Lectures 2 & 3: Examples of distributed systems

- Simple PDEs
  - ⋆ Diffusion equation
  - ⋆ Linear transport equation
  - ⋆ Wave equation
  - ⋆ Evolution of population equation

- Not-so-simple PDEs
  - ⋆ Reaction-diffusion equation
  - ⋆ Swift-Hohenberg equation
  - ⋆ Navier-Stokes equations

- Networks of dynamic systems
  - ⋆ Coordinated/cooperative control
  - ⋆ Leader selection in dynamic networks
  - ⋆ Micro-cantilever arrays
  - ⋆ Biochemical networks
  - ⋆ Wind farms

- Distributed control
  - ⋆ Feedback-based
  - ⋆ Sensor-free

Diffusion equation

$$\frac{\partial \phi(x,t)}{\partial t} \;=\; \frac{\partial^2 \phi(x,t)}{\partial x^2} \;+\; u(x,t) \;\Leftrightarrow\; \phi_t(x,t) \;=\; \phi_{xx}(x,t) \;+\; u(x,t)$$

$\phi(x,t)$ – temperature at position x and time t

$u(x,t)$ – heat addition along the bar

- Need to specify initial and boundary conditions

  ⋆ One IC: $\phi(x,0) \;=\; \phi_0(x)$

  ⋆ Two BCs:

  Homogeneous Dirichlet: $\phi(\pm 1,t) \;=\; 0$

  Homogeneous Neumann: $\phi_x(\pm 1,t) \;=\; 0$

  Homogeneous Robin:
$$\begin{aligned} a\,\phi(-1,t) \;+\; b\,\phi_x(-1,t) &\;=\; 0 \\ c\,\phi(+1,t) \;+\; d\,\phi_x(+1,t) &\;=\; 0 \end{aligned}$$

- In higher spatial dimensions

$$\phi_t(x,t) \;=\; \Delta\phi(x,t) \;+\; u(x,t)$$

$x \;=\; \left[\begin{array}{ccc} x_1 & \cdots & x_n \end{array}\right]^T$ – vector of spatial coordinates

$\Delta \;=\; \dfrac{\partial^2}{\partial x_1^2} \;+\; \cdots \;+\; \dfrac{\partial^2}{\partial x_n^2}$ – Laplacian

- Boundary actuation in 1D

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) \;+\; d(x,t) \\ \phi(x,0) &= \phi_0(x) \\ \phi(-1,t) &= u(t), \;\; \phi(+1,t) \;=\; 0 \end{aligned}$$

A finite dimensional example

- Mass-spring system

$$m\,\ddot{\phi}(t) \;+\; k\,\phi(t) \;=\; u(t)$$

$\phi(t)$ – position of a mass at time t

$u(t)$ – force acting on a mass

- A state-space representation

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} \;=\; \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \;+\; \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

$$\phi(t) \;=\; \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}$$

$\psi_1(t) \;=\; \phi(t)$ – position at time t

$\psi_2(t) \;=\; \dot{\phi}(t)$ – velocity at time t

State-space (evolution) representation

$$\begin{aligned}\dot{\psi}(t) &= A\,\psi(t) \;+\; B\,u(t)\\ \phi(t) &= C\,\psi(t)\end{aligned}$$

- Finite dimensional state space: $\psi(t) \in \mathbb{R}^n$

- Variations of constants formula

$$\psi(t) \;=\; \mathrm{e}^{At}\psi(0) \;+\; \int_0^t \mathrm{e}^{A(t\,-\,\tau)}\,B\,u(\tau)\,\mathrm{d}\tau$$

- Can we do something similar for infinite dimensional systems?

Linear transport equation

$$\begin{aligned} \phi_t(x,t) &= -a\,\phi_x(x,t) \\ \phi(x,0) &= f(x), \quad x \in \mathbb{R} \end{aligned}$$

Spatial Fourier transform

$$\hat{\phi}(\kappa,t) = \int_{-\infty}^{\infty} \phi(x,t)\,\mathrm{e}^{-\mathrm{j}\kappa\,x}\,\mathrm{d}x$$

yields

$$\left.\begin{aligned} \dot{\hat{\phi}}(\kappa,t) &= -\,(a\,\mathrm{j}\kappa)\,\hat{\phi}(\kappa,t) \\ \hat{\phi}(\kappa,0) &= \hat{f}(\kappa), \quad \kappa \in \mathbb{R} \end{aligned}\right\} \Rightarrow \hat{\phi}(\kappa,t) = \mathrm{e}^{-a\,\mathrm{j}\kappa\,t}\,\hat{f}(\kappa)$$

