EE 8235: Modeling, Dynamics, and Control of Distributed Systems

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Lecture Slides; Fall 2011

Lecture 1: Overview of topics; Course mechanics

- Modeling, Dynamics, and Control of Distributed Systems
  - ⋆ Course description
  - ⋆ Overview of topics
  - ⋆ Prerequisites and requirements
  - ⋆ References and software
  - ⋆ Online resources
- All class info
  - ⋆ Course web page
    www.umn.edu/∼mihailo//courses/f11/ee8235.html

Lectures 2 & 3: Examples of distributed systems

- Simple PDEs
  - ⋆ Diffusion equation
  - ⋆ Linear transport equation
  - ⋆ Wave equation
  - ⋆ Evolution of population equation
- Not-so-simple PDEs
  - ⋆ Reaction-diffusion equation
  - ⋆ Swift-Hohenberg equation
  - ⋆ Navier-Stokes equations

- Networks of dynamic systems
  - ⋆ Coordinated/cooperative control
  - ⋆ Leader selection in dynamic networks
  - ⋆ Micro-cantilever arrays
  - ⋆ Biochemical networks
  - ⋆ Wind farms
- Distributed control
  - ⋆ Feedback-based
  - ⋆ Sensor-free

Diffusion equation

$$\frac{\partial \phi(x,t)}{\partial t} = \frac{\partial^2 \phi(x,t)}{\partial x^2} + u(x,t) \;\Leftrightarrow\; \phi_t(x,t) = \phi_{xx}(x,t) + u(x,t)$$

$\phi(x,t)$ – temperature at position x and time t

$u(x,t)$ – heat addition along the bar

- Need to specify initial and boundary conditions

  ⋆ One IC: $\phi(x,0) = \phi_0(x)$

  ⋆ Two BCs:

  Homogeneous Dirichlet: $\phi(\pm 1, t) = 0$

  Homogeneous Neumann: $\phi_x(\pm 1, t) = 0$

  Homogeneous Robin:
$$\begin{aligned} a\,\phi(-1,t) + b\,\phi_x(-1,t) &= 0 \\ c\,\phi(+1,t) + d\,\phi_x(+1,t) &= 0 \end{aligned}$$

- In higher spatial dimensions

$$\phi_t(x,t) \;=\; \Delta\phi(x,t) \;+\; u(x,t)$$

$x \;=\; \left[\begin{array}{ccc} x_1 & \cdots & x_n \end{array}\right]^T$ – vector of spatial coordinates

$\Delta \;=\; \dfrac{\partial^2}{\partial x_1^2} \;+\; \cdots \;+\; \dfrac{\partial^2}{\partial x_n^2}$ – Laplacian

- Boundary actuation in 1D

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) \;+\; d(x,t) \\ \phi(x,0) &= \phi_0(x) \\ \phi(-1,t) &= u(t), \;\; \phi(+1,t) \;=\; 0 \end{aligned}$$

A finite dimensional example

- Mass-spring system

$$m\,\ddot{\phi}(t) \;+\; k\,\phi(t) \;=\; u(t)$$

$\phi(t)$ – position of a mass at time t

$u(t)$ – force acting on a mass

- A state-space representation

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} \;=\; \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \;+\; \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

$$\phi(t) \;=\; \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}$$

$\psi_1(t) \;=\; \phi(t)$ – position at time t

$\psi_2(t) \;=\; \dot{\phi}(t)$ – velocity at time t

State-space (evolution) representation

$$\begin{aligned} \dot{\psi}(t) &= A\,\psi(t) \;+\; B\,u(t) \\ \phi(t) &= C\,\psi(t) \end{aligned}$$

- Finite dimensional state space: $\psi(t) \in \mathbb{R}^n$
- Variations of constants formula

$$\psi(t) \;=\; \mathrm{e}^{At}\psi(0) \;+\; \int_0^t \mathrm{e}^{A(t\,-\,\tau)}\,B\,u(\tau)\,\mathrm{d}\tau$$

- Can we do something similar for infinite dimensional systems?

Linear transport equation

$$\begin{aligned}\phi_t(x,t) &= -a\,\phi_x(x,t)\\ \phi(x,0) &= f(x), \quad x \in \mathbb{R}\end{aligned}$$

Spatial Fourier transform

$$\hat{\phi}(\kappa,t) = \int_{-\infty}^{\infty} \phi(x,t)\,\mathrm{e}^{-\mathrm{j}\kappa\,x}\,\mathrm{d}x$$

yields

$$\left.\begin{aligned}\dot{\hat{\phi}}(\kappa,t) &= -\,(a\,\mathrm{j}\kappa)\,\hat{\phi}(\kappa,t)\\ \hat{\phi}(\kappa,0) &= \hat{f}(\kappa), \quad \kappa \in \mathbb{R}\end{aligned}\right\} \;\Rightarrow\; \hat{\phi}(\kappa,t) = \mathrm{e}^{-a\,\mathrm{j}\kappa\,t}\,\hat{f}(\kappa)$$

- Back to physical space

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(\kappa,t)\,\mathrm{e}^{\mathrm{j}\kappa\,x}\,\mathrm{d}\kappa = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\kappa)\,\mathrm{e}^{\mathrm{j}\kappa(x\,-\,at)}\,\mathrm{d}\kappa = f(x - at)$$

Solution doesn't appear to be of the form: $\text{"}\mathrm{e}^{-a\,\partial_x}\text{"} \times f(x)$

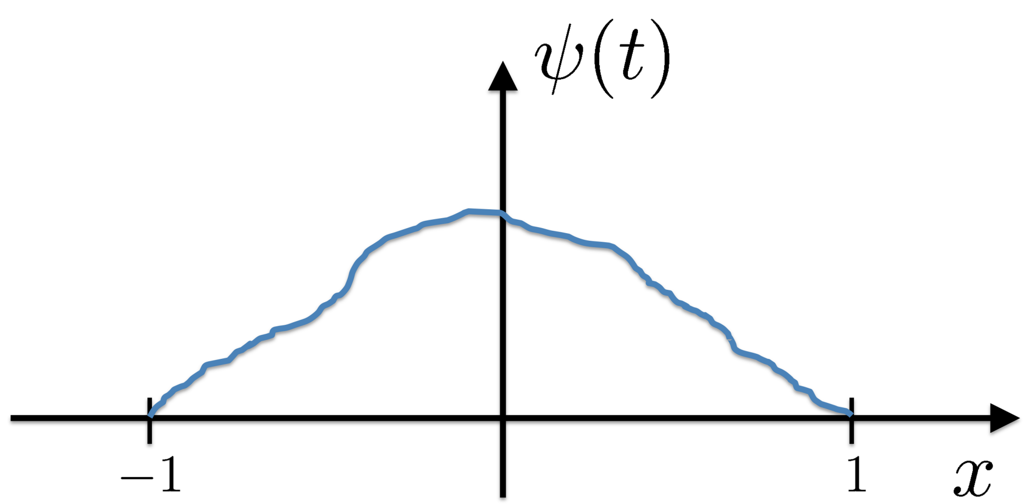
Diffusion equation

$$
\begin{aligned}
\phi_t(x,t) &= \phi_{xx}(x,t) + u(x,t) \\
\phi(x,0) &= \phi_0(x) \\
\phi(\pm 1,t) &= 0
\end{aligned}
$$

Define $\psi(t) = \phi(\cdot,t)$ and write an abstract evolution equation:

$$
\begin{aligned}
\dot{\psi}(t) &= \mathcal{A}\,\psi(t) + u(t) \\
\phi(t) &= \psi(t)
\end{aligned}
$$

- Infinite dimensional state-space: $\psi(t) \in \mathbb{H}$



- A candidate for state-space

square-integrable functions: $\mathbb{H} \;=\; L_2\,[-1,\,1] \;=\; \left\{f,\;\; \int_{-1}^{1} f^*(x)\,f(x)\,\mathrm{d}x \;<\; \infty\right\}$

- $\mathcal{A} \;=\; \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} \;+$ boundary conditions (contained in the domain of $\mathcal{A}$)

$$\mathcal{D}(\mathcal{A}) \;=\; \left\{f \,\in\, L_2\,[-1,\,1],\; \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \in\, L_2\,[-1,\,1],\;\, f(\pm 1) \;=\; 0\right\}$$

Wave equation

$$\begin{aligned} \phi_{tt}(x,t) &= \phi_{xx}(x,t) \;+\; u(x,t) \\ \phi(x,0) &= \phi_{10}(x), \quad \phi_t(x,0) \;=\; \phi_{20}(x), \\ \phi(\pm 1,t) &= 0 \end{aligned}$$

Define $\psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} = \begin{bmatrix} \phi(\cdot,t) \\ \phi_t(\cdot,t) \end{bmatrix}$ and write an abstract evolution equation:

$$\begin{aligned} \begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ \mathrm{d}^2/\mathrm{d}x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u(t) \\ \phi(t) &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \end{aligned}$$

- Energy of a wave: $\left\{ \begin{aligned} E(t) &= \frac{1}{2}\int_{-1}^{1} \left(\phi_x^2(x,t) \;+\; \phi_t^2(x,t)\right) \mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^{1} \left(\psi_{1x}^2(x,t) \;+\; \psi_2^2(x,t)\right) \mathrm{d}x \end{aligned} \right.$

- Selection of state-space: more subtle than for diffusion equation!

Evolution of population equation

$$
\begin{aligned}
\phi_t(x,t) &= -\phi_x(x,t) \; - \; \mu(x,t)\,\phi(x,t) \\
\phi(x,0) &= \phi_0(x) \;\; x \, \geq \, 0, \\
\phi(0,t) &= u(t), \qquad t \, \geq \, 0
\end{aligned}
$$

$\phi(x,t)$ – number of people of age x at time t

$\mu(x,t)$ – mortality function

$\phi_0(x)$ – initial age distribution

$u(t)$ – number of people born at time t

- Control problem: design u to achieve desired age profile $\phi_d(x)$ at time T

Reaction-diffusion equations

$$\boldsymbol{\phi}_t(x,t) \;=\; D\,\Delta\boldsymbol{\phi}(x,t) \;+\; \mathbf{f}(\boldsymbol{\phi}(x,t))$$

ϕ – vector-valued field of interest

$\mathbf{f}(\phi)$ – nonlinear reaction term

Δ – Laplacian

D – matrix of positive diffusion constants

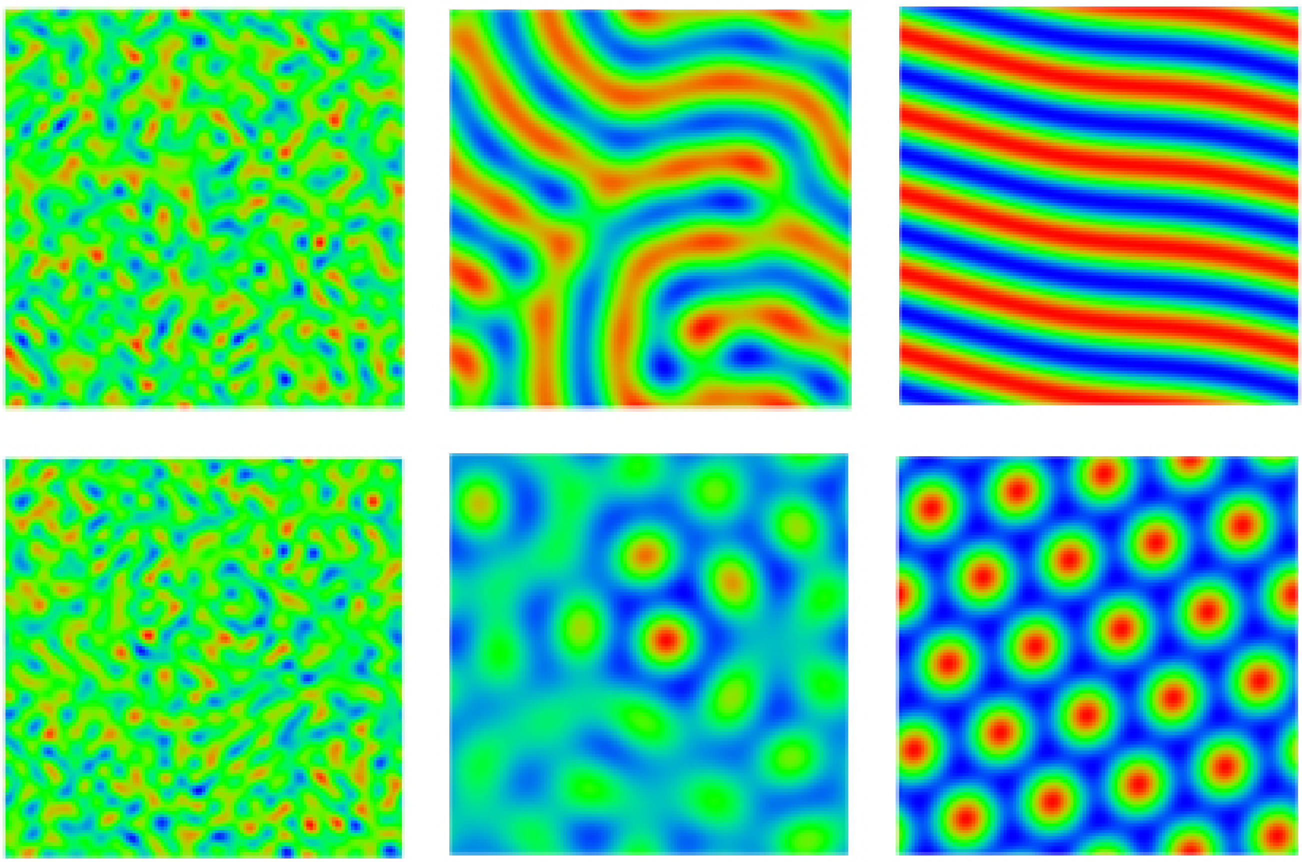
- MAPK CASCADES: responsible for cell proliferation and growth

$$\begin{aligned}
\phi_{1t} &= 0.001\,\phi_{1xx} \;-\; \frac{\phi_1}{1\,+\,\phi_1} \;+\; \frac{0.4}{1\,+\,\phi_3} \\
\phi_{2t} &= 0.001\,\phi_{2xx} \;-\; \frac{\phi_2}{1\,+\,\phi_2} \;+\; 0.4\phi_1 \\
\phi_{3t} &= 0.001\,\phi_{3xx} \;-\; \frac{\phi_3}{1\,+\,\phi_3} \;+\; 0.4\phi_2
\end{aligned}$$

