$T(x, \xi; t = 0.01)$:



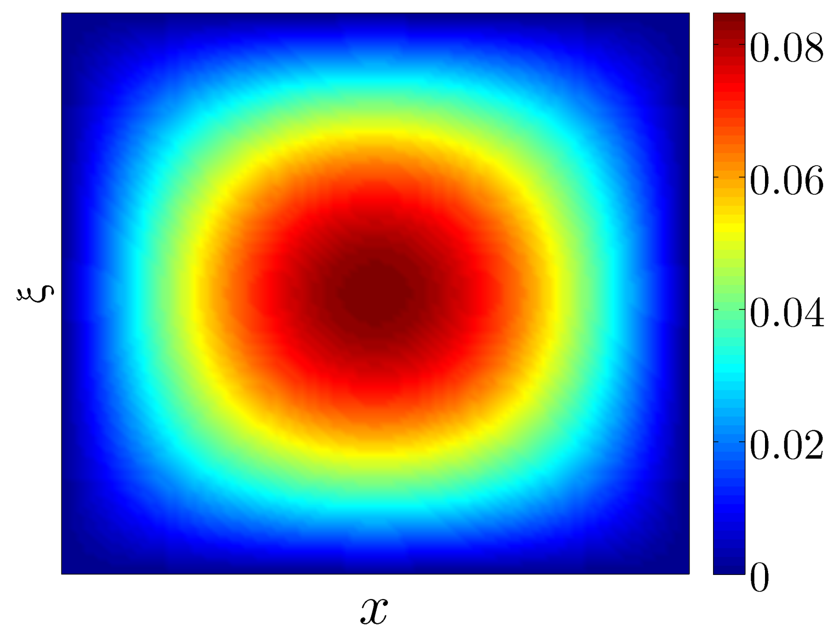
$T(x, \xi; t = 0.1)$:



$T(x, \xi; t = 0.3)$:



$T(x, \xi; t = 1)$:

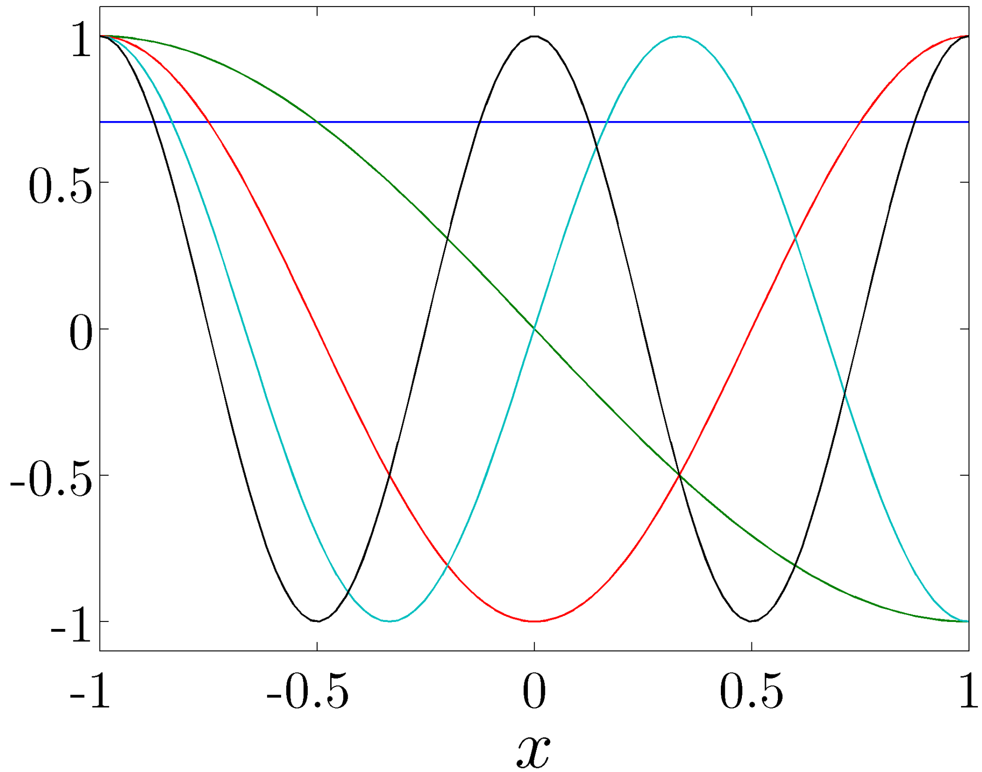


Diffusion equation on $L_2[-1,1]$ with Neumann BCs

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) + u(x,t) \\ \phi(x,0) &= f(x) \\ \phi_x(\pm 1,t) &= 0 \end{aligned}$$