- Back to physical space

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(\kappa,t)\,\mathrm{e}^{\mathrm{j}\kappa\,x}\,\mathrm{d}\kappa = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\kappa)\,\mathrm{e}^{\mathrm{j}\kappa(x\,-\,at)}\,\mathrm{d}\kappa = f(x - at)$$

Solution doesn't appear to be of the form: $\text{"}\mathrm{e}^{-a\,\partial_x}\text{"} \times f(x)$

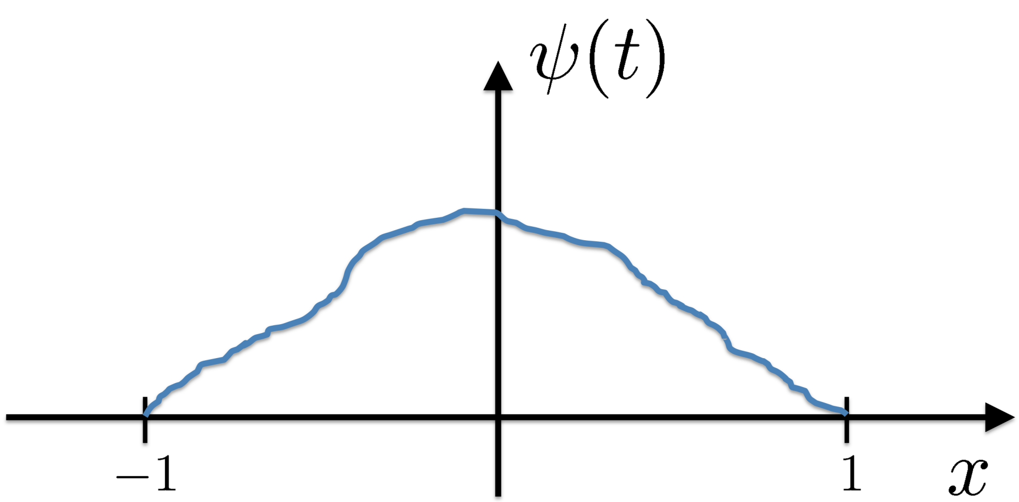
Diffusion equation

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) + u(x,t) \\ \phi(x,0) &= \phi_0(x) \\ \phi(\pm 1,t) &= 0 \end{aligned}$$

Define $\psi(t) = \phi(\cdot, t)$ and write an abstract evolution equation:

$$\begin{aligned} \dot{\psi}(t) &= \mathcal{A}\,\psi(t) + u(t) \\ \phi(t) &= \psi(t) \end{aligned}$$

- Infinite dimensional state-space: $\psi(t) \in \mathbb{H}$



- A candidate for state-space

  square-integrable functions: $\mathbb{H} \;=\; L_2\,[-1,\,1] \;=\; \left\{ f,\;\; \int_{-1}^{1} f^*(x)\, f(x)\, \mathrm{d}x \;<\; \infty \right\}$

- $\mathcal{A} \;=\; \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} \;+$ boundary conditions (contained in the domain of $\mathcal{A}$)

$$\mathcal{D}(\mathcal{A}) \;=\; \left\{ f \,\in\, L_2\,[-1,\,1],\, \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \in\, L_2\,[-1,\,1],\;\, f(\pm 1) \,=\, 0 \right\}$$

Wave equation

$$
\begin{aligned}
\phi_{tt}(x,t) &= \phi_{xx}(x,t) \;+\; u(x,t) \\
\phi(x,0) &= \phi_{10}(x), \quad \phi_t(x,0) \;=\; \phi_{20}(x), \\
\phi(\pm 1,t) &= 0
\end{aligned}
$$

Define $\psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} = \begin{bmatrix} \phi(\cdot,t) \\ \phi_t(\cdot,t) \end{bmatrix}$ and write an abstract evolution equation:

$$
\begin{aligned}
\begin{bmatrix} \dot\psi_1(t) \\ \dot\psi_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ \mathrm{d}^2/\mathrm{d}x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \;+\; \begin{bmatrix} 0 \\ I \end{bmatrix} u(t) \\
\phi(t) &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}
\end{aligned}
$$

- Energy of a wave: $\left\{ \begin{aligned} E(t) &= \frac{1}{2} \int_{-1}^{1} \left(\phi_x^2(x,t) \;+\; \phi_t^2(x,t) \right) \mathrm{d}x \\ &= \frac{1}{2} \int_{-1}^{1} \left(\psi_{1x}^2(x,t) \;+\; \psi_2^2(x,t) \right) \mathrm{d}x \end{aligned} \right.$

- Selection of state-space: more subtle than for diffusion equation!