Swift-Hohenberg equation

$$\phi_t = \epsilon\,\phi - (\Delta + 1)^2\,\phi + c\,\phi^2 - \phi^3$$

Nonlinear: first order in time, fourth order in space



- Web-site of Michael Cross at Caltech contains interactive demonstrations

Navier-Stokes equations

conservation of momentum: $\mathbf{v}_t \;=\; -\,(\mathbf{v}\cdot\nabla)\,\mathbf{v} \;-\; \nabla p \;+\; (1/Re)\,\Delta\mathbf{v} \;+\; \mathbf{d}$

conservation of mass: $0 \;=\; \nabla\cdot\mathbf{v}$

Describe the fluid motion

Nonlinear system of equations for:

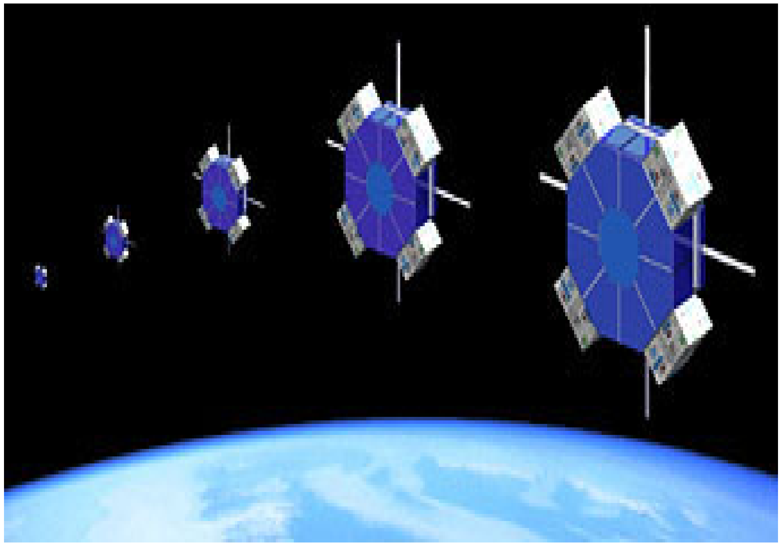
- pressure: $p(x_1, x_2, x_3, t)$
- velocity: $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$

"del" operator: $\nabla \;=\; \dfrac{\partial}{\partial x_1}\mathbf{e}_1 \;+\; \dfrac{\partial}{\partial x_2}\mathbf{e}_2 \;+\; \dfrac{\partial}{\partial x_3}\mathbf{e}_3$

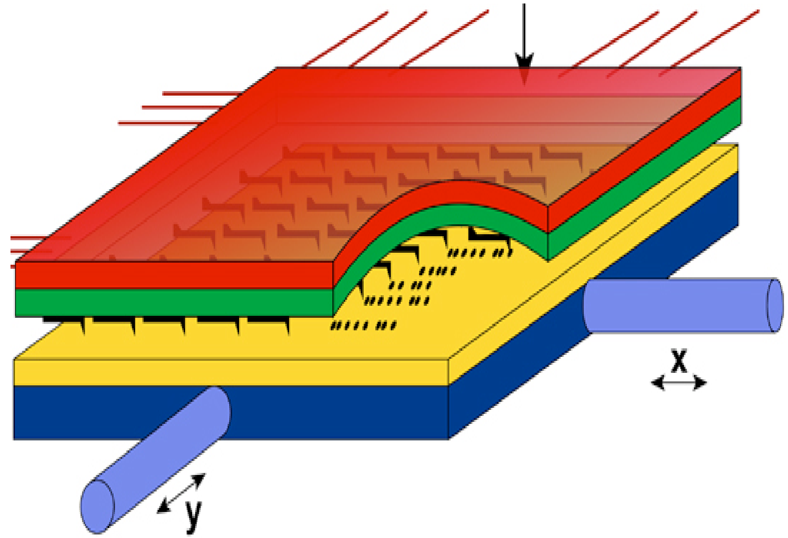
Reynolds number: $Re \;=\; \dfrac{\text{inertial forces}}{\text{viscous forces}}$

Networks of dynamic systems

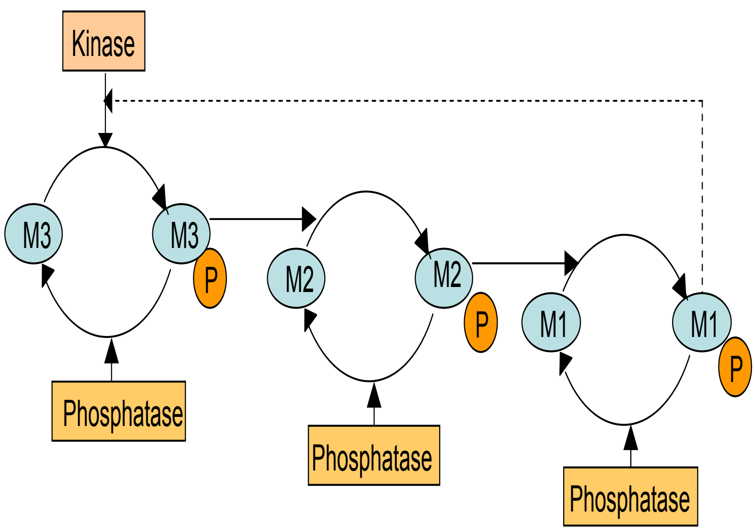
Coordinated control



Micro-cantilever arrays

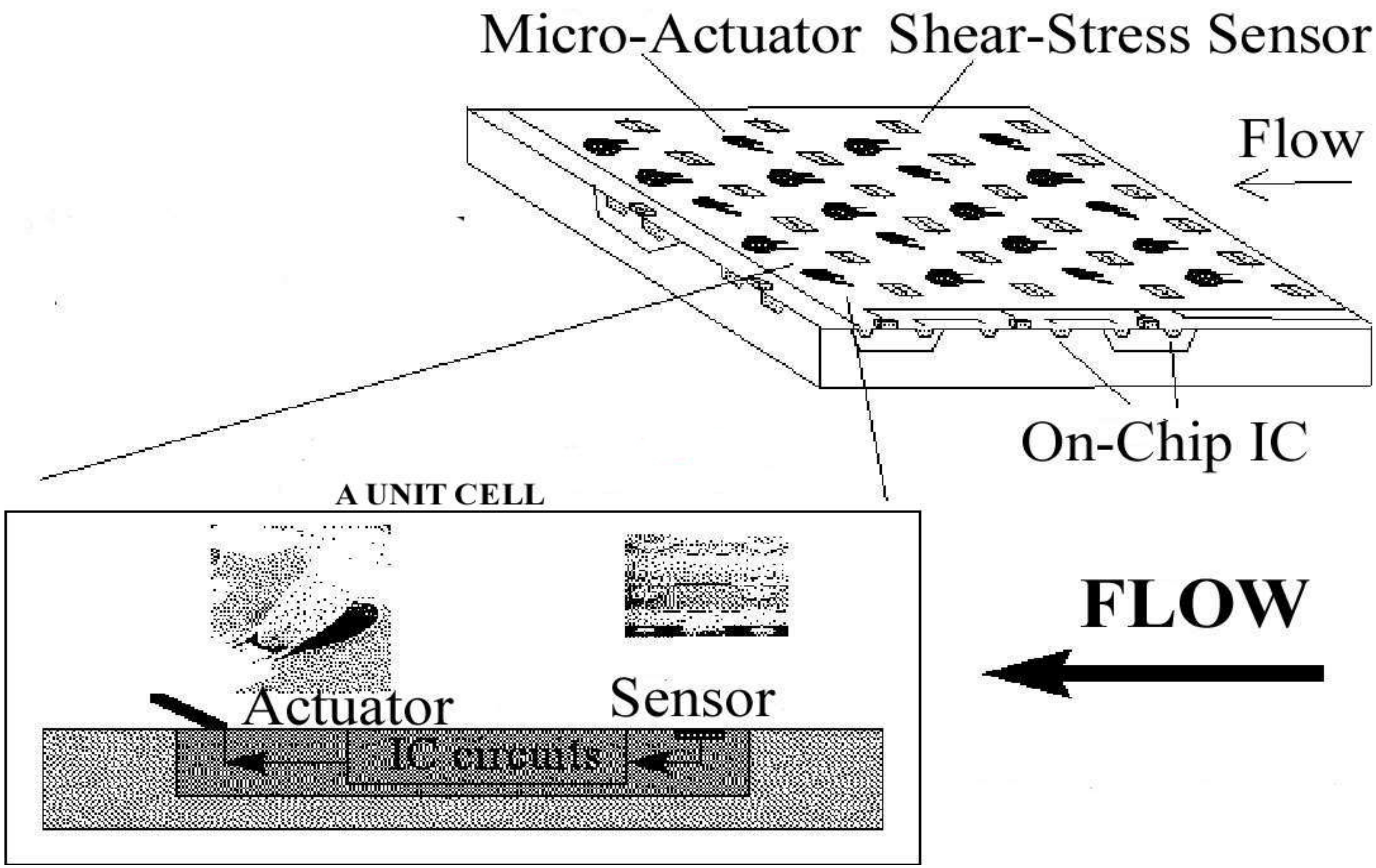


Biochemical networks



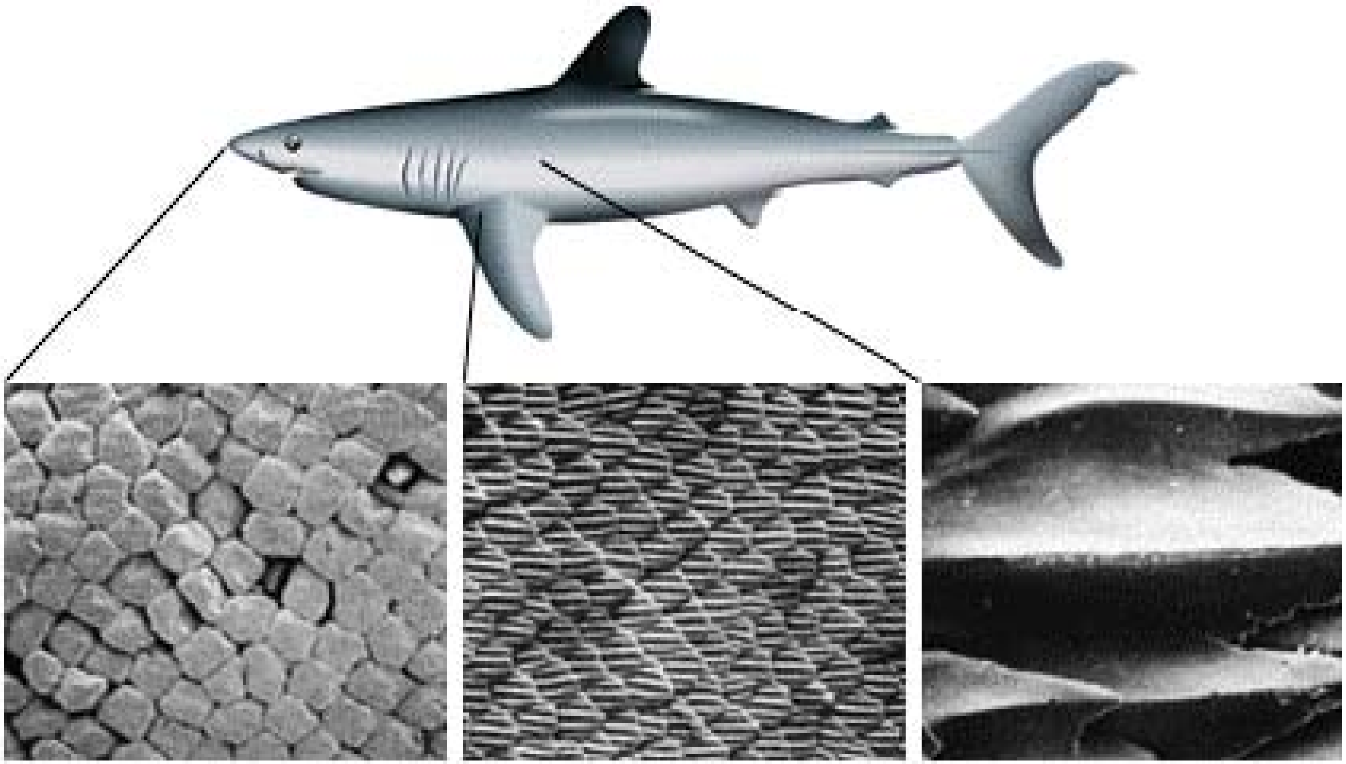
Wind farms

Feedback flow control

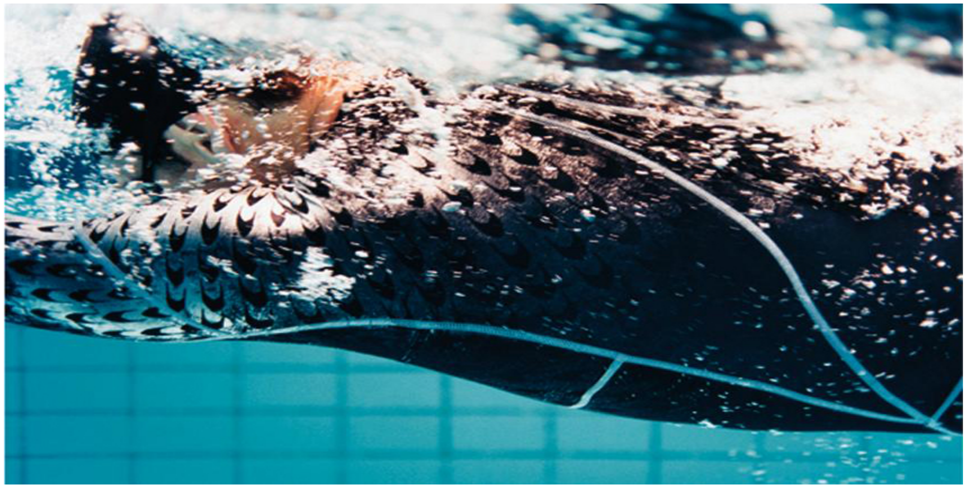


- CHALLENGES
  - ⋆ **control-oriented modeling of turbulent flows**
  - ⋆ **design of estimators for turbulent flows**
  - ⋆ **design of spatially localized distributed controllers**
  - ⋆ **design of controllers of low dynamical order**

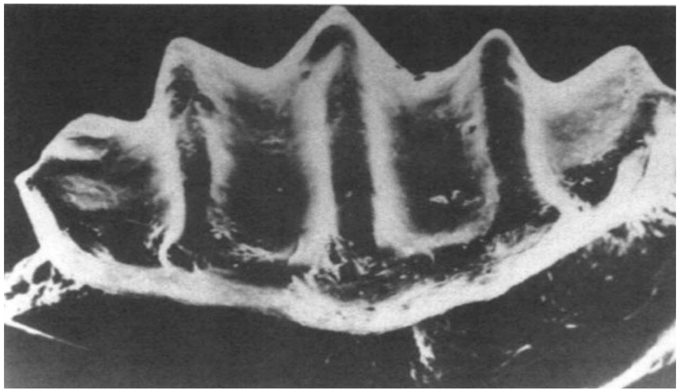
Flow control in nature . . .

