Notation

$$\phi_t(x,t) = \phi_{xx}(x,t) + u(x,t)$$

$\phi(x,t)$ ..... field (function) ϕ evaluated at time t and position x

$\phi(x,t) \in \mathbb{R}$ .... scalar

$\Psi(t) = \phi(\cdot,t)$ .... at any fixed t $\Psi(t)$ is a function in a certain Hilbert space.

Abstract notation

$$\frac{d\Psi(t)}{dt} = \mathcal{A}\Psi(t) + \mathcal{B}u(t)$$

$$\Psi(t) \in \mathcal{H} \overset{e.g.}{=} \left\{ f,\ \int_{-\infty}^{\infty} f^*(x) f(x)\, dx < +\infty \right\}$$

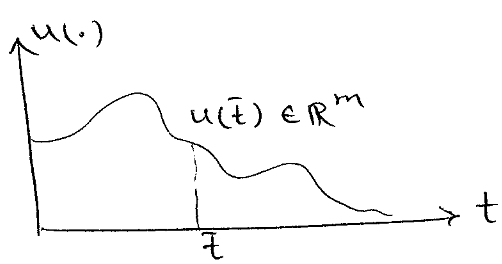
$$\Psi(t) = \phi(\cdot,t) \iff [\Psi(t)](x) = \phi(x,t)$$

Side note:

e.g. finite-dimensional system

$\dot{\Psi} = A\Psi + Bu$

input $u \in L_2(0,\infty)$



Wave equation

$$\phi_{tt} = \phi_{xx} + u$$

Define states of the system such that we are left with a first order system of equations in time.

$$\psi_1 = \phi$$
$$\psi_2 = \phi_t$$

Then

- $\psi_{1t} = \phi_t$
- $\psi_{2t} = \phi_{tt}$

So, $\psi_{1t} = \psi_2$

$$\psi_{2t} = \phi_{tt} = \phi_{xx} + u = \psi_{1xx} + u$$

$$\Rightarrow \quad \psi_{1t} = 0 \cdot \psi_1 + I \cdot \psi_2 + 0 \cdot u \qquad (1)$$

$$\psi_{2t} = \frac{d^2}{dx^2} \psi_1 + 0 \cdot \psi_2 + I \cdot u \qquad (2)$$

$$\phi = I \cdot \psi_1 + 0 \cdot \psi_2 + 0 \cdot u \qquad (3)$$

(1) & (2) ... state equations
(1st order in time diff. equations)

(3) eq. ------ output equation
(static in time equation that tells you how to obtain output of interest from the states and inputs)

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u(t)$$

$$\phi(t) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}$$

Note :

Energy of wave : $$E(t) = \frac{1}{2}\int_{-1}^{1} \left(\phi_x^2(x,t) + \phi_t^2(x,t)\right) dx$$
$$= \frac{1}{2}\int_{-1}^{1} \left(\psi_{1x}^2(x,t) + \psi_2^2(x,t)\right) dx$$

We will show (in HW) that

$\begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \in \begin{bmatrix} L_2[-1,1] \\ L_2[-1,1] \end{bmatrix}$ is not a good candidate for state space.

So, selection of state space for wave equation is more subtle than for diffusion equation.

L_2 norm of $\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$

$$\left\| \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \right\|_{L^2}^2 = \left\langle \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \right\rangle_{L^2}$$

$$= \int_{-1}^{1} \begin{bmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{bmatrix}^* \begin{bmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{bmatrix} dx$$

$$= \int_{-1}^{1} \begin{bmatrix} \bar{\psi}_1(x,t) & \bar{\psi}_2(x,t) \end{bmatrix} \begin{bmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{bmatrix} dx$$

$$= \int_{-1}^{1} \left(\psi_1^2(x,t) + \psi_2^2(x,t)\right) dx$$

Even though the above norm is a valid L_2 norm of the state, this norm is not the norm $\underbrace{\text{(wave energy)}}_{E(t)\text{ above}}$ that we are interested in.

Notation

$$(\Delta + I)^2 \phi = (\Delta + I)(\Delta + I)\phi$$

in 1D :

$$\Delta = \frac{\partial^2}{\partial x^2}$$

Then: $(\Delta+1)^2 = \left(\frac{\partial^2}{\partial x^2}+1\right)\left(\frac{\partial^2}{\partial x^2}+1\right) =$

$$\frac{\partial^4}{\partial x^4} + 2\frac{\partial^2}{\partial x^2} + 1$$

$$(\Delta + I)^2 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^2 \phi}{\partial x^2} + \phi$$

$x \in (-\infty, +\infty)$

Fourier Transform :

$$\hat{\phi} - 2K^2 \hat{\phi} + K^4 \hat{\phi} = (1 - K^2)^2 \hat{\phi}$$