Evolution of population equation

$$\begin{aligned}\phi_t(x,t) &= -\phi_x(x,t) - \mu(x,t)\,\phi(x,t) \\ \phi(x,0) &= \phi_0(x) \quad x \geq 0, \\ \phi(0,t) &= u(t), \qquad t \geq 0\end{aligned}$$

$\phi(x,t)$ – number of people of age x at time t

$\mu(x,t)$ – mortality function

$\phi_0(x)$ – initial age distribution

$u(t)$ – number of people born at time t

- Control problem: design u to achieve desired age profile $\phi_d(x)$ at time T

Reaction-diffusion equations

$$\boldsymbol{\phi}_t(x,t) \;=\; D\,\Delta\boldsymbol{\phi}(x,t) \;+\; \mathbf{f}(\boldsymbol{\phi}(x,t))$$

ϕ – vector-valued field of interest

$\mathbf{f}(\phi)$ – nonlinear reaction term

Δ – Laplacian

D – matrix of positive diffusion constants

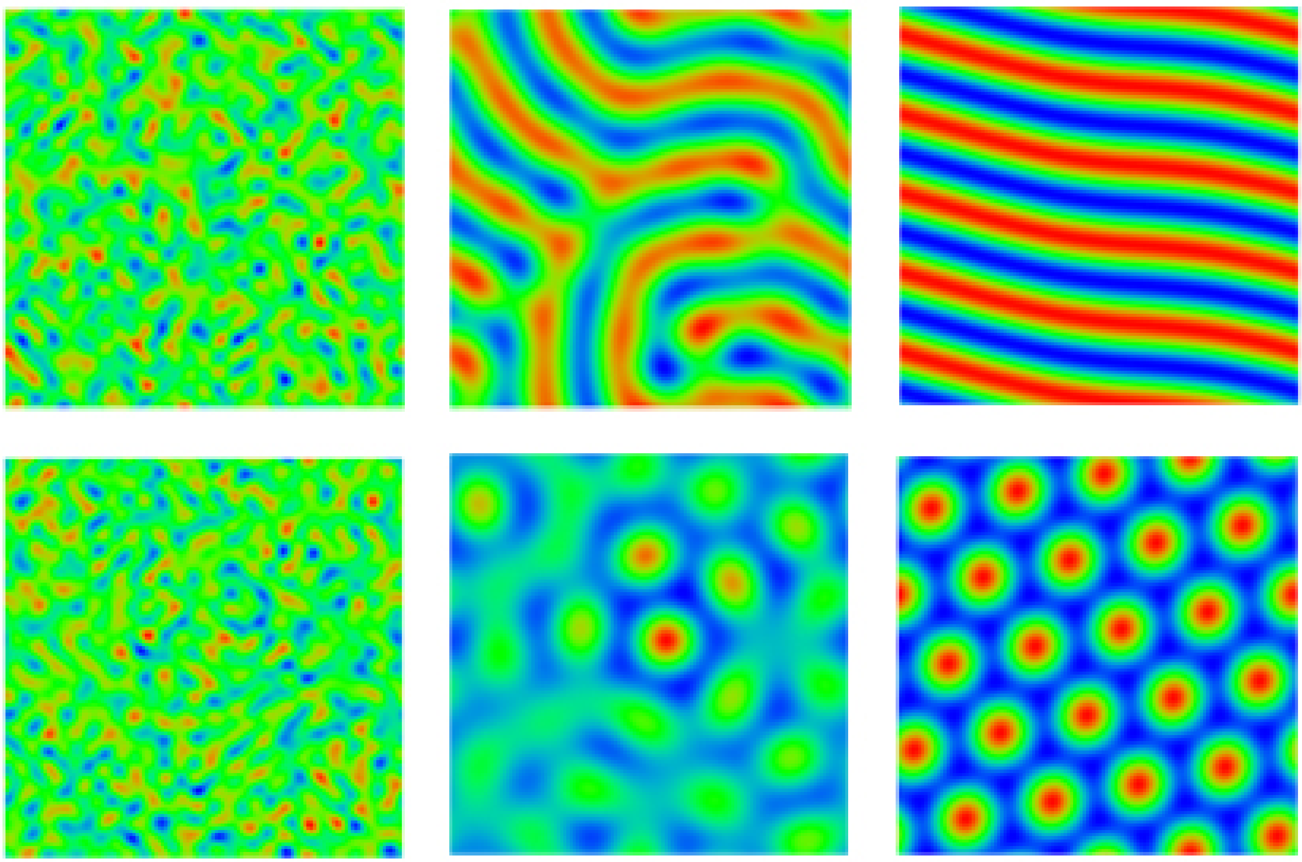
MAPK CASCADES: responsible for cell proliferation and growth