- Orthonormal basis

$$\left\{ v_0(x) = \frac{1}{\sqrt{2}};\ v_n(x) = \cos\left(\frac{n\pi}{2}(x+1)\right) \right\}_{n \in \mathbb{N}}$$



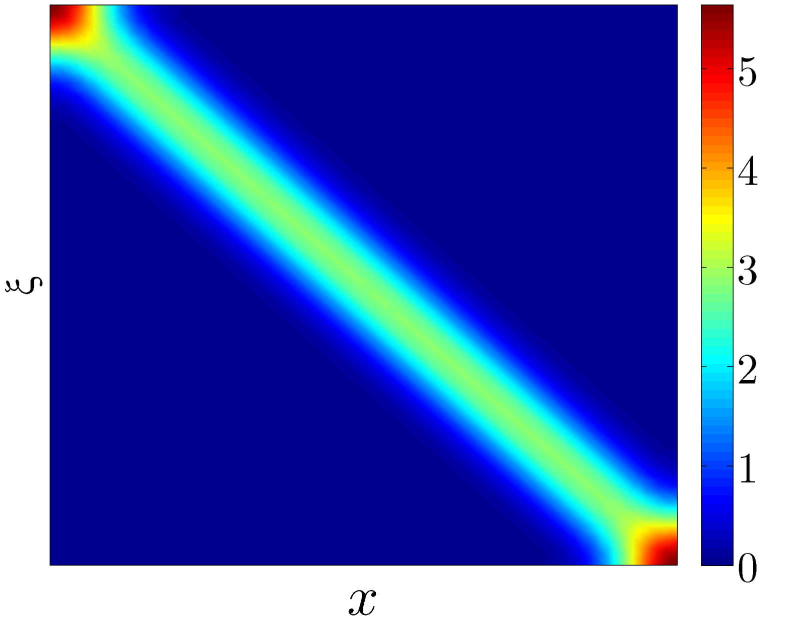
- Eigenfunctions of $\dfrac{\mathrm{d}^2}{\mathrm{d}\,x^2}$ with Neumann BCs

$$\frac{\mathrm{d}^2\, v_n(x)}{\mathrm{d}\,x^2} \;=\; \lambda_n\, v_n(x), \quad \left\{\lambda_0 \;=\; 0;\; \lambda_n \;=\; -\left(\frac{n\pi}{2}\right)^2\right\}_{n\,\in\,\mathbb{N}}$$