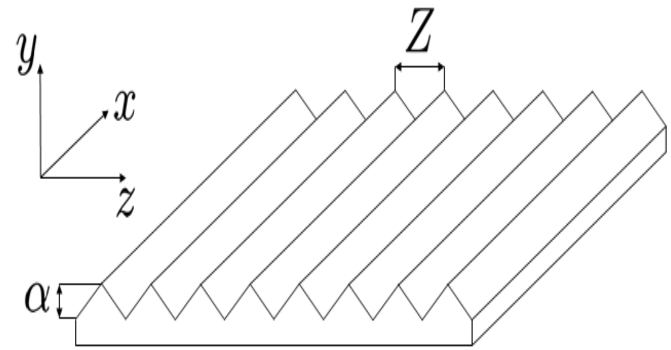


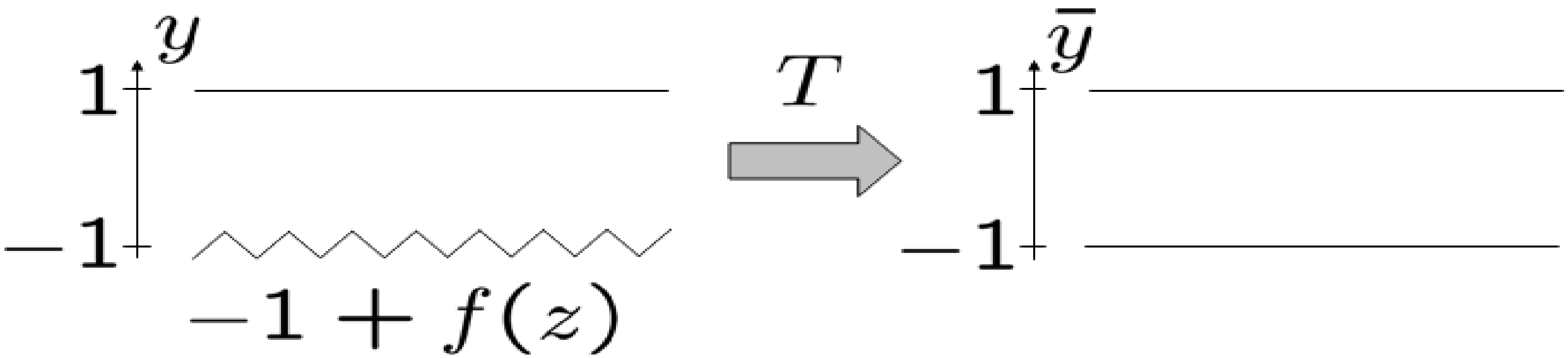
. . . and in swimming competitions



Riblets

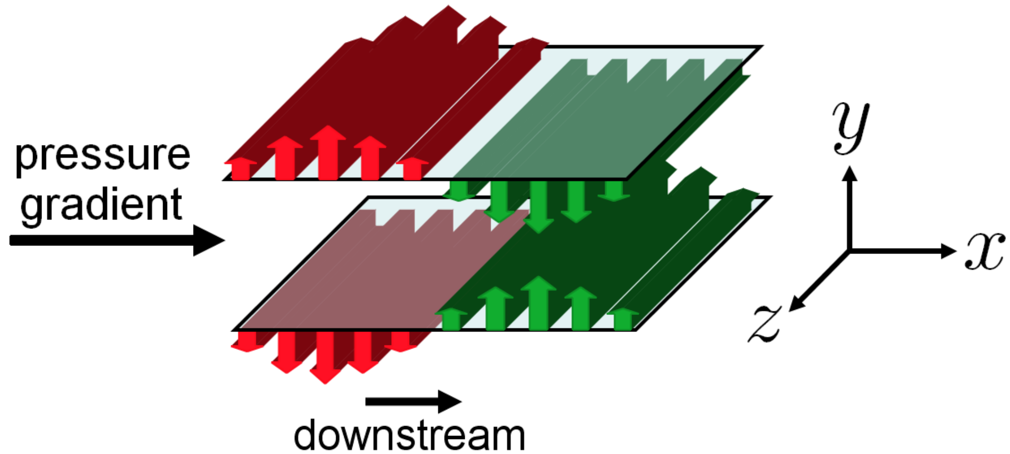






PDEs with spatially periodic coefficients

Blowing and suction along the walls



NORMAL VELOCITY: $V(y = \pm 1) \; = \; \mp \alpha \cos\left(\omega_x(x - ct)\right)$

- TRAVELING WAVE PARAMETERS:

spatial frequency: ω_x

speed: c $\quad \begin{cases} c > 0 & \text{downstream} \\ c < 0 & \text{upstream} \end{cases}$

amplitude: α

- INVESTIGATE THE EFFECTS OF c, ω_x, α ON:

  ⋆ **cost of control**

  ⋆ **onset of turbulence**

Lectures 4 & 5: Solutions to simple infinite dimensional systems

- Notion of a Hilbert space
  - ⋆ Complete linear vector space with an inner product

- Examples of solutions to infinite dimensional systems
  - ⋆ Infinite number of decoupled scalar states
  - ⋆ Continuum of decoupled states
  - ⋆ 1D heat equation
  - ⋆ 1D wave equation

- Informal discussion
  - ⋆ Serves as a motivation for formal developments (later in the course)

Hilbert space

- Hilbert space $\mathbb{H}$: a linear vector space
  - ⋆ complete (i.e., Cauchy sequences in $\mathbb{H}$ converge to an element in $\mathbb{H}$)
  - ⋆ has an inner product
- Inner product $\langle\cdot,\cdot\rangle$: $\mathbb{H} \times \mathbb{H} \to \mathbb{C}$
  - ⋆ $\langle u, v\rangle \;=\; \overline{\langle v, u\rangle}$
  - ⋆ $\langle u, v + w\rangle \;=\; \langle u, v\rangle \,+\, \langle u, w\rangle$
  - ⋆ $\langle u, \alpha\, v\rangle \;=\; \alpha\,\langle u, v\rangle; \quad \langle \alpha\, u, v\rangle \;=\; \overline{\alpha}\,\langle u, v\rangle$
- $\langle\cdot,\cdot\rangle$: induces a norm on $\mathbb{H}$: for $v \in \mathbb{H}$, $\|v\|^2 = \langle v, v\rangle$
  - ⋆ $\|v\| \;\geq\; 0$, for all $v \in \mathbb{H}$
  - ⋆ $\|v\| \;=\; 0 \quad \Leftrightarrow \quad v \;=\; 0$
  - ⋆ $\|\alpha\, v\| \;=\; |\alpha|\,\|v\|$
  - ⋆ $\|u \,+\, v\| \;\leq\; \|u\| \,+\, \|v\|$

Examples of Hilbert spaces

- $\mathbb{R}^n$, $\mathbb{C}^n$

- $\ell_2\,(\mathbb{Z})$, $\ell_2\,(\mathbb{N})$, $\ell_2\,(\mathbb{N}_0)$

$$\ell_2\,(\mathbb{Z}) \;=\; \left\{ \{f_n\}_{n\,\in\,\mathbb{Z}}, \;\; \sum_{n\,=\,-\infty}^{\infty} f_n^*\, f_n \;<\; \infty \right\}$$

- $L_2\,(-\infty, \infty)$, $L_2\,(0, \infty)$, $L_2\,[a, b]$

$$L_2(-\infty, \infty) \;=\; \left\{ f, \;\; \int_{-\infty}^{\infty} f^*(x)\, f(x)\, \mathrm{d}x \;<\; \infty \right\}$$

- The geometries of ℓ_2 and L_2 are similar to the geometry of $\mathbb{C}^n$

$\mathbb{C}^n$ vs. $L_2(-\infty, \infty)$

	$\mathbb{C}^n$	$L_2(-\infty, \infty)$
addition	$w = u + v$ $\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$	$w = u + v$ $\begin{bmatrix} w_1(x) \\ \vdots \\ w_n(x) \end{bmatrix} = \begin{bmatrix} u_1(x) \\ \vdots \\ u_n(x) \end{bmatrix} + \begin{bmatrix} v_1(x) \\ \vdots \\ v_n(x) \end{bmatrix}$
inner product	$\langle u, v \rangle = u^* v = \sum_{i=1}^{n} \overline{u}_i v_i$	$\langle u, v \rangle = \int_{-\infty}^{\infty} u^*(x) v(x) \, \mathrm{d}x = \int_{-\infty}^{\infty} \sum_{i=1}^{n} \overline{u}_i(x) v_i(x) \, \mathrm{d}x$
norm	$\|v\|^2 = \langle v, v \rangle = v^* v$	$\|v\|^2 = \langle v, v \rangle = \int_{-\infty}^{\infty} v^*(x) v(x) \, \mathrm{d}x$

Infinite number of decoupled scalar states

$$\dot{\psi}_n(t) \;=\; a_n\,\psi_n(t), \;\; n \,\in\, \mathbb{N}$$

- Abstract evolution equation on $\ell_2\,(\mathbb{N})$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\begin{bmatrix}\psi_1(t)\\ \psi_2(t)\\ \vdots\end{bmatrix} \;=\; \begin{bmatrix}a_1 & & \\ & a_2 & \\ & & \ddots\end{bmatrix}\begin{bmatrix}\psi_1(t)\\ \psi_2(t)\\ \vdots\end{bmatrix} \;\;\Leftrightarrow\;\; \frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} \;=\; \mathcal{A}\,\psi(t)$$

Solution

$$\begin{bmatrix}\psi_1(t)\\ \psi_2(t)\\ \vdots\end{bmatrix} \;=\; \begin{bmatrix}\mathrm{e}^{a_1\,t} & & \\ & \mathrm{e}^{a_2\,t} & \\ & & \ddots\end{bmatrix}\begin{bmatrix}\psi_1(0)\\ \psi_2(0)\\ \vdots\end{bmatrix} \quad \text{looks like} \quad \psi(t) \;=\; \mathrm{e}^{\mathcal{A}\,t}\,\psi(0)$$

- Later: conditions for well-posedness on $\ell_2\,(\mathbb{N})$

Continuum of decoupled scalar states

$$\dot{\psi}(\kappa, t) \;=\; a(\kappa)\, \psi(\kappa, t), \quad \kappa \in \mathbb{R}$$

- Generator of the dynamics

multiplication operator: $\left[M_a\, \psi(\cdot, t)\right](\kappa) \;=\; a(\kappa)\, \psi(\kappa, t)$

Solution

$\psi(\kappa, t) \;=\; \mathrm{e}^{a(\kappa)\, t}\, \psi(\kappa, 0)$ looks like $\psi(\kappa, t) \;=\; \left[\mathrm{e}^{M_a\, t}\, \psi(\cdot, 0)\right](\kappa)$

- Later: conditions for well-posedness on $L_2\,(-\infty, \infty)$

Diffusion equation on $L_2\,(-\infty, \infty)$

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) \,+\, u(x,t) \\ \phi(x,0) &= f(x), \;\; x \in \mathbb{R} \end{aligned}$$

Spatial Fourier transform:

$$\left.\begin{aligned} \dot{\hat{\phi}}(\kappa,t) &= -\kappa^2\,\hat{\phi}(\kappa,t) \,+\, \hat{u}(\kappa,t) \\ \hat{\phi}(\kappa,0) &= \hat{f}(\kappa), \;\; \kappa \in \mathbb{R} \end{aligned}\right\} \Rightarrow \hat{\phi}(\kappa,t) \,=\, \mathrm{e}^{-\kappa^2\,t}\,\hat{f}(\kappa) + \int_0^t \mathrm{e}^{-\kappa^2\,(t\,-\,\tau)}\,\hat{u}(\kappa,\tau)\,\mathrm{d}\tau$$

- Abstractly

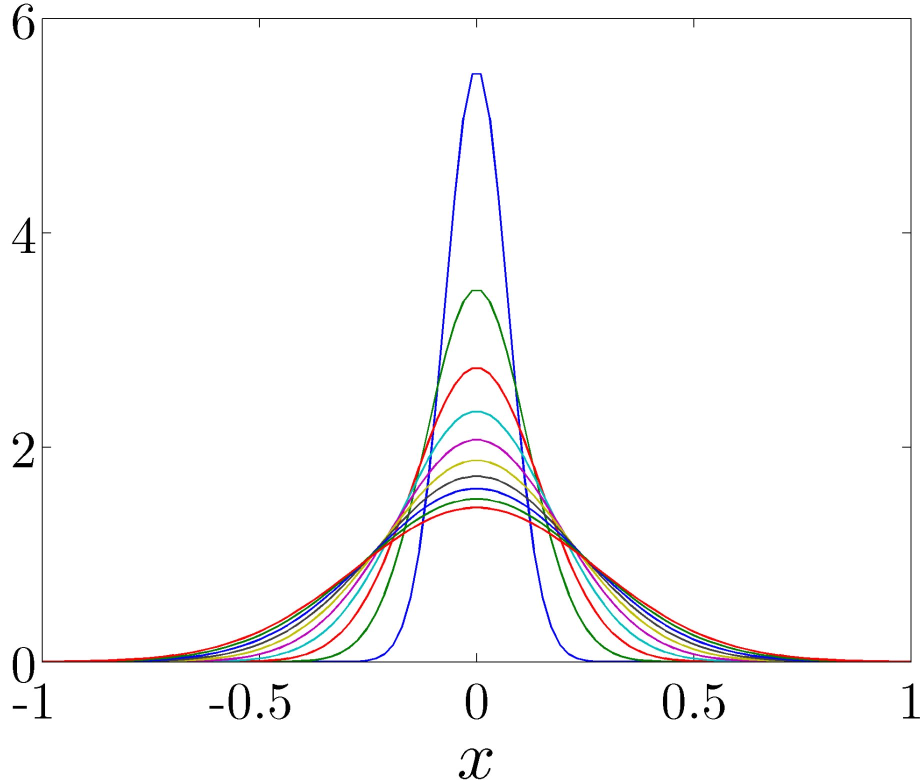
$$\hat{\phi}(\kappa,t) \,=\, \hat{T}(\kappa,t)\,\hat{f}(\kappa) \,+\, \int_0^t \hat{T}(\kappa,t-\tau)\,\hat{u}(\kappa,\tau)\,\mathrm{d}\tau$$

$$\Updownarrow$$

$$\phi(x,t) \,=\, \int_{-\infty}^{\infty} T(x-\xi,t)\,f(\xi)\,\mathrm{d}\xi \,+\, \int_0^t \int_{-\infty}^{\infty} T(x-\xi,t-\tau)\,u(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