$$\begin{aligned}
\phi_{1t} &= 0.001\,\phi_{1xx} - \frac{\phi_1}{1+\phi_1} + \frac{0.4}{1+\phi_3}\\
\phi_{2t} &= 0.001\,\phi_{2xx} - \frac{\phi_2}{1+\phi_2} + 0.4\phi_1\\
\phi_{3t} &= 0.001\,\phi_{3xx} - \frac{\phi_3}{1+\phi_3} + 0.4\phi_2
\end{aligned}$$

Swift-Hohenberg equation

$$\phi_t = \epsilon\,\phi - (\Delta + 1)^2\,\phi + c\,\phi^2 - \phi^3$$

Nonlinear: first order in time, fourth order in space



- Web-site of Michael Cross at Caltech contains interactive demonstrations

Navier-Stokes equations

conservation of momentum: $\mathbf{v}_t = -(\mathbf{v} \cdot \nabla)\,\mathbf{v} - \nabla p + (1/Re)\,\Delta \mathbf{v} + \mathbf{d}$

conservation of mass: $0 = \nabla \cdot \mathbf{v}$

Describe the fluid motion

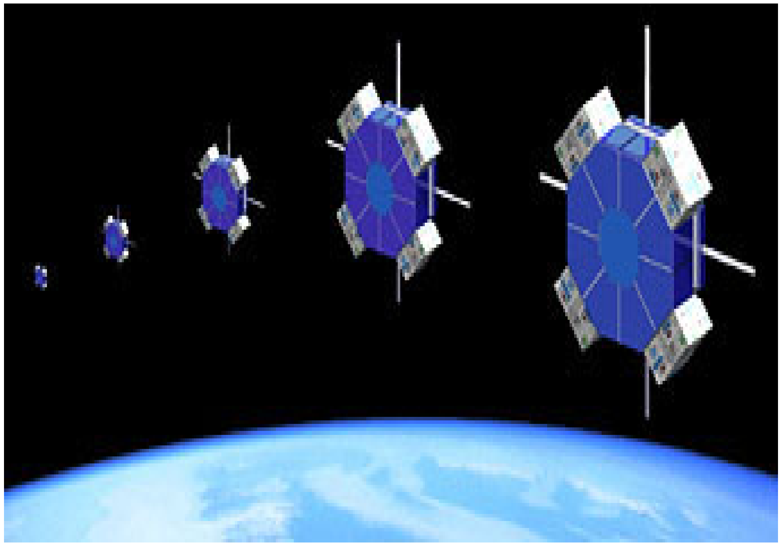
Nonlinear system of equations for:
- pressure: $p(x_1, x_2, x_3, t)$
- velocity: $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$

"del" operator: $\nabla = \dfrac{\partial}{\partial x_1}\mathbf{e}_1 + \dfrac{\partial}{\partial x_2}\mathbf{e}_2 + \dfrac{\partial}{\partial x_3}\mathbf{e}_3$

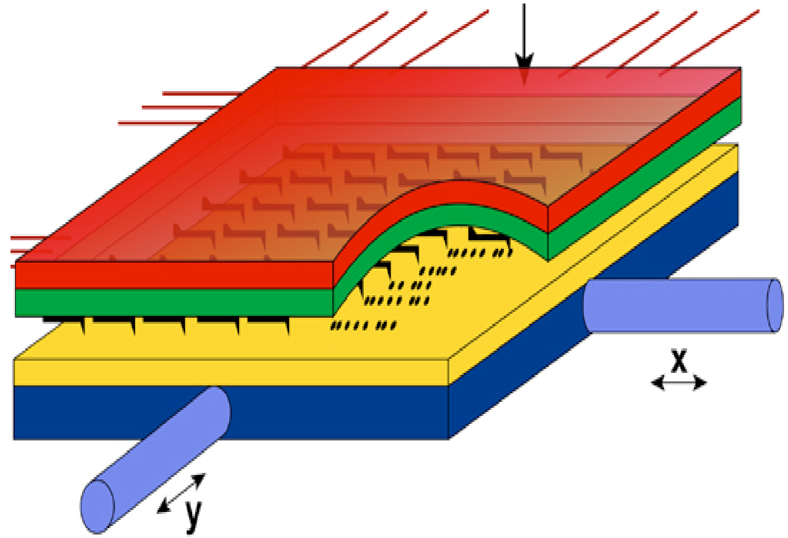
Reynolds number: $Re = \dfrac{\text{inertial forces}}{\text{viscous forces}}$

Networks of dynamic systems

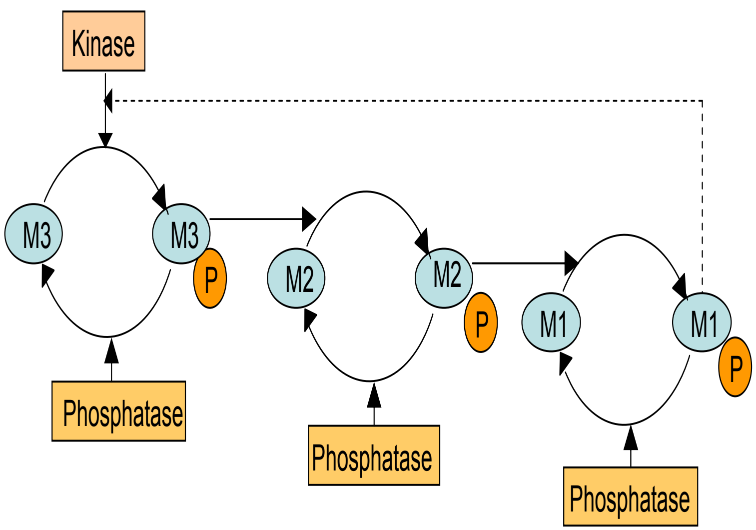
Coordinated control



Micro-cantilever arrays

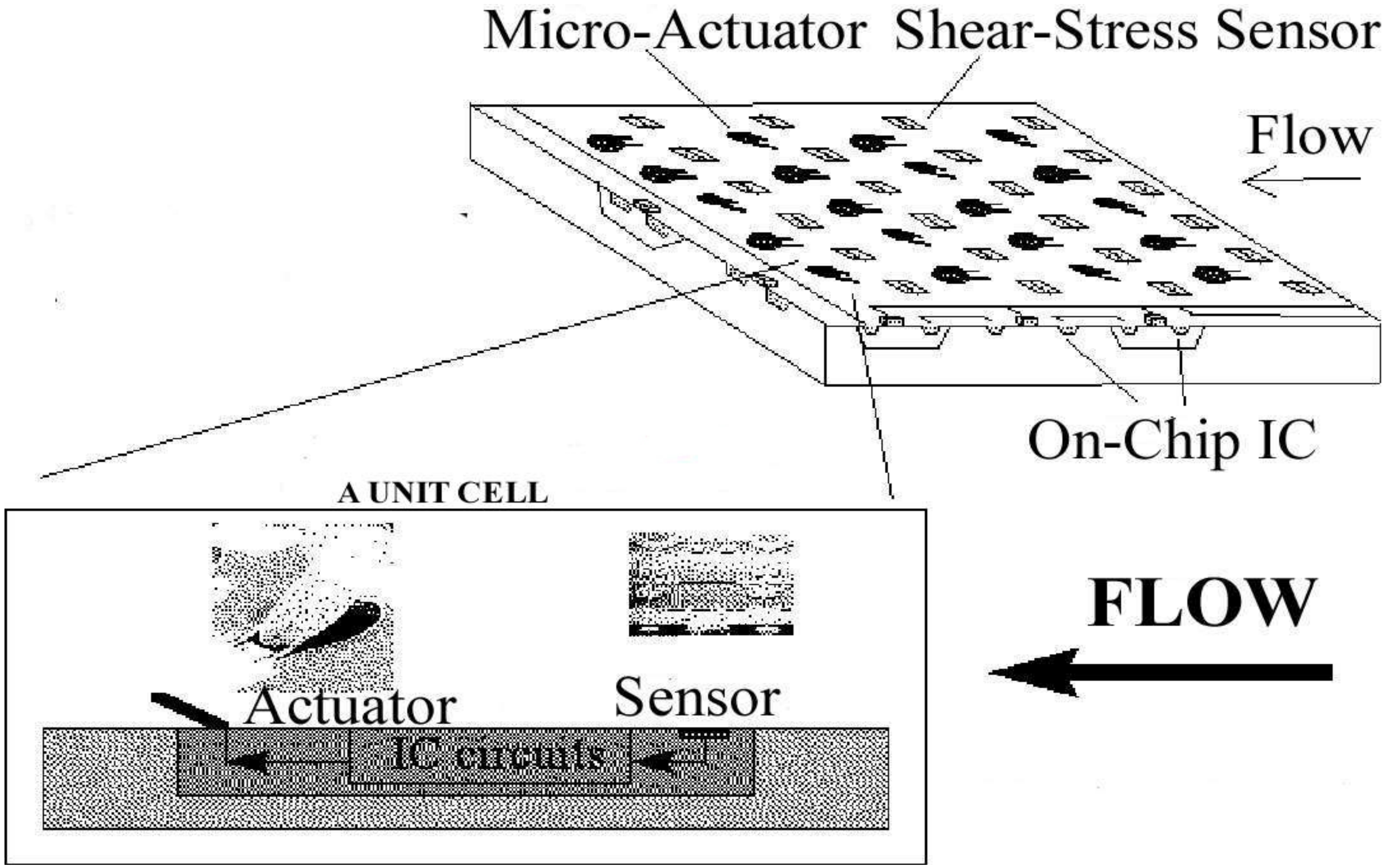


Biochemical networks



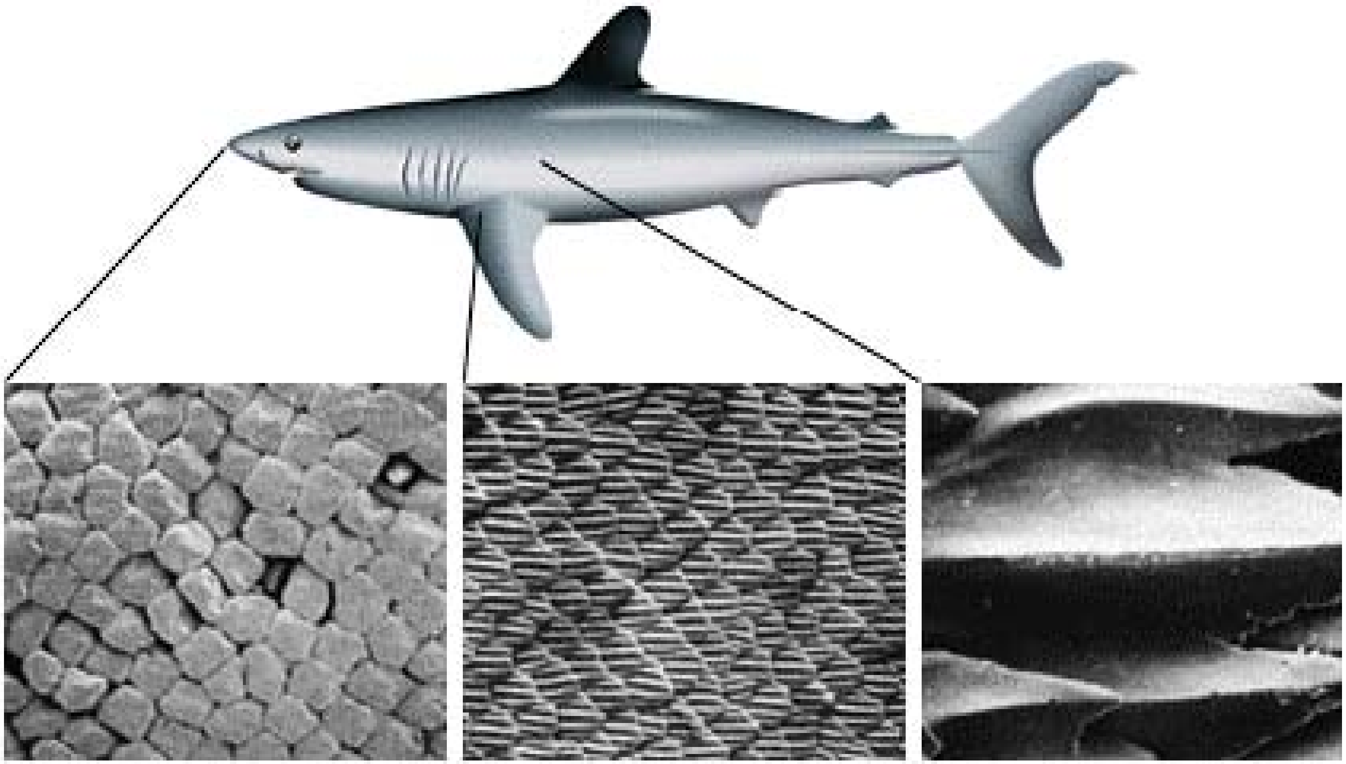
Wind farms

Feedback flow control

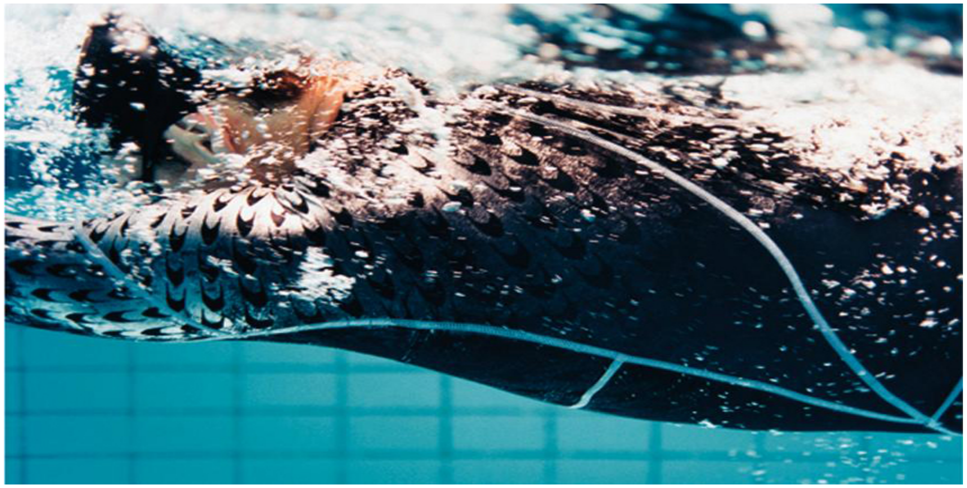


- CHALLENGES
  - ⋆ **control-oriented modeling of turbulent flows**
  - ⋆ **design of estimators for turbulent flows**
  - ⋆ **design of spatially localized distributed controllers**
  - ⋆ **design of controllers of low dynamical order**

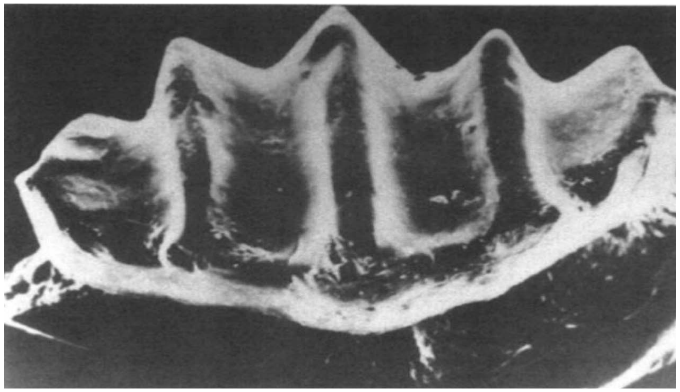
Flow control in nature . . .

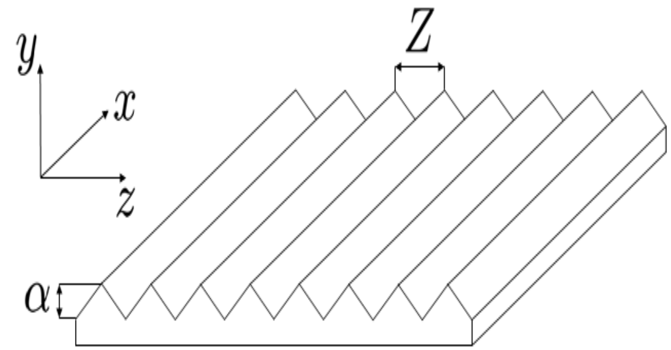


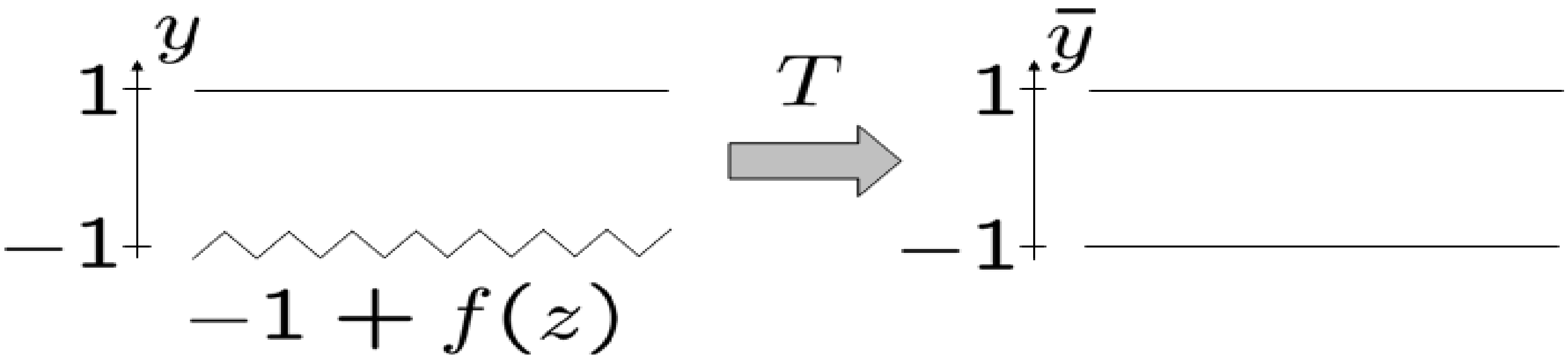
. . . and in swimming competitions



Riblets

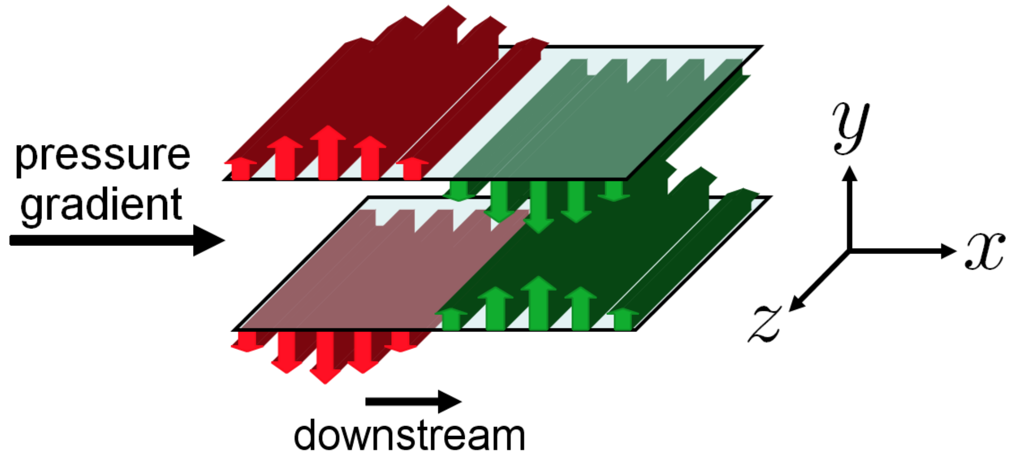






PDEs with spatially periodic coefficients

Blowing and suction along the walls



NORMAL VELOCITY: $V(y = \pm 1) \;=\; \mp\, \alpha \cos\left(\omega_x (x \,-\, ct)\right)$

- TRAVELING WAVE PARAMETERS:

spatial frequency: ω_x

speed: c $\quad \begin{cases} c > 0 & \text{downstream} \\ c < 0 & \text{upstream} \end{cases}$

amplitude: α

- INVESTIGATE THE EFFECTS OF c, ω_x, α ON:

⋆ **cost of control**

⋆ **onset of turbulence**