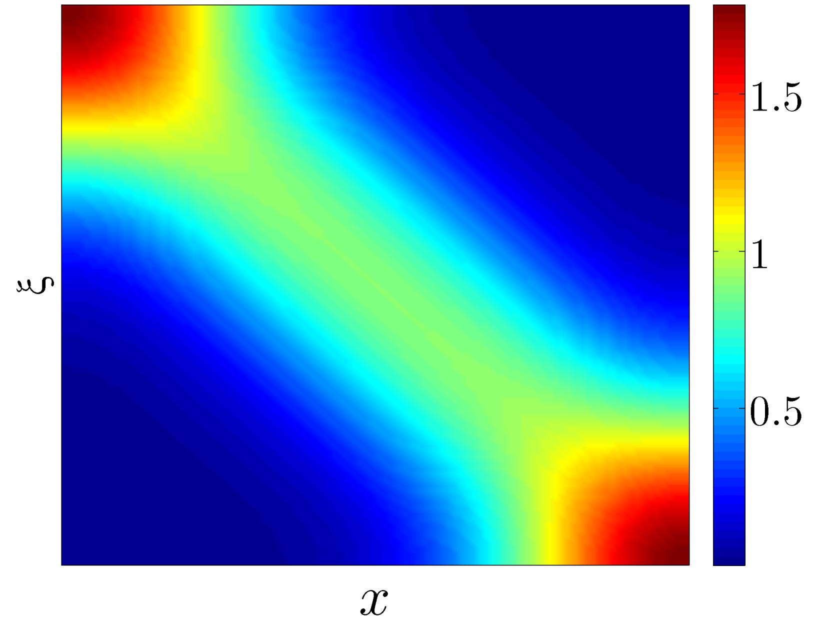
- Green's function

$$\begin{aligned} T(x,\xi;t) \;&=\; \sum_{n\,=\,0}^{\infty} \mathrm{e}^{\lambda_n\, t}\, v_n(x)\, v_n^*(\xi) \\ &=\; \frac{1}{2} \;+\; \sum_{n\,=\,1}^{\infty} \mathrm{e}^{-\left(\frac{n\pi}{2}\right)^2 t}\, \cos\left(\frac{n\pi}{2}\,(x\;+\;1)\right) \cos\left(\frac{n\pi}{2}\,(\xi\;+\;1)\right) \end{aligned}$$

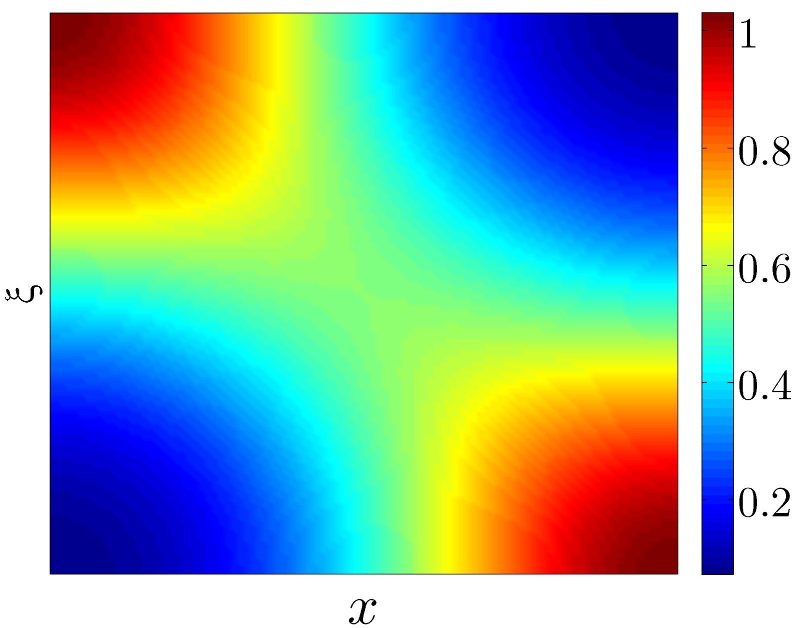
$T(x, \xi; t = 0.01)$:



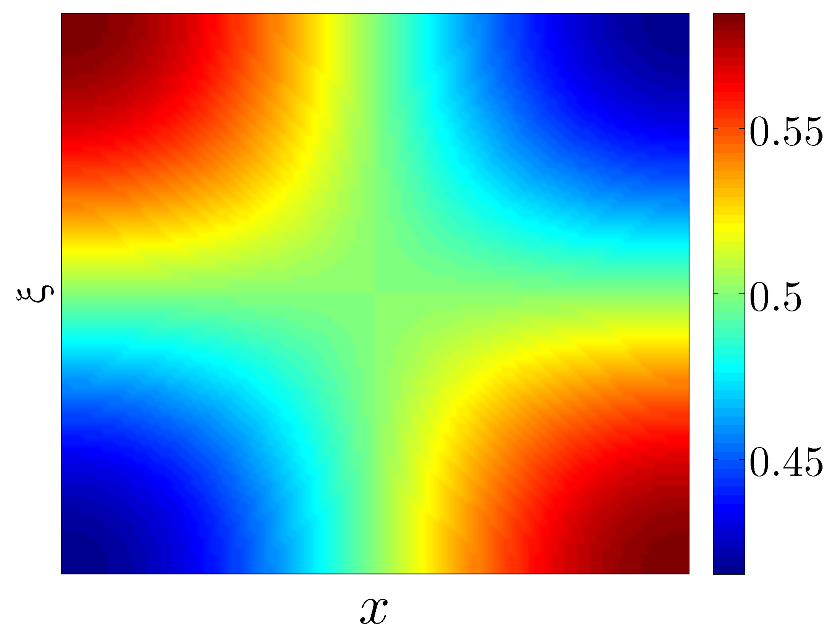
$T(x, \xi; t = 0.1)$:



$T(x, \xi; t = 0.3)$:



$T(x, \xi; t = 1)$:



Finite dimensional analogy

$$\dot{\psi}(t) \; = \; A \, \psi(t)$$

Let A have a full set of linearly independent orthonormal e-vectors

$$A \, v_i \; = \; \lambda_i \, v_i \quad \Leftrightarrow \quad A \underbrace{\left[\, v_1 \quad \cdots \quad v_n \, \right]}_{V} \; = \; \underbrace{\left[\, v_1 \quad \cdots \quad v_n \, \right]}_{V} \underbrace{\left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array} \right]}_{\Lambda}$$

- A – diagonalizable by a unitary coordinate transformation

$$A \; = \; \left[\, v_1 \quad \cdots \quad v_n \, \right] \left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array} \right] \left[\begin{array}{c} v_1^* \\ \vdots \\ v_n^* \end{array} \right]$$

$$\mathrm{e}^{A\,t} \; = \; \left[\, v_1 \quad \cdots \quad v_n \, \right] \left[\begin{array}{ccc} \mathrm{e}^{\lambda_1 \, t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_n \, t} \end{array} \right] \left[\begin{array}{c} v_1^* \\ \vdots \\ v_n^* \end{array} \right]$$

Dyadic decomposition of matrix A

- Action of A on $u \in \mathbb{C}^n$

$$
\begin{aligned}
A\,u \;&=\; \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right] \left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array}\right] \left[\begin{array}{c} v_1^* \\ \vdots \\ v_n^* \end{array}\right] u \\
&=\; \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right] \left[\begin{array}{c} \lambda_1\, v_1^*\, u \\ \vdots \\ \lambda_n\, v_n^*\, u \end{array}\right] \\
&=\; \lambda_1\, v_1\, v_1^*\, u \,+\, \cdots \,+\, \lambda_n\, v_n\, v_n^*\, u \\
&=\; \sum_{i\,=\,1}^{n} \lambda_i\, v_i\, \langle v_i, u \rangle
\end{aligned}
$$

- Solution to $\dot{\psi}(t) \;=\; A\,\psi(t)$

$$
\psi(t) \;=\; \mathrm{e}^{A\,t}\,\psi(0) \;=\; \sum_{i\,=\,1}^{n} \mathrm{e}^{\lambda_i\,t}\, v_i\, \langle v_i, \psi(0) \rangle
$$

Dyadic decomposition of operator $\mathcal{A}$

- Action of operator $\mathcal{A}$ (with a full set of orthonormal e-functions) on $u \in \mathbb{H}$

$$[\mathcal{A}\, u]\,(x) \;=\; \sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\, \langle v_n, u \rangle$$

- For the heat equation with Dirichlet BCs

$$\left[\frac{\mathrm{d}^2\, u}{\mathrm{d}\, x^2}\right](x) \;=\; \sum_{n\,=\,1}^{\infty} -\left(\frac{n\pi}{2}\right)^2 v_n(x)\, \langle v_n, u \rangle$$

- Solution to $\dot{\psi}(t) \;=\; \mathcal{A}\, \psi(t)$

$$[\psi(t)]\,(x) \;=\; [\mathcal{T}(t)\, \psi(0)]\,(x) \;=\; \sum_{n\,=\,1}^{\infty} \mathrm{e}^{-\left(\frac{n\pi}{2}\right)^2 t}\, v_n(x)\, \langle v_n, \psi(0) \rangle$$

A few additional notes

- Orthonormal basis $\{v_n\}_{n \in \mathbb{N}}$

$$\phi(x) \;=\; \sum_{n=1}^{\infty} \alpha_n \, v_n(x) \;=\; \sum_{n=1}^{\infty} \langle v_n, \phi \rangle \; v_n(x)$$

$$\psi(x) \;=\; \sum_{n=1}^{\infty} \beta_n \, v_n(x) \;=\; \sum_{n=1}^{\infty} \langle v_n, \psi \rangle \; v_n(x)$$

- Properties

1. $\langle \psi, \phi \rangle \;=\; \sum_{n=1}^{\infty} \overline{\langle v_n, \psi \rangle} \; \langle v_n, \phi \rangle \;=\; \sum_{n=1}^{\infty} \bar{\beta}_n \, \alpha_n$