- Back to physical space

$$T(x,t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{T}(\kappa,t)\,\mathrm{e}^{\mathrm{j}\kappa\, x}\,\mathrm{d}\kappa \;=\; \frac{1}{2\sqrt{\pi\, t}}\,\mathrm{e}^{-x^2/(4t)}$$



Solution can be represented as:

$$\phi(x,t) \;=\; [\mathcal{T}(t)\,f(\cdot)]\,(x) \;+\; \left[\int_0^t \mathcal{T}(t-\tau)\,u(\cdot,\tau)\,\mathrm{d}\tau\right](x)$$

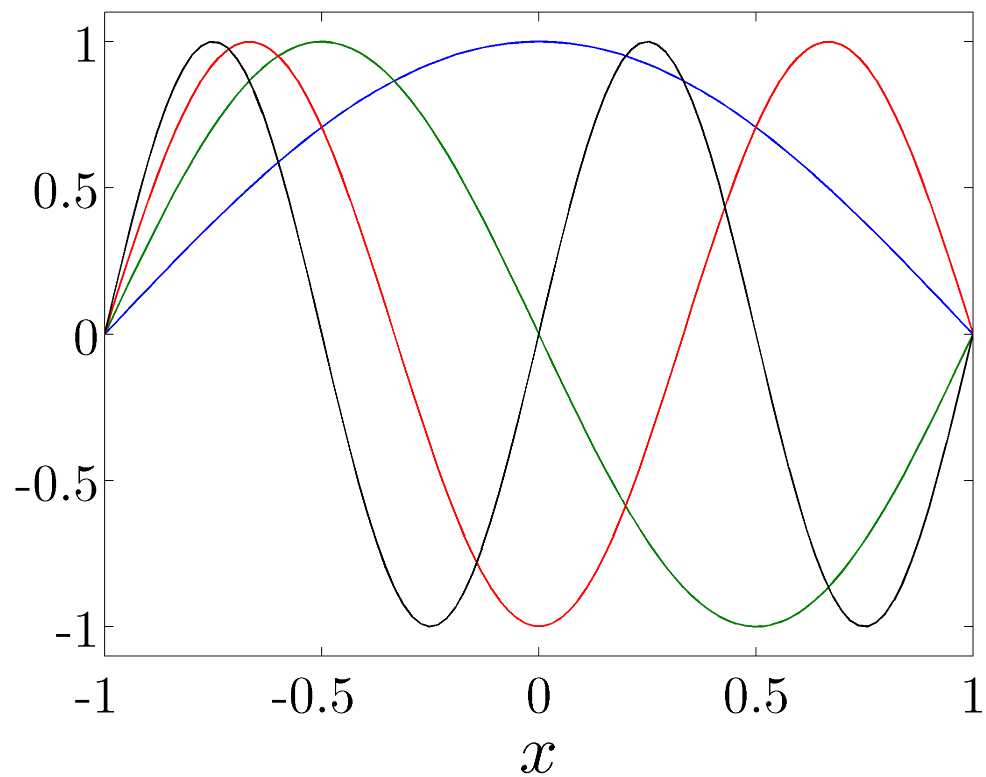
$$[\mathcal{T}(t)\,f(\cdot)]\,(x) \;=\; \int_{-\infty}^{\infty} T(x-\xi,t)\,f(\xi)\,\mathrm{d}\xi$$

Diffusion equation on $L_2\,[-1, 1]$ with Dirichlet BCs

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) + u(x,t) \\ \phi(x,0) &= f(x) \\ \phi(\pm 1,t) &= 0 \end{aligned}$$

- Consider

$$\left\{ v_n(x) = \sin\left(\frac{n\pi}{2}(x+1)\right)\right\}_{n \in \mathbb{N}}$$



- Properties of $\left\{v_n(x) \;=\; \sin\left(\frac{n\pi}{2}\,(x\,+\,1)\right)\right\}_{n\,\in\,\mathbb{N}}$

1. Satisfy BCs
$$v_n(\pm 1) \;=\; 0$$

2. Of unit length and mutually orthogonal (i.e., orthonormal)
$$\langle v_n, v_m \rangle \;=\; \delta_{nm} \;=\; \begin{cases} 1, & n \;=\; m \\ 0, & n \;\neq\; m \end{cases}$$

3. Complete basis of $L_2\,[-1, 1]$
$$\overline{\mathrm{span}\,\{v_n\}_{n\,\in\,\mathbb{N}}} \;=\; L_2\,[-1, 1]$$

4. Eigenfunctions of $\dfrac{\mathrm{d}^2}{\mathrm{d}\,x^2}$ with Dirichlet BCs
$$\frac{\mathrm{d}^2\,v_n(x)}{\mathrm{d}\,x^2} \;=\; \lambda_n\,v_n(x), \;\;\; \lambda_n \;=\; -\left(\frac{n\pi}{2}\right)^2$$

Solution technique

1. Represent the solution as

$$\phi(x,t) \;=\; \sum_{n=1}^{\infty} \alpha_n(t)\, v_n(x)$$

$$\alpha_n(t) \;=\; \langle v_n, \phi \rangle$$

2. Substitute into the PDE and use $v_n''(x) \;=\; \lambda_n\, v_n(x)$

$$\sum_{n=1}^{\infty} \dot{\alpha}_n(t)\, v_n(x) \;=\; \sum_{n=1}^{\infty} \lambda_n\, \alpha_n(t)\, v_n(x) \;+\; u(x,t)$$

3. Take an inner product with v_m

$$\left\langle v_m, \sum_{n=1}^{\infty} \dot{\alpha}_n(t)\, v_n \right\rangle \;=\; \left\langle v_m, \sum_{n=1}^{\infty} \lambda_n\, \alpha_n(t)\, v_n \right\rangle \;+\; \langle v_m, u \rangle$$

4. Use orthonormality of $\{v_n(x)\}_{n \in \mathbb{N}}$

$$\dot{\alpha}_m(t) \;=\; \lambda_m\, \alpha_m(t) \;+\; u_m(t)$$

$$\Downarrow$$

$$\alpha_m(t) \;=\; \mathrm{e}^{\lambda_m t}\, \underbrace{\alpha_m(0)}_{\langle v_m, f\rangle} \;+\; \int_0^t \mathrm{e}^{\lambda_m (t-\tau)}\, \underbrace{u_m(\tau)}_{\langle v_m, u\rangle}\, \mathrm{d}\tau$$

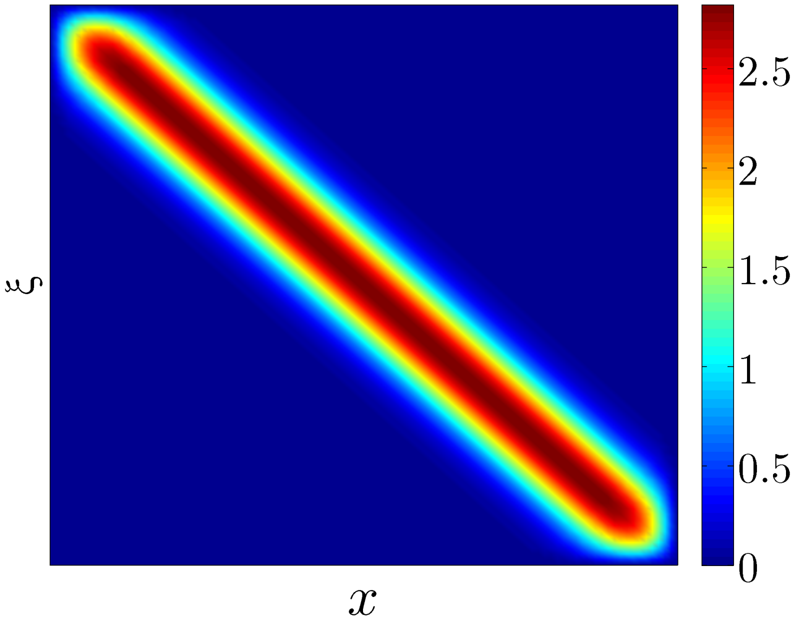
5. Express solution as

$$\begin{aligned}\phi(x,t) &= \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n t}\, v_n(x)\, \langle v_n, f\rangle \;+\; \int_0^t \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n (t-\tau)}\, v_n(x)\, \langle v_n, u(\cdot,\tau)\rangle \,\mathrm{d}\tau \\ &= \int_{-1}^{1} \underbrace{\sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n t}\, v_n(x)\, v_n^*(\xi)}_{T(x,\xi;t)}\, f(\xi)\,\mathrm{d}\xi \;+\; \int_0^t \int_{-1}^{1} \underbrace{\sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n (t-\tau)}\, v_n(x)\, v_n^*(\xi)}_{T(x,\xi;t-\tau)}\, u(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau\end{aligned}$$

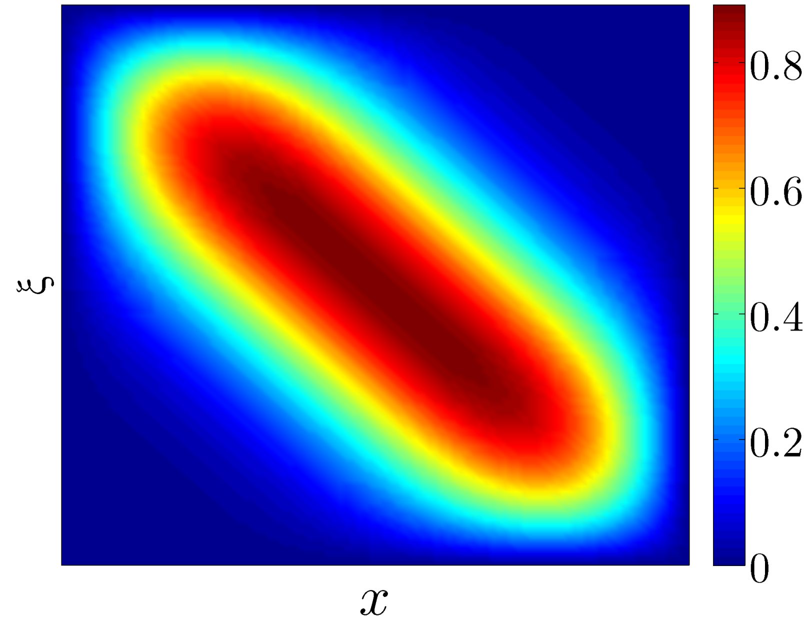
- Green's function for diffusion equation on $L_2\,[-1,1]$ with Dirichlet BCs

$$\begin{aligned}T(x,\xi;t) &= \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n t}\, v_n(x)\, v_n^*(\xi) \\ &= \sum_{n=1}^{\infty} \mathrm{e}^{-\left(\frac{n\pi}{2}\right)^2 t}\, \sin\left(\frac{n\pi}{2}\,(x+1)\right) \sin\left(\frac{n\pi}{2}\,(\xi+1)\right)\end{aligned}$$

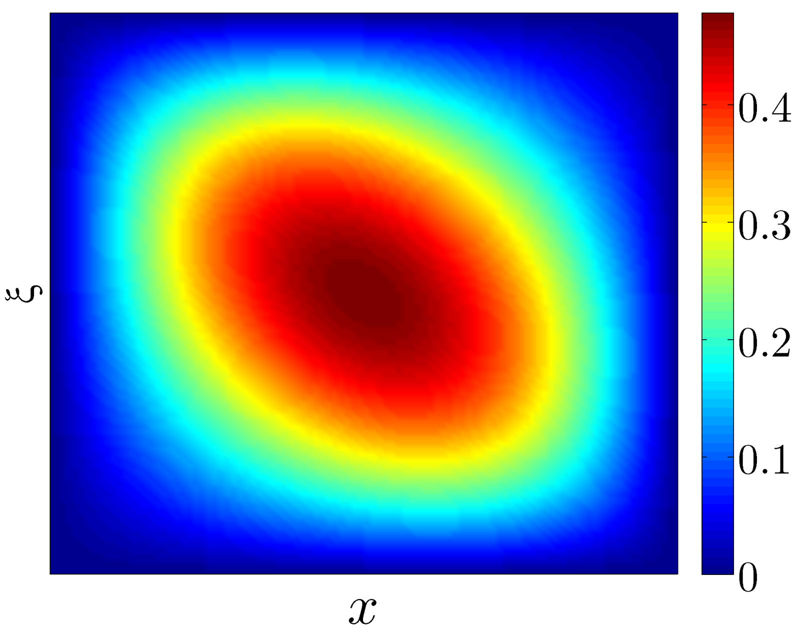
$T(x, \xi; t = 0.01)$:



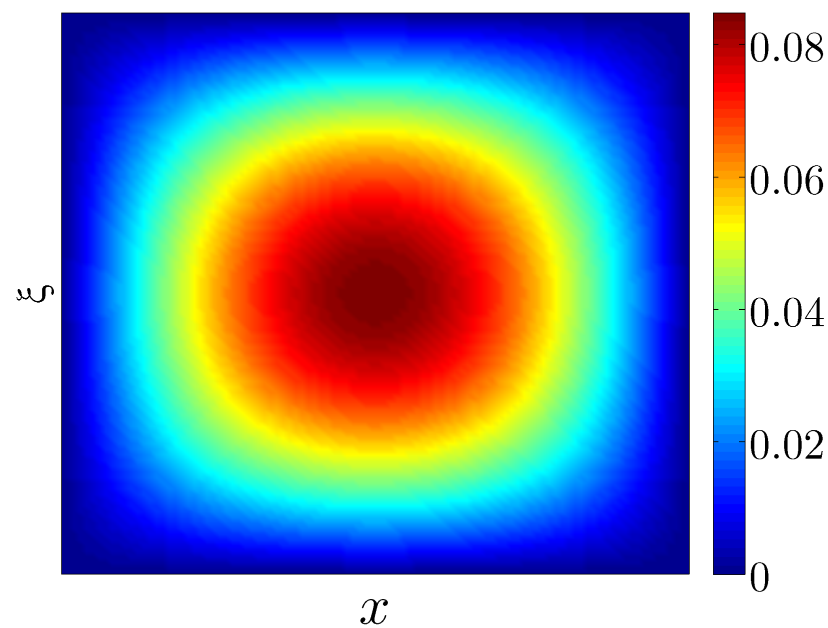
$T(x, \xi; t = 0.1)$:



$T(x, \xi; t = 0.3)$:



$T(x, \xi; t = 1)$:

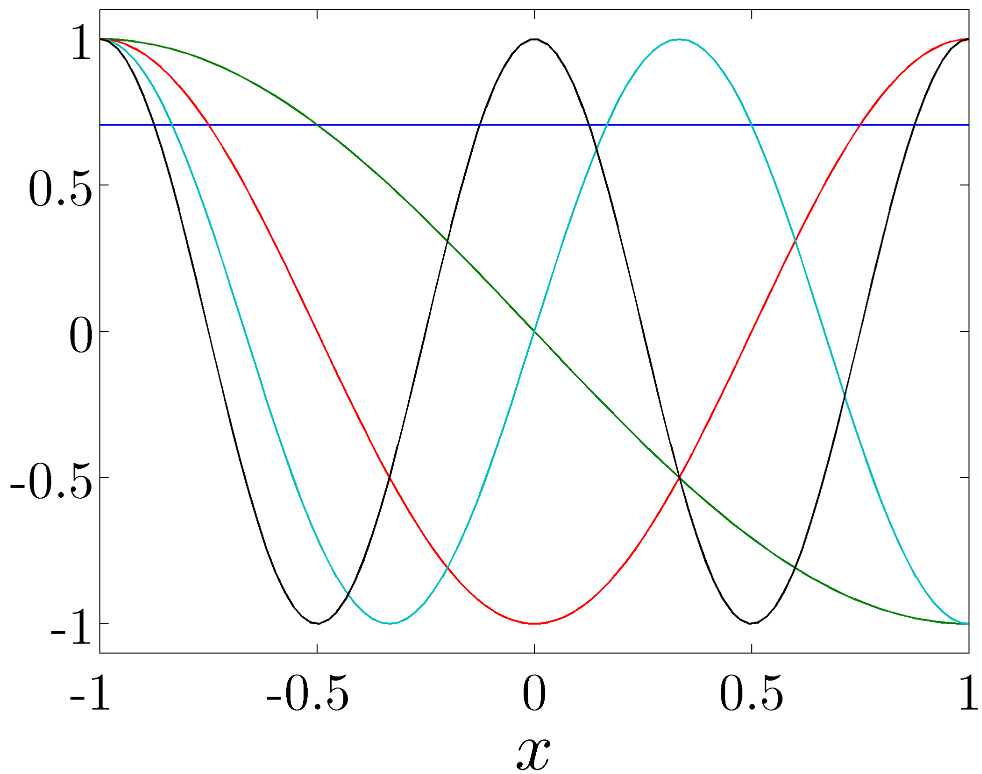


Diffusion equation on $L_2\,[-1,1]$ with Neumann BCs

$$
\begin{aligned}
\phi_t(x,t) &= \phi_{xx}(x,t) + u(x,t) \\
\phi(x,0) &= f(x) \\
\phi_x(\pm 1,t) &= 0
\end{aligned}
$$

- Orthonormal basis

$$
\left\{ v_0(x) = \frac{1}{\sqrt{2}};\; v_n(x) = \cos\left(\frac{n\pi}{2}\,(x+1)\right)\right\}_{n \in \mathbb{N}}
$$



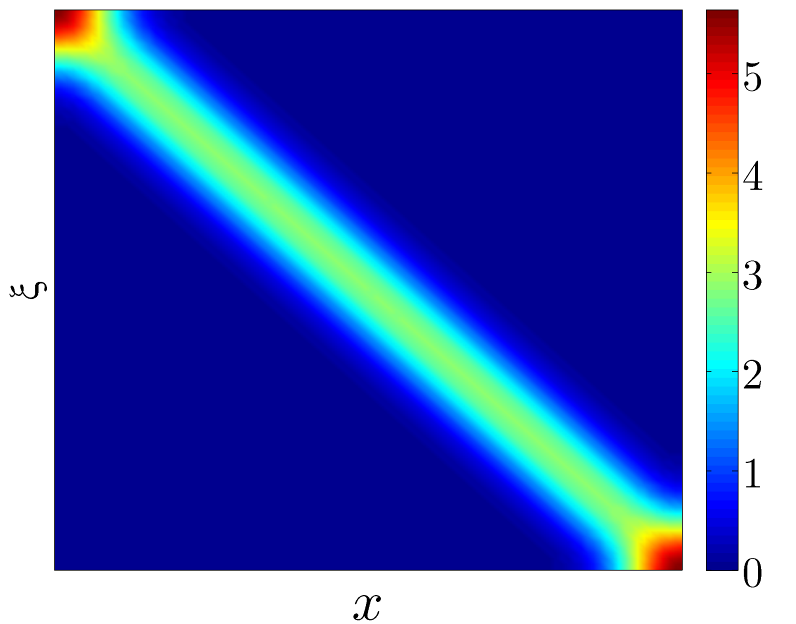
- Eigenfunctions of $\dfrac{\mathrm{d}^2}{\mathrm{d}\,x^2}$ with Neumann BCs

$$\frac{\mathrm{d}^2\, v_n(x)}{\mathrm{d}\,x^2} \;=\; \lambda_n\, v_n(x), \quad \left\{\lambda_0 \;=\; 0;\; \lambda_n \;=\; -\left(\frac{n\pi}{2}\right)^2\right\}_{n\,\in\,\mathbb{N}}$$

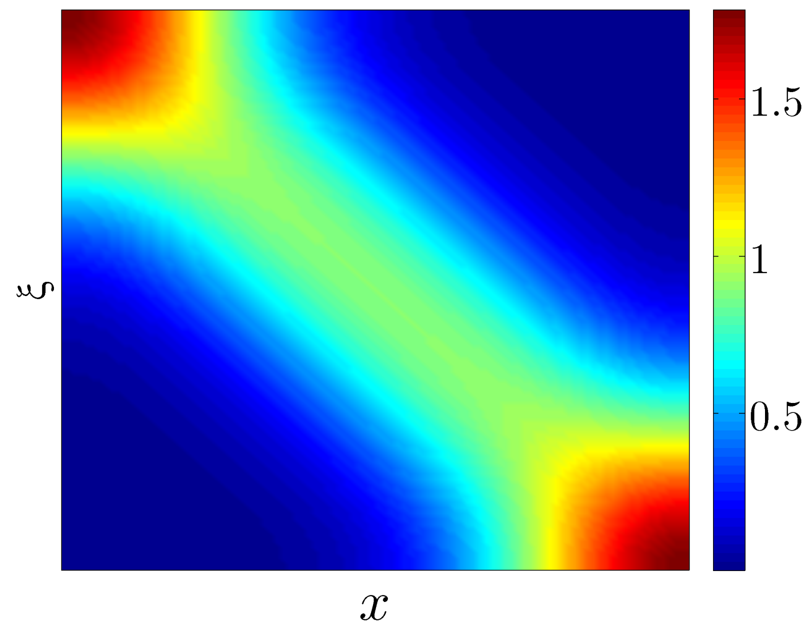
- Green's function

$$\begin{aligned} T(x,\xi;t) \;&=\; \sum_{n\,=\,0}^{\infty} \mathrm{e}^{\lambda_n\, t}\, v_n(x)\, v_n^*(\xi) \\ &=\; \frac{1}{2} \,+\, \sum_{n\,=\,1}^{\infty} \mathrm{e}^{-\left(\frac{n\pi}{2}\right)^2 t}\, \cos\left(\frac{n\pi}{2}\,(x\,+\,1)\right) \cos\left(\frac{n\pi}{2}\,(\xi\,+\,1)\right) \end{aligned}$$

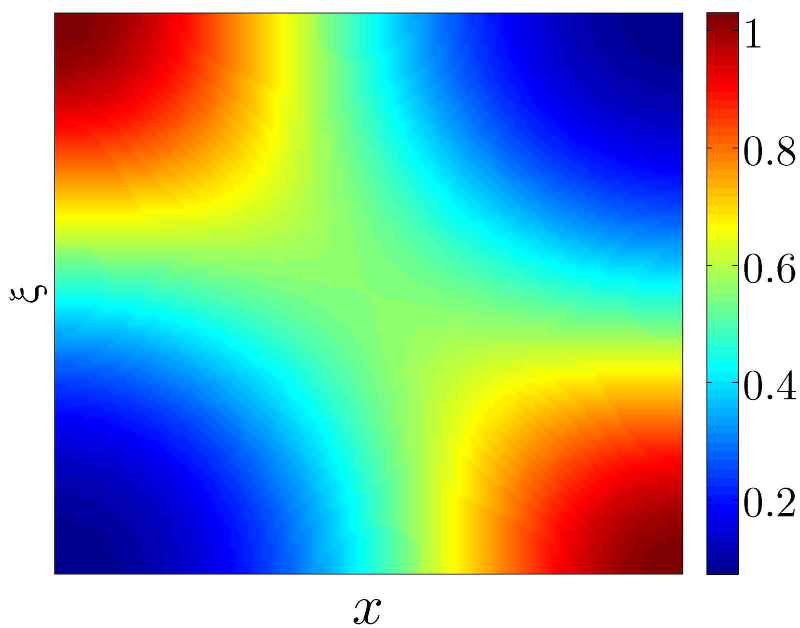
$T(x, \xi; t = 0.01)$:



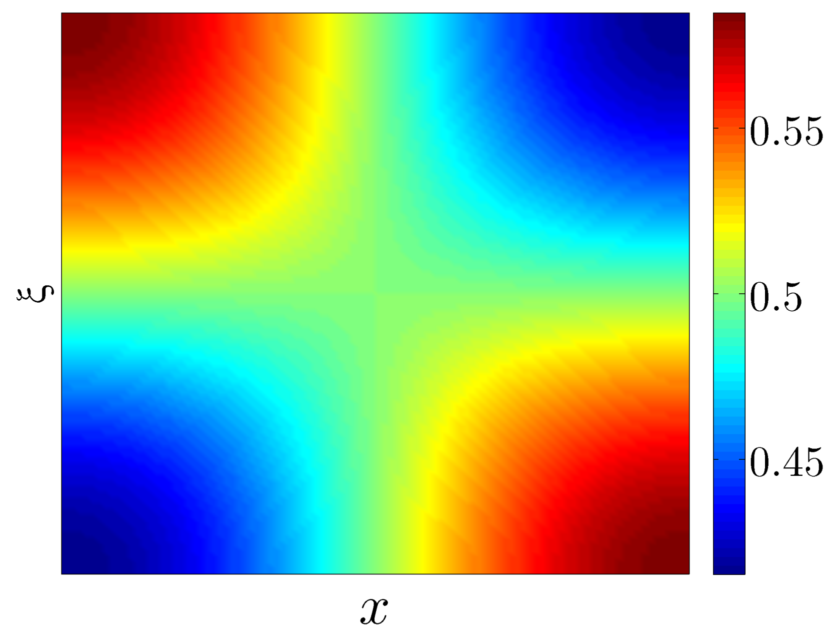
$T(x, \xi; t = 0.1)$:



$T(x, \xi; t = 0.3)$:



$T(x, \xi; t = 1)$:



Finite dimensional analogy

$$\dot{\psi}(t) \;=\; A\,\psi(t)$$

Let A have a full set of linearly independent orthonormal e-vectors

$$A\,v_i \;=\; \lambda_i\,v_i \quad\Leftrightarrow\quad A\underbrace{\left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]}_{V} \;=\; \underbrace{\left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]}_{V}\underbrace{\left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array}\right]}_{\Lambda}$$

- A – diagonalizable by a unitary coordinate transformation

$$A \;=\; \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]\left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array}\right]\left[\begin{array}{c} v_1^* \\ \vdots \\ v_n^* \end{array}\right]$$

$$\mathrm{e}^{A\,t} \;=\; \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]\left[\begin{array}{ccc} \mathrm{e}^{\lambda_1\,t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_n\,t} \end{array}\right]\left[\begin{array}{c} v_1^* \\ \vdots \\ v_n^* \end{array}\right]$$

Dyadic decomposition of matrix A

- Action of A on $u \in \mathbb{C}^n$

$$
\begin{aligned}
A\,u \;&=\; \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^* \\ \vdots \\ v_n^* \end{bmatrix} u \\
&=\; \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1\, v_1^*\, u \\ \vdots \\ \lambda_n\, v_n^*\, u \end{bmatrix} \\
&=\; \lambda_1\, v_1\, v_1^*\, u \;+\; \cdots \;+\; \lambda_n\, v_n\, v_n^*\, u \\
&=\; \sum_{i\,=\,1}^{n} \lambda_i\, v_i\, \langle v_i, u \rangle
\end{aligned}
$$

- Solution to $\quad \dot{\psi}(t) \;=\; A\,\psi(t)$

$$
\psi(t) \;=\; \mathrm{e}^{A\,t}\,\psi(0) \;=\; \sum_{i\,=\,1}^{n} \mathrm{e}^{\lambda_i\,t}\, v_i\, \langle v_i, \psi(0) \rangle
$$

Dyadic decomposition of operator $\mathcal{A}$

- Action of operator $\mathcal{A}$ (with a full set of orthonormal e-functions) on $u \in \mathbb{H}$

$$[\mathcal{A}\, u]\,(x) \;=\; \sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\, \langle v_n, u \rangle$$

- For the heat equation with Dirichlet BCs

$$\left[\frac{\mathrm{d}^2 u}{\mathrm{d}\, x^2}\right](x) \;=\; \sum_{n\,=\,1}^{\infty} -\left(\frac{n\pi}{2}\right)^2 v_n(x)\, \langle v_n, u \rangle$$

- Solution to $\dot{\psi}(t) \;=\; \mathcal{A}\, \psi(t)$

$$[\psi(t)]\,(x) \;=\; [\mathcal{T}(t)\, \psi(0)]\,(x) \;=\; \sum_{n\,=\,1}^{\infty} \mathrm{e}^{-\left(\frac{n\pi}{2}\right)^2 t}\, v_n(x)\, \langle v_n, \psi(0) \rangle$$

A few additional notes

- Orthonormal basis $\{v_n\}_{n \in \mathbb{N}}$

$$\phi(x) = \sum_{n=1}^{\infty} \alpha_n v_n(x) = \sum_{n=1}^{\infty} \langle v_n, \phi \rangle \, v_n(x)$$

$$\psi(x) = \sum_{n=1}^{\infty} \beta_n v_n(x) = \sum_{n=1}^{\infty} \langle v_n, \psi \rangle \, v_n(x)$$

- Properties

1. $\langle \psi, \phi \rangle = \sum_{n=1}^{\infty} \overline{\langle v_n, \psi \rangle} \, \langle v_n, \phi \rangle = \sum_{n=1}^{\infty} \overline{\beta}_n \, \alpha_n$

2. $\|\psi\|^2 = \langle \psi, \psi \rangle = \sum_{n=1}^{\infty} | \langle v_n, \psi \rangle |^2 = \sum_{n=1}^{\infty} |\beta_n|^2$

3. ψ orthogonal to each $v_n \;\Rightarrow\; \psi = 0$

4. Convergence in L_2-sense $\|\psi - \sum_{n=1}^{N} \langle v_n, \psi \rangle \, v_n\| \xrightarrow{N \longrightarrow \infty} 0$

Projection theorem

- $\mathbb{H}$: Hilbert space; V: closed subspace of $\mathbb{H}$
  - ⋆ For each $\psi \in \mathbb{H}$, there is a unique $v_0 \in V$ such that

$$\|\psi - v_0\| = \min_{v \in V} \|\psi - v\|$$

  - ⋆ $v_0 \in V$ minimizing vector $\Leftrightarrow$ $(\psi - v_0) \perp V$

- Consequence: the best approximation of ψ using N orthonormal vectors v_n

$$\psi_N = \sum_{n=1}^{N} \langle v_n, \psi \rangle \, v_n$$

Proof: follows directly from Projection theorem

$$\left\langle v_n, \psi - \sum_{m=1}^{N} \alpha_m v_m \right\rangle = 0, \;\; n = \{1, \ldots, N\} \;\; \Rightarrow \;\; \alpha_m = \langle v_m, \psi \rangle$$

Orthonormality: approximation improved by adding $\langle v_{N+1}, \psi \rangle \, v_{N+1}$

Wave equation on infinite spatial extent

$$
\begin{aligned}
\phi_{tt}(x,t) &= c^2\,\phi_{xx}(x,t) \;+\; u(x,t) \\
\phi(x,0) &= f(x), \quad \phi_t(x,0) \;=\; g(x), \quad x \in \mathbb{R}
\end{aligned}
$$

- Evolution equation

$$
\begin{aligned}
\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ c^2\,\mathrm{d}^2/\mathrm{d}x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \;+\; \begin{bmatrix} 0 \\ I \end{bmatrix} u(t) \\
\phi(t) &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}
\end{aligned}
$$

$$\Big\downarrow \textbf{Fourier transform}$$

$$
\begin{aligned}
\begin{bmatrix} \dot{\hat{\psi}}_1(\kappa,t) \\ \dot{\hat{\psi}}_2(\kappa,t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -c^2\,\kappa^2 & 0 \end{bmatrix} \begin{bmatrix} \hat{\psi}_1(\kappa,t) \\ \hat{\psi}_2(\kappa,t) \end{bmatrix} \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(\kappa,t) \\
\hat{\phi}(\kappa,t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\psi}_1(\kappa,t) \\ \hat{\psi}_2(\kappa,t) \end{bmatrix}
\end{aligned}
$$

D'Alembert's formula

- Solution to the unforced problem

$$
\begin{aligned}
\hat{\phi}(\kappa, t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\psi}_1(\kappa, t) \\ \hat{\psi}_2(\kappa, t) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(c\,\kappa\,t) & \sin(c\,\kappa\,t)/(c\,\kappa) \\ -c\,\kappa\,\sin(c\,\kappa\,t) & \cos(c\,\kappa\,t) \end{bmatrix} \begin{bmatrix} \hat{f}(\kappa) \\ \hat{g}(\kappa) \end{bmatrix} \\
&= \cos(c\,\kappa\,t)\,\hat{f}(\kappa) \;+\; \frac{\sin(c\,\kappa\,t)}{c\,\kappa}\,\hat{g}(\kappa) \\
&= \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}c\,\kappa\,t} + \mathrm{e}^{-\mathrm{j}c\,\kappa\,t}\right)\hat{f}(\kappa) \;+\; t\,\mathrm{sinc}\,(c\,\kappa\,t)\,\hat{g}(\kappa)
\end{aligned}
$$

$\downarrow$ **inverse Fourier transform**

$$
\begin{aligned}
\phi(x, t) &= \frac{1}{2}\left(f\,(x + ct) + f\,(x - ct)\right) \;+\; \frac{1}{2c}\int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{x - \xi}{ct}\right) g(\xi)\,\mathrm{d}\xi \\
&= \frac{1}{2}\left(f\,(x + ct) + f\,(x - ct)\right) \;+\; \frac{1}{2c}\int_{x - ct}^{x + ct} g(\xi)\,\mathrm{d}\xi
\end{aligned}
$$

Lectures 6, 7, 8: { Kernel representation of linear operators; Hilbert space adjoint of a linear operator }

- Kernel representation of an integral operator
  - ⋆ Generalization of matrix/vector multiplication
  - ⋆ Represents action of integral operators and linear dynamical systems

- Adjoint of an operator
  - ⋆ Generalizes notion of complex-conjugate-transpose to operators
  - ⋆ Useful in linear algebra and functional analysis (solutions of linear equations, optimization, . . .)

- Self-adjoint operators
  - ⋆ Can be used to characterize complete orthonormal basis of a Hilbert space

Kernel representation

- Recall: Solution of diffusion equation on $L_2\,[-1,1]$ with Dirichlet BCs

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) \\ \phi(x,0) &= f(x) \\ \phi(\pm 1,t) &= 0 \end{aligned}$$

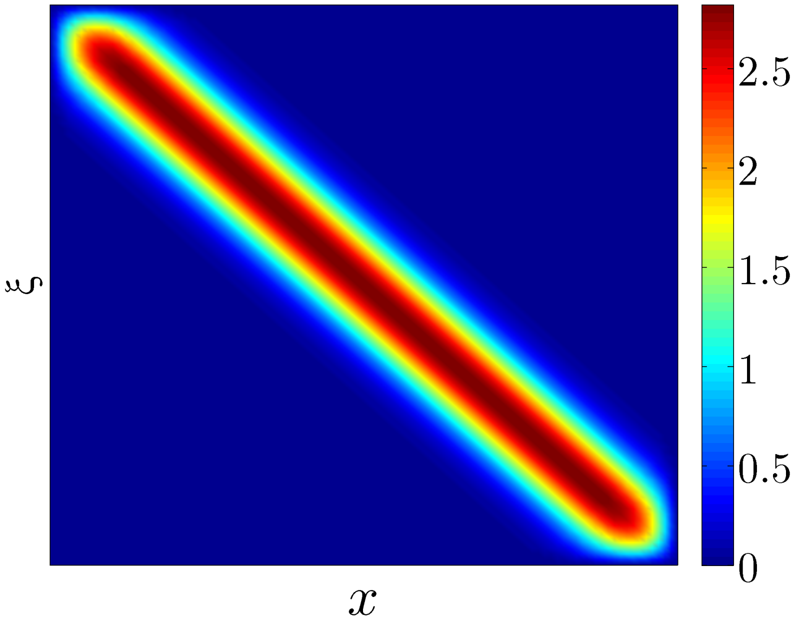
given by

$$\phi(x,t) \;=\; \left[\mathcal{T}(t)\,f\right](x) \;=\; \int_{-1}^{1} T(x,\xi;t)\,f(\xi)\,\mathrm{d}\xi$$

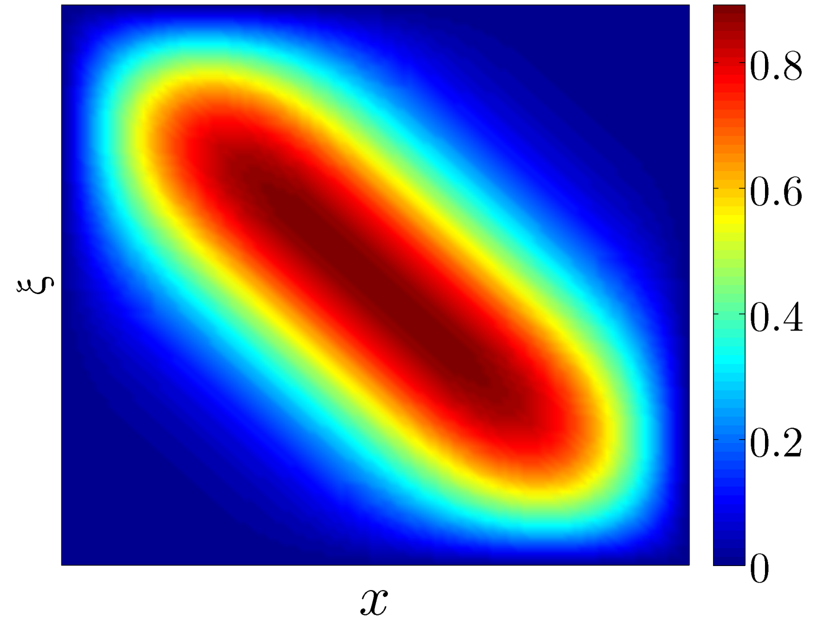
- Kernel representation of operator $\mathcal{T}(t)$: $L_2\,[-1,1] \longrightarrow L_2\,[-1,1]$

$$T(x,\xi;t) \;=\; \sum_{n\,=\,1}^{\infty} \mathrm{e}^{-\left(\frac{n\pi}{2}\right)^2 t}\,\sin\left(\frac{n\pi}{2}\,(x\,+\,1)\right)\,\sin\left(\frac{n\pi}{2}\,(\xi\,+\,1)\right)$$

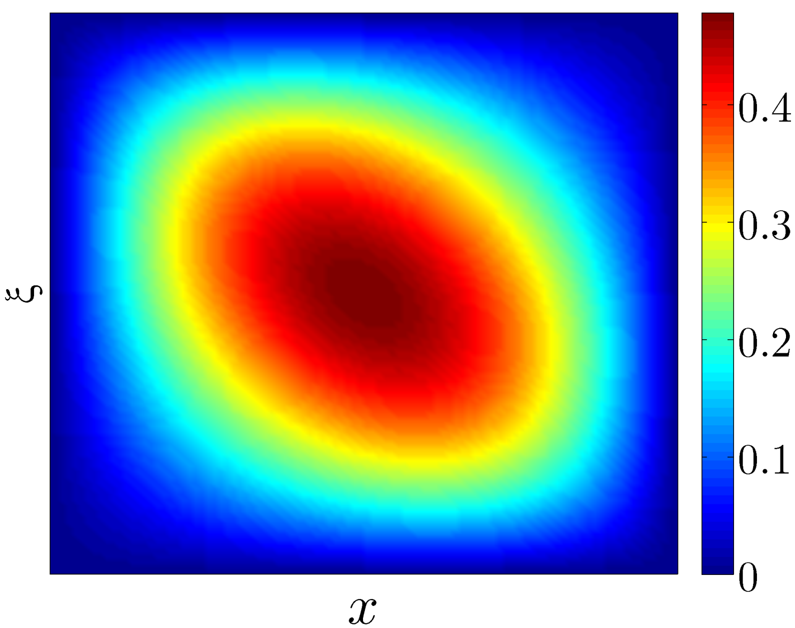
$T(x, \xi; t = 0.01)$:



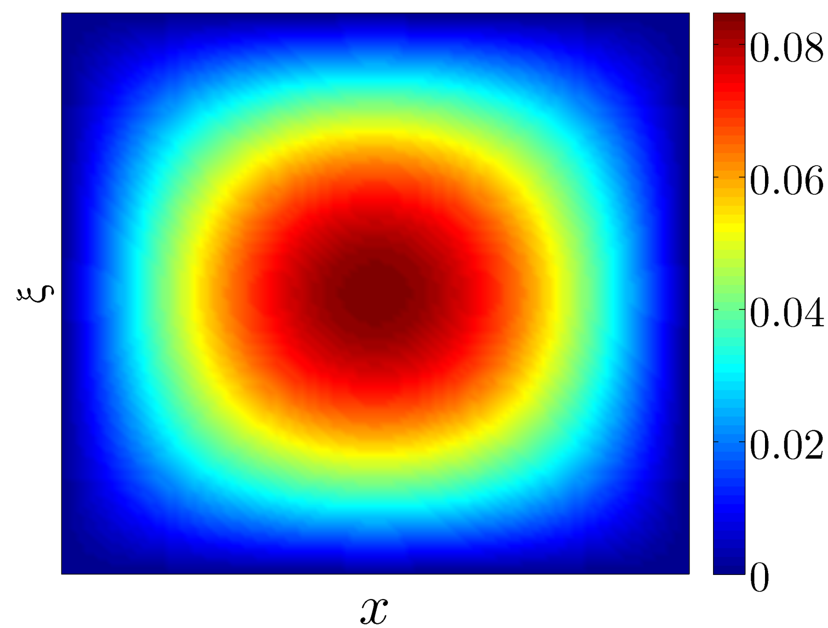
$T(x, \xi; t = 0.1)$:



$T(x, \xi; t = 0.3)$:



$T(x, \xi; t = 1)$:



- For operator $\mathcal{T}$: $f \longrightarrow g$ given by

$$g(x) \;=\; [\mathcal{T} f]\,(x) \;=\; \int_a^b T(x,\xi)\, f(\xi)\,\mathrm{d}\xi$$

- Vector-valued f and $g \quad \Rightarrow \quad$ matrix-valued $T(\cdot,\cdot)$

$$g(x) \;=\; \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}, \; f(x) \;=\; \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} \;\Rightarrow\; T(\cdot,\cdot) \;=\; \begin{bmatrix} T_{11}(\cdot,\cdot) & T_{12}(\cdot,\cdot) & T_{13}(\cdot,\cdot) \\ T_{21}(\cdot,\cdot) & T_{22}(\cdot,\cdot) & T_{23}(\cdot,\cdot) \end{bmatrix}$$

- Kernels of identity and multiplication operators are distributions

$$g(x) \;=\; [I\, f]\,(x) \;=\; f(x) \;=\; \int_a^b \delta(x-\xi)\, f(\xi)\,\mathrm{d}\xi$$

$$g(x) \;=\; [M_a\, f]\,(x) \;=\; a(x)\, f(x) \;=\; \int_a^b a(x)\,\delta(x-\xi)\, f(\xi)\,\mathrm{d}\xi$$

- Kernel of M_a: $\begin{cases} \text{impulse sheet supported along the line } x = \xi \text{ in } [a,\,b] \times [a,\,b] \\ \text{strength "modulated" by the function } a(\cdot) \end{cases}$

Generalizations

- Can be generalized to $\mathcal{T}$: $L_2(\Omega) \longrightarrow L_2(\Omega)$, $\Omega \subset \mathbb{R}^n$

$$g(x) \;=\; [\mathcal{T} f]\,(x) \;=\; \int_\Omega T(x,\xi)\, f(\xi)\,\mathrm{d}\xi$$

- Examples of bounded $\mathcal{T}$: $L_2(\Omega) \longrightarrow L_2(\Omega)$
  - $\star$ Ω compact; $T(\cdot,\cdot)$ has no distributions; $T(\cdot,\cdot)$ bounded
  - $\star$ Ω compact; $\sup\limits_{x\,\in\,\Omega} \displaystyle\int_\Omega |T(x,\xi)|\,\mathrm{d}\xi \;<\; \infty$; $\sup\limits_{\xi\,\in\,\Omega} \displaystyle\int_\Omega |T(x,\xi)|\,\mathrm{d}x \;<\; \infty$
  - $\star$ $\mathcal{T}$ Hilbert-Schmidt, i.e., $\displaystyle\int_\Omega\int_\Omega |T(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi \;<\; \infty$

- $\mathcal{T}$: discrete spectrum and complete set of orthonormal e-functions

$$[\mathcal{T} f]\,(x) \;=\; \sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\,\langle v_n, f\rangle \;=\; \int_\Omega \underbrace{\left(\sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\, v_n^*(\xi)\right)}_{T(x,\xi)} f(\xi)\,\mathrm{d}\xi$$

Hilbert space adjoint

- The adjoint of a bounded operator $\mathcal{A}$: $\mathbb{H}_1 \longrightarrow \mathbb{H}_2$
  - the operator $\mathcal{A}^\dagger$: $\mathbb{H}_2 \longrightarrow \mathbb{H}_1$ defined by

$$\langle \psi_2, \mathcal{A}\,\psi_1 \rangle_2 \;=\; \left\langle \mathcal{A}^\dagger\,\psi_2, \psi_1 \right\rangle_1, \quad \text{for all } \psi_1 \in \mathbb{H}_1 \text{ and } \psi_2 \in \mathbb{H}_2$$

- Examples
  - $\mathcal{A}$: $\mathbb{C}^n \longrightarrow \mathbb{C}^m$ with standard inner product $\quad \mathcal{A}^\dagger = \mathcal{A}^*$
  - $\mathcal{A}$: $\mathbb{C}^n \longrightarrow \mathbb{C}^m$ with $\{\langle f_i, g_i \rangle_i = f_i^*\,Q_i\,g_i;\, Q_i = Q_i^* > 0\}$ $\quad \mathcal{A}^\dagger = Q_1^{-1}\,\mathcal{A}^*\,Q_2$
  - $\mathcal{A}$: $\mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}$, $\mathcal{A}(Q) = \displaystyle\int_0^\infty \mathrm{e}^{At}\,Q\,\mathrm{e}^{A^*t}\,\mathrm{d}t$ with $\langle R, Q \rangle = \mathrm{trace}\,(R^*\,Q)$

$$\mathcal{A}^\dagger(R) = \int_0^\infty \mathrm{e}^{A^*t}\,R\,\mathrm{e}^{At}\,\mathrm{d}t$$

  - $\mathcal{A}$: $L_2[0,t] \longrightarrow \mathbb{C}^n$, $[\mathcal{A}\,u]\,(t) = \displaystyle\int_0^t \mathrm{e}^{A(t-\tau)}\,B\,u(\tau)\,\mathrm{d}\tau$ with standard inner products on $L_2[0,t]$ and $\mathbb{C}^n$

$$\left[\mathcal{A}^\dagger\,x(t)\right](\tau) = B^*\,\mathrm{e}^{A^*(t-\tau)}\,x(t)$$

⋆ $\mathcal{A}$: $L_2[a,b] \longrightarrow L_2[a,b]$, $[\mathcal{A}\, f]\,(x) \;=\; \displaystyle\int_a^b A(x,\xi)\, f(\xi)\, \mathrm{d}\xi$ with standard inner product on $L_2[0,t]$ ▮

$$\left[\mathcal{A}^\dagger\, g\right]\,(x) \;=\; \int_a^b A^*(\xi,x)\, g(\xi)\, \mathrm{d}\xi$$

⋆ $\mathcal{A}$: $L_2[a,b] \longrightarrow L_2[a,b]$, $[\mathcal{A}\, f]\,(x) \;=\; \displaystyle\int_a^x A(x,\xi)\, f(\xi)\, \mathrm{d}\xi$ with standard inner product on $L_2[0,t]$ ▮

$$\left[\mathcal{A}^\dagger\, g\right]\,(x) \;=\; \int_x^b A^*(\xi,x)\, g(\xi)\, \mathrm{d}\xi$$

▮

- For bounded $\mathcal{A}$: $\mathbb{H}_1 \longrightarrow \mathbb{H}_2$, $\mathcal{B}$: $\mathbb{H}_2 \longrightarrow \mathbb{H}_3$, $\alpha \in \mathbb{C}$

$$I^\dagger \;=\; I, \quad (\alpha\,\mathcal{A})^\dagger \;=\; \overline{\alpha}\,\mathcal{A}^\dagger, \quad \|\mathcal{A}^\dagger\| \;=\; \|\mathcal{A}\|$$

$$(\mathcal{A}_1 \,+\, \mathcal{A}_2)^\dagger \;=\; \mathcal{A}_1^\dagger \,+\, \mathcal{A}_2^\dagger, \quad (\mathcal{B}\,\mathcal{A})^\dagger \;=\; \mathcal{A}^\dagger\,\mathcal{B}^\dagger, \quad \|\mathcal{A}^\dagger\,\mathcal{A}\| \;=\; \|\mathcal{A}\|^2$$

Fundamental subspaces

- The range space of $\mathcal{A} : \mathbb{H}_1 \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}_2$

$$\mathcal{R}(\mathcal{A}) \;=\; \{g \in \mathbb{H}_2;\;\; g \;=\; \mathcal{A}f,\; f \in \mathcal{D}(\mathcal{A})\}$$

- The null space of $\mathcal{A} : \mathbb{H}_1 \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}_2$

$$\mathcal{N}(\mathcal{A}) \;=\; \{f \in \mathbb{H}_1;\;\; \mathcal{A}f \;=\; 0\}$$

- For a bounded $\mathcal{A}$: $\mathbb{H}_1 \longrightarrow \mathbb{H}_2$
  - ⋆ $[\mathcal{R}(\mathcal{A})]^\perp = \mathcal{N}(\mathcal{A}^\dagger); \quad \overline{[\mathcal{R}(\mathcal{A})]} = [\mathcal{N}(\mathcal{A}^\dagger)]^\perp$
  - ⋆ $[\mathcal{R}(\mathcal{A}^\dagger)]^\perp = \mathcal{N}(\mathcal{A}); \quad \overline{[\mathcal{R}(\mathcal{A}^\dagger)]} = [\mathcal{N}(\mathcal{A})]^\perp$

- For bounded $\mathcal{A}$: $\mathbb{H}_1 \longrightarrow \mathbb{H}_2$, $\mathcal{B}$: $\mathbb{H}_2 \longrightarrow \mathbb{H}_3$
  - ⋆ $\mathcal{N}(\mathcal{B}\,\mathcal{A}) \supseteq \mathcal{N}(\mathcal{A})$ but $\mathcal{N}(\mathcal{A}) = \mathcal{N}(\mathcal{A}^\dagger\,\mathcal{A})$
  - ⋆ $\mathcal{R}(\mathcal{B}\,\mathcal{A}) \subseteq \mathcal{R}(\mathcal{B})$ but $\overline{\mathcal{R}(\mathcal{A})} = \overline{\mathcal{R}(\mathcal{A}\,\mathcal{A}^\dagger)}$

Adjoint of an unbounded operator

- The adjoint of an unbounded operator $\left\{\begin{array}{l} \mathcal{A}: \mathbb{H}_1 \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}_2 \\ \mathcal{D}(\mathcal{A}) \text{ dense in } \mathbb{H}_1 \end{array}\right.$

  ⋆ the operator $\mathcal{A}^\dagger$: $\mathbb{H}_2 \supset \mathcal{D}(\mathcal{A}^\dagger) \longrightarrow \mathbb{H}_1$ defined by

$$\left\{\begin{array}{rcl} \mathcal{D}(\mathcal{A}^\dagger) & = & \{\psi_2 \in \mathbb{H}_2;\ \exists\, \phi_1 \in \mathbb{H}_1 \text{ s.t. } \langle \psi_2, \mathcal{A}\,\psi_1 \rangle_2 = \langle \phi_1, \psi_1 \rangle_1 \text{ for all } \psi_1 \in \mathcal{D}(\mathcal{A})\} \\ \mathcal{A}^\dagger\,\psi_2 & = & \phi_1 \end{array}\right.$$

- Informally

$$\langle \psi_2, \mathcal{A}\,\psi_1 \rangle_2 = \left\langle \mathcal{A}^\dagger\,\psi_2, \psi_1 \right\rangle_1 \left\{\begin{array}{l} \text{for all } \psi_1 \in \mathcal{D}(\mathcal{A}) \text{ and } \psi_2 \text{ for which the RHS is finite} \\ \text{such } \psi_2 \in \mathbb{H}_2 \text{ determine } \mathcal{D}(\mathcal{A}^\dagger) \end{array}\right.$$

Examples (to be solved in class)

- $$\left\{ \begin{array}{rcl} [\mathcal{A}\, f]\,(x) & = & \left[\dfrac{\mathrm{d} f}{\mathrm{d} x}\right](x) \\ \mathcal{D}(\mathcal{A}) & = & \left\{ f \,\in\, L_2\,[-1,\,1],\; \dfrac{\mathrm{d} f}{\mathrm{d} x} \in\, L_2\,[-1,\,1],\; f(-1) \,=\, 0 \right\} \end{array} \right.$$

- $$\left\{ \begin{array}{rcl} [\mathcal{A}\, f]\,(x) & = & \left[\dfrac{\mathrm{d}^2 f}{\mathrm{d} x^2}\right](x) \\ \mathcal{D}(\mathcal{A}) & = & \left\{ f \,\in\, L_2\,[-1,\,1],\; \dfrac{\mathrm{d}^2 f}{\mathrm{d} x^2} \in\, L_2\,[-1,\,1],\; f(\pm 1) \,=\, 0 \right\} \end{array} \right.$$

Useful property

- $\begin{cases} \mathcal{A}: & \text{unbounded densely defined operator with domain } \mathcal{D}(\mathcal{A}) \subset \mathbb{H} \\ \mathcal{B}: & \text{bounded operator defined on the whole } \mathbb{H} \end{cases}$

  ⋆ $(\alpha\,\mathcal{A})^\dagger = \overline{\alpha}\,\mathcal{A}^\dagger; \quad \mathcal{D}\left((\alpha\,\mathcal{A})^\dagger\right) = \begin{cases} \mathcal{D}\left(\mathcal{A}^\dagger\right), & \alpha \neq 0 \\ \mathbb{H}, & \alpha = 0 \end{cases}$

  ⋆ $(\mathcal{A} + \mathcal{B})^\dagger = \mathcal{A}^\dagger + \mathcal{B}^\dagger$, with domain $\mathcal{D}\left((\mathcal{A} + \mathcal{B})^\dagger\right) = \mathcal{D}\left(\mathcal{A}^\dagger\right)$

  ⋆ $\mathcal{A}$ has bounded inverse $\Rightarrow$ $\mathcal{A}^\dagger$ has bounded inverse: $\left(\mathcal{A}^\dagger\right)^{-1} = \left(\mathcal{A}^{-1}\right)^\dagger$

- Examples on $L_2[-1, 1]$

$$\left.\begin{array}{rcl} f'(x) & = & g(x) \\ f(-1) & = & 0 \end{array}\right\} \Rightarrow f(x) = \int_{-1}^{x} g(\xi)\,\mathrm{d}\xi = \int_{-1}^{1} \mathbb{1}(x - \xi)\,g(\xi)\,\mathrm{d}\xi$$

$$\left.\begin{array}{rcl} f''(x) & = & g(x) \\ f(\pm 1) & = & 0 \end{array}\right\} \Rightarrow f(x) = \int_{-1}^{1} \left((x - \xi)\,\mathbb{1}(x - \xi) + \frac{(x + 1)(\xi - 1)}{2} \right) g(\xi)\,\mathrm{d}\xi$$

Self-adjoint operators

$$\begin{cases} \langle \psi_2, \mathcal{A}\,\psi_1 \rangle_2 &= \langle \mathcal{A}\,\psi_2, \psi_1 \rangle_1 \text{ for all } \psi_1,\ \psi_2 \in \mathcal{D}(\mathcal{A}) \\ \mathcal{D}(\mathcal{A}^\dagger) &= \mathcal{D}(\mathcal{A}) \end{cases}$$

$\mathcal{A}$ self-adjoint $\Rightarrow$ $\begin{cases} \text{all e-values of } \mathcal{A} \text{ are real} \\ v_n,\ v_m\text{: e-vectors corresponding to } \lambda_n \neq \lambda_m \ \Rightarrow\ \langle v_n, v_m \rangle = 0 \end{cases}$

- $\mathcal{A}$: densely defined self-adjoint operator in $\mathbb{H}$ with discrete spectrum

  $\Downarrow$

  $\mathcal{A}$ has an orthonormal set of e-functions that span $\mathbb{H}$

Example (to be solved in class)

- E-value decomposition of $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $L_2[-1,1]$ with Dirichlet BCs

$$\left\{ \begin{array}{rcl} [\mathcal{A}\, f]\,(x) & = & \left[\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right](x) \\ \mathcal{D}(\mathcal{A}) & = & \left\{ f \,\in\, L_2\,[-1,\,1],\; \dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} \,\in\, L_2\,[-1,\,1],\; f(\pm 1) \,=\, 0 \right\} \end{array} \right.$$

- Need to solve

$$\left\{ \begin{array}{rcl} \left[\dfrac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right](x) & = & \lambda\, v(x) \\ v(\pm 1) & = & 0 \end{array} \right.$$

$$\downarrow$$

$$\left\{ v_n(x) \;=\; \sin\left(\frac{n\pi}{2}\,(x\,+\,1)\right);\; \lambda_n \;=\; -\left(\frac{n\pi}{2}\right)^2 \right\}_{n\,\in\,\mathbb{N}}$$

Lecture 9: Spectral theory for compact normal operators

- Resolvent and spectrum of an operator

- Compact operators
  - ⋆ Direct extension of matrices

- Normal operators
  - ⋆ Commute with its adjoint

- Compact normal operators
  - ⋆ Unitarily diagonalizable
  - ⋆ E-functions provide a complete orthonormal basis of $\mathbb{H}$

- Riesz-spectral operators

Resolvent

- Want to study equations of the form

$$(\lambda I - \mathcal{A})\,\psi = u, \quad \{\mathcal{A} : \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H};\ \lambda \in \mathbb{C};\ \psi, u \in \mathbb{H}\}$$

Determine conditions under which $\mathcal{A}_\lambda = (\lambda I - \mathcal{A})$ is boundedly invertible

Relevant conditions:
$$\begin{cases} (1) & \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ exists} \\ (2) & \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ is bounded} \\ (3) & \text{The domain of } \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ is dense in } \mathbb{H} \end{cases}$$

- The resolvent set of $\mathcal{A}$:

$$\rho(\mathcal{A}) := \{\lambda \in \mathbb{C};\ \text{(1), (2), (3) hold}\}$$

- The spectrum of $\mathcal{A}$:

$$\sigma(\mathcal{A}) := \mathbb{C} \setminus \rho(\mathcal{A})$$

Spectrum

$$(1) \quad \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ exists}$$

$$(2) \quad \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ is bounded}$$

$$(3) \quad \text{The domain of } \mathcal{R}_\lambda = (\lambda I - \mathcal{A})^{-1} \text{ is dense in } \mathbb{H}$$

- $\sigma(\mathcal{A})$ can be decomposed into

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_c(\mathcal{A}) \cup \sigma_r(\mathcal{A})$$

⋆ Point spectrum

$$\sigma_p(\mathcal{A}) := \{\lambda \in \mathbb{C};\ (\lambda I - \mathcal{A}) \text{ is not one-to-one}\}$$

⋆ Continuous spectrum

$$\sigma_c(\mathcal{A}) := \{\lambda \in \mathbb{C};\ \text{(1) and (3) hold, but (2) doesn't}\}$$

⋆ Residual spectrum

$$\sigma_r(\mathcal{A}) := \{\lambda \in \mathbb{C};\ \text{(1) holds but (3) doesn't}\}$$

Examples

- Point spectrum

$$\{\lambda \in \sigma_p(\mathcal{A}): \text{ e-values; } \; v \in \mathcal{N}(\lambda I - \mathcal{A}): \text{ e-functions}\}$$

- Continuous spectrum

$$\text{multiplication operator on } L_2[a,b]: \;\; [M_a \, f(\cdot)]\,(x) \; = \; a(x)\, f(x)$$

- Residual spectrum

$$\text{right-shift operator on } \ell_2(\mathbb{N}): \;\; [S_r \, f(\cdot)]\,(n) \; = \; f_{n-1}$$

Spectral decomposition of compact normal operators

- compact, normal operator $\mathcal{A}$ on $\mathbb{H}$ admits a dyadic decomposition

$$\left.\begin{array}{rcl} [\mathcal{A}\, v_n]\,(x) & = & \lambda_n\, v_n(x) \\ \langle v_n, v_m \rangle & = & \delta_{nm} \end{array}\right\} \;\Rightarrow\; [\mathcal{A}\, f]\,(x) \;=\; \sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\, \langle v_n, f \rangle \quad \text{for all } f \in \mathbb{H}$$

$$\mathcal{A}\colon \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}, \quad \text{with compact and normal } \mathcal{A}^{-1}$$

$$\downarrow$$

$$\left.\begin{array}{rcl} \left[\mathcal{A}^{-1}\, v_n\right](x) & = & \lambda_n^{-1}\, v_n(x) \\ \langle v_n, v_m \rangle & = & \delta_{nm} \end{array}\right\} \;\Rightarrow\; \left[\mathcal{A}^{-1} f\right](x) \;=\; \sum_{n\,=\,1}^{\infty} \lambda_n^{-1}\, v_n(x)\, \langle v_n, f \rangle\,, \quad f \in \mathbb{H}$$

$$[\mathcal{A}\, f]\,(x) \;=\; \sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\, \langle v_n, f \rangle\,, \quad f \in \mathcal{D}(\mathcal{A})$$

$$\mathcal{D}(\mathcal{A}) \;=\; \left\{ f \in \mathbb{H};\;\; \sum_{n\,=\,1}^{\infty} |\lambda_n|^2\, |\langle v_n, f \rangle|^2 \;<\; \infty \right\}$$

- compact, normal operator $\mathcal{A}$ on $\mathbb{H}$

$$\left.\begin{array}{rcl}[\mathcal{A}\, v_n]\,(x) & = & \lambda_n\, v_n(x), \quad \lambda_n \neq 0 \\ \langle v_n, v_m\rangle & = & \delta_{nm}\end{array}\right\} \quad \left\{\begin{array}{rcl} u & = & u_{\mathcal{R}(\mathcal{A})} + u_{\mathcal{N}(\mathcal{A})} \\ & = & \displaystyle\sum_{n=1}^{\infty} v_n\, \langle v_n, u\rangle + u_{\mathcal{N}(\mathcal{A})}\end{array}\right.$$

- Solutions to

$$(\lambda I - \mathcal{A})\,\psi = u, \quad \lambda \neq 0$$

1. λ – not an eigenvalue of $\mathcal{A}$ $\Rightarrow$ unique solution

$$\psi = \sum_{n=1}^{\infty} \frac{\langle v_n, u\rangle}{\lambda - \lambda_n}\, v_n + \frac{1}{\lambda}\, u_{\mathcal{N}(\mathcal{A})}$$

2. $\left.\begin{array}{l}\lambda \text{ – eigenvalue of } \mathcal{A} \\ J \text{ – index set s.t. } \lambda_j = \lambda\end{array}\right\}$ $\Rightarrow$ there is a solution iff $\langle v_j, u\rangle = 0$ for all $j \in J$

$$\psi = \sum_{j \in J} c_j\, v_j + \sum_{j \in \mathbb{N}\backslash J} \frac{\langle v_j, u\rangle}{\lambda - \lambda_j}\, v_j + \frac{1}{\lambda}\, u_{\mathcal{N}(\mathcal{A})}$$

Singular Value Decomposition of compact operators

- compact operator $\mathcal{A}$: $\mathbb{H}_1 \longrightarrow \mathbb{H}_2$ admits a Schmidt Decomposition (i.e., an SVD)

$$[\mathcal{A} f](x) \;=\; \sum_{n=1}^{\infty} \sigma_n u_n(x) \langle v_n, f \rangle$$

$$\left[\mathcal{A}\mathcal{A}^{\dagger} u_n\right](x) \;=\; \sigma_n^2 u_n(x) \;\Rightarrow\; \{u_n\}_{n \in \mathbb{N}} \text{ orthonormal basis of } \mathbb{H}_2$$

$$\left[\mathcal{A}^{\dagger}\mathcal{A} v_n\right](x) \;=\; \sigma_n^2 v_n(x) \;\Rightarrow\; \{v_n\}_{n \in \mathbb{N}} \text{ orthonormal basis of } \mathbb{H}_1$$

- matrix M: $\mathbb{C}^n \longrightarrow \mathbb{C}^m$

$$M \;=\; U \Sigma V^* \;=\; \sum_{i=1}^{r} \sigma_i u_i v_i^* \;\Rightarrow\; M f \;=\; \sum_{i=1}^{r} \sigma_i u_i \langle v_i, f \rangle$$

$$M M^* u_i \;=\; \sigma_i^2 u_i$$

$$M^* M v_i \;=\; \sigma_i^2 v_i$$

Riesz basis

- $\{v_n\}_{n \in \mathbb{N}}$: Riesz basis of $\mathbb{H}$ if
  - $\overline{\mathrm{span}\,\{v_n\}_{n \in \mathbb{N}}} \;=\; \mathbb{H}$
  - there are $m, M > 0$ such that for any $N \in \mathbb{N}$ and any $\{\alpha_n\}$, $n = 1, \ldots, N$

$$m \sum_{n=1}^{N} |\alpha_n|^2 \;\leq\; \|\sum_{n=1}^{N} \alpha_n v_n\|^2 \;\leq\; M \sum_{n=1}^{N} |\alpha_n|^2$$

- closed $\mathcal{A}: \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}$

$$[\mathcal{A}\, v_n]\,(x) \;=\; \lambda_n\, v_n(x) \qquad \begin{cases} \{\lambda_n\}_{n\,\in\,\mathbb{N}} & \text{simple e-values} \\ \{v_n\}_{n\,\in\,\mathbb{N}} & \text{Riesz basis of } \mathbb{H} \end{cases}$$

⋆ $\left[\mathcal{A}^{\dagger}\, w_n\right](x) \;=\; \bar{\lambda}_n\, w_n(x) \;\;\Rightarrow\;\; \{w_n\}_{n\,\in\,\mathbb{N}}$ can be scaled s.t. $\langle w_n, v_m\rangle \;=\; \delta_{nm}$

⋆ every $f \in \mathbb{H}$ can be represented uniquely by

$$f(x) \;=\; \sum_{n\,=\,1}^{\infty} v_n(x)\,\langle w_n, f\rangle$$

$$m \sum_{n\,=\,1}^{\infty} |\,\langle w_n, f\rangle\,|^2 \;\leq\; \|f\|^2 \;\leq\; M \sum_{n\,=\,1}^{\infty} |\,\langle w_n, f\rangle\,|^2$$

or by

$$f(x) \;=\; \sum_{n\,=\,1}^{\infty} w_n(x)\,\langle v_n, f\rangle$$

$$\frac{1}{M} \sum_{n\,=\,1}^{\infty} |\,\langle v_n, f\rangle\,|^2 \;\leq\; \|f\|^2 \;\leq\; \frac{1}{m} \sum_{n\,=\,1}^{\infty} |\,\langle v_n, f\rangle\,|^2$$

Riesz-spectral operator

- closed $\mathcal{A} : \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}$ is Riesz-spectral operator if

$$[\mathcal{A}\, v_n]\,(x) \;=\; \lambda_n\, v_n(x) \quad \left\{ \begin{array}{ll} \{\lambda_n\}_{n\,\in\,\mathbb{N}} & \text{simple e-values} \\ \{v_n\}_{n\,\in\,\mathbb{N}} & \text{Riesz basis of } \mathbb{H} \\ \overline{\{\lambda_n\}_{n\,\in\,\mathbb{N}}} & \text{totally disconnected} \end{array} \right.$$

$\mathcal{A}$ – Riesz-spectral operator with e-pair $\{(\lambda_n, v_n)\}_{n\,\in\,\mathbb{N}}$

$\{w_n\}_{n\,\in\,\mathbb{N}}$ – e-functions of $\mathcal{A}^\dagger$ s.t. $\langle w_n, v_m \rangle \;=\; \delta_{nm}$

$$\downarrow$$

$$\left\{ \begin{array}{l} \sigma(\mathcal{A}) \;=\; \overline{\{\lambda_n\}_{n\,\in\,\mathbb{N}}}, \quad \rho(\mathcal{A}) \;=\; \{\lambda_n \,\in\, \mathbb{C}; \;\; \inf_{n\,\in\,\mathbb{N}} |\lambda \,-\, \lambda_n| \,>\, 0\} \\ \lambda \,\in\, \rho(\mathcal{A}) \;\Rightarrow\; \left[(\lambda I \,-\, \mathcal{A})^{-1} f\right](x) \;=\; \displaystyle\sum_{n\,=\,1}^{\infty} \frac{1}{\lambda \,-\, \lambda_n}\, v_n(x)\, \langle w_n, f \rangle \\ [\mathcal{A}\, f]\,(x) \;=\; \displaystyle\sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\, \langle w_n, f \rangle\,, \;\; \mathcal{D}(\mathcal{A}) \;=\; \left\{ f \in \mathbb{H};\; \sum_{n\,=\,1}^{\infty} |\lambda_n|^2\, |\langle w_n, f \rangle|^2 \,<\, \infty \right\} \end{array} \right.$$

Lectures 10 & 11: Semigroup Theory

- Want to generalize matrix exponential to infinite dimensional setting

- Strongly continuous (C_0) semigroup
  - ⋆ Extension of matrix exponential

- Infinitesimal generator of a C_0-semigroup

- Examples and conditions

Solution to abstract evolution equation

- Abstract evolution equation on a Hilbert space $\mathbb{H}$

$$\frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} \;=\; \mathcal{A}\,\psi(t), \quad \psi(0) \,\in\, \mathbb{H}$$

Dilemma: how to define "$\mathrm{e}^{\mathcal{A}\,t}$"?

Finite dimensional case:

$$M \,\in\, \mathbb{C}^{n\times n} \quad \Rightarrow \quad \mathrm{e}^{M\,t} \;=\; \sum_{k\,=\,1}^{\infty} \frac{(M\,t)^k}{k!}$$

$$\frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} \;=\; \mathcal{A}\,\psi(t), \quad \psi(0) \,\in\, \mathbb{H}$$

- Assume:
  - ⋆ For each $\psi(0) \in \mathbb{H}$, there is a unique solution $\psi(t)$
  - ⋆ There is a well defined mapping $T(t)$: $\mathbb{H} \longrightarrow \mathbb{H}$

$$\psi(t) \;=\; T(t)\,\psi(0)$$

    $T(t)$ - time-parameterized family of linear operators on $\mathbb{H}$

  - ⋆ Solution varies continuously with initial state

    $T(t)$: a bounded operator (on $\mathbb{H}$)

$$\|T(t)\| \;=\; \sup_{f\,\in\,\mathbb{H}} \frac{\|T(t)\,f\|}{\|f\|} \;<\; \infty$$

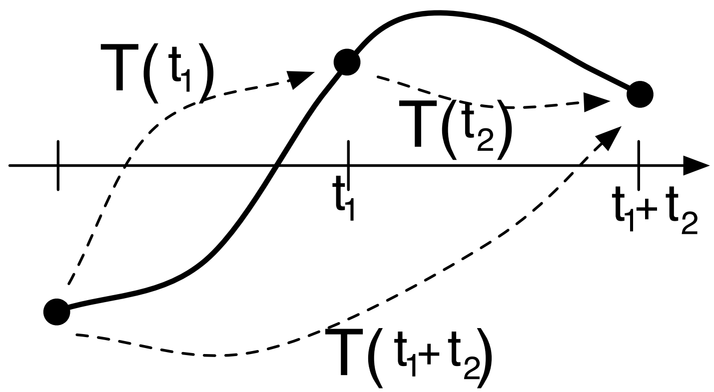
Strongly continuous semigroups

- Properties of $T(t)$: $$\psi(t) \;=\; T(t)\,\psi(0)$$

- Initial condition: $$T(0) \;=\; I$$

- Semigroup property:

$$T(t_1 + t_2) \;=\; T(t_2)\,T(t_1) \;=\; T(t_1)\,T(t_2), \quad \text{for all } t_1, t_2 \geq 0$$



- Strong continuity:

$$\lim_{t \to 0^+} \|T(t)\,\psi(0) \;-\; \psi(0)\| \;=\; 0, \;\; \text{for all } \;\psi(0) \,\in\, \mathbb{H}$$

a weaker condition than:

$$\lim_{t \to 0^+} \|T(t) \;-\; I\| \;=\; \lim_{t \to 0^+} \sup_{f \,\in\, \mathbb{H}} \frac{\|\,(T(t) \;-\; I)\,f\|}{\|f\|} \;=\; 0$$

Examples

- Linear transport equation

$$\left.\begin{aligned} \phi_t(x,t) &= \pm c\,\phi_x(x,t) \\ \phi(x,0) &= f(x), \;\; x \in \mathbb{R} \end{aligned}\right\} \;\Rightarrow\; \left\{\begin{aligned} \frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} &= \pm c\,\frac{\mathrm{d}}{\mathrm{d}\,x}\,\psi(t) \\ \psi(0