09-15-11

Cauchy Sequence

$$\lim_{m,n\to\infty} \| v_m - v_n \| \to 0$$

$\| \cdot \|$ ... notion of distance between elements of the space.

Finite dimensional space :

$\mathbb{C}^n$ or $\mathbb{R}^n$

$$v \in \mathbb{C}^n \iff v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \; ; \; v_i \in \mathbb{C}$$

Inner product on $\mathbb{C}^n$:

$$\langle u, v \rangle_{\mathbb{C}^n} = \underbrace{u^*}_{\text{Complex-Conjugate transpose of } u} v = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}^* \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= [\bar{u}_1 \; \dots \; \bar{u}_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} \bar{u}_i v_i$$

Inner product is Linear in second argument:

$$\langle u, v + w \rangle_{\mathbb{C}^n} = \langle u, v \rangle_{\mathbb{C}^n} + \langle u, w \rangle_{\mathbb{C}^n}$$

It is Conjugate Linear in its first argument:

$$\langle \alpha u_1, v \rangle_{\mathbb{C}^n} = \alpha^* \langle u_1, v \rangle_{\mathbb{C}^n}$$

$$= \bar{\alpha} \langle u, v \rangle_{\mathbb{C}^n}$$

$$\langle u, v \rangle_{\mathbb{C}^n} = \overline{\langle v, u \rangle}_{\mathbb{C}^n}$$

Banach Space : Complete normed space

(But, there is no notion of inner product that induces the norm).

example of Banach space : ℓ_p.

$$\ell_p = \left\{ \{f_n\}_{n \in \mathbb{Z}} ,\ \sum_{n=-\infty}^{\infty} |f_n|^p < +\infty \right\}$$

$$p = \infty : \ \ell_\infty \longrightarrow \sup |f_n|$$

$$\dot{\Psi}_n(t) = a_n \Psi_n(t)$$

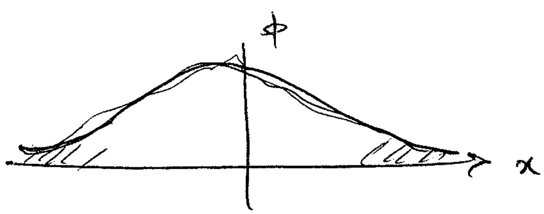
$$\Rightarrow \Psi_n(t) = e^{a_n t}\, \Psi_n(0) \quad \text{(solution)}$$

$$\begin{bmatrix} \Psi_1(t) \\ \vdots \\ \Psi_n(t) \end{bmatrix} = \begin{bmatrix} e^{a_1 t} & & \\ & \ddots & \\ & & e^{a_n t} \end{bmatrix} \begin{bmatrix} \Psi_1(0) \\ \vdots \\ \Psi_n(0) \end{bmatrix}$$

$$\phi_t(x,t) = \phi_{xx}(x,t) + u(x,t)$$

$$\phi \in L_2(-\infty, \infty)$$

Need two boundary conditions, but the fact that the operators act on elements of $L_2(-\infty,\infty)$ means that ϕ should vanish at $\pm\infty$. Otherwise it can't be square-integrable.



$$\left\langle v_m \,,\ \sum_{n=1}^{\infty} \dot{\alpha}_n(t)\, v_n \right\rangle$$

$$= \sum_{n=1}^{\infty} \dot{\alpha}_n(t)\, \underbrace{\langle v_m \,, v_n \rangle}_{\delta_{m,n} \,=\, \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}}$$

$$= \dot{\alpha}_m(t)$$