2. $\|\psi\|^2 \;=\; \langle \psi, \psi \rangle \;=\; \sum_{n=1}^{\infty} | \, \langle v_n, \psi \rangle \, |^2 \;=\; \sum_{n=1}^{\infty} |\beta_n|^2$

3. ψ orthogonal to each $v_n \;\; \Rightarrow \;\; \psi = 0$

4. Convergence in L_2-sense $\|\psi \; - \; \sum_{n=1}^{N} \langle v_n, \psi \rangle \; v_n\| \;\xrightarrow{N \,\longrightarrow\, \infty}\; 0$

Projection theorem

- $\mathbb{H}$: Hilbert space; V: closed subspace of $\mathbb{H}$
  - ⋆ For each $\psi \in \mathbb{H}$, there is a unique $v_0 \in V$ such that

$$\|\psi - v_0\| = \min_{v \in V} \|\psi - v\|$$

  - ⋆ $v_0 \in V$ minimizing vector $\Leftrightarrow$ $(\psi - v_0) \perp V$

- Consequence: the best approximation of ψ using N orthonormal vectors v_n

$$\psi_N = \sum_{n=1}^{N} \langle v_n, \psi \rangle \, v_n$$

Proof: follows directly from Projection theorem

$$\left\langle v_n, \psi - \sum_{m=1}^{N} \alpha_m v_m \right\rangle = 0, \;\; n = \{1, \ldots, N\} \;\; \Rightarrow \;\; \alpha_m = \langle v_m, \psi \rangle$$

Orthonormality: approximation improved by adding $\langle v_{N+1}, \psi \rangle \, v_{N+1}$

Wave equation on infinite spatial extent

$$\begin{aligned} \phi_{tt}(x,t) &= c^2\,\phi_{xx}(x,t) \;+\; u(x,t) \\ \phi(x,0) &= f(x), \;\; \phi_t(x,0) \;=\; g(x), \;\; x \in \mathbb{R} \end{aligned}$$

- Evolution equation

$$\begin{aligned} \begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ c^2\,\mathrm{d}^2/\mathrm{d}x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u(t) \\ \phi(t) &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \end{aligned}$$

$$\Big\downarrow \textbf{Fourier trasform}$$

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\psi}}_1(\kappa,t) \\ \dot{\hat{\psi}}_2(\kappa,t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -c^2\,\kappa^2 & 0 \end{bmatrix} \begin{bmatrix} \hat{\psi}_1(\kappa,t) \\ \hat{\psi}_2(\kappa,t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(\kappa,t) \\ \hat{\phi}(\kappa,t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\psi}_1(\kappa,t) \\ \hat{\psi}_2(\kappa,t) \end{bmatrix} \end{aligned}$$

D'Alembert's formula

- Solution to the unforced problem

$$
\begin{aligned}
\hat{\phi}(\kappa, t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\psi}_1(\kappa, t) \\ \hat{\psi}_2(\kappa, t) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(c\,\kappa\, t) & \sin(c\,\kappa\, t)/(c\,\kappa) \\ -c\,\kappa\, \sin(c\,\kappa\, t) & \cos(c\,\kappa\, t) \end{bmatrix} \begin{bmatrix} \hat{f}(\kappa) \\ \hat{g}(\kappa) \end{bmatrix} \\
&= \cos(c\,\kappa\, t)\, \hat{f}(\kappa) \;+\; \frac{\sin(c\,\kappa\, t)}{c\,\kappa}\, \hat{g}(\kappa) \\
&= \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}c\,\kappa\, t} + \mathrm{e}^{-\mathrm{j}c\,\kappa\, t}\right) \hat{f}(\kappa) \;+\; t\, \mathrm{sinc}\,(c\,\kappa\, t)\, \hat{g}(\kappa)
\end{aligned}
$$

$\downarrow$ **inverse Fourier trasform**

$$
\begin{aligned}
\phi(x, t) &= \frac{1}{2}\left(f\,(x + ct) + f\,(x - ct)\right) \;+\; \frac{1}{2c} \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{x - \xi}{ct}\right) g(\xi)\, \mathrm{d}\xi \\
&= \frac{1}{2}\left(f\,(x + ct) + f\,(x - ct)\right) \;+\; \frac{1}{2c} \int_{x - ct}^{x + ct} g(\xi)\, \mathrm{d}\xi
\end{aligned}
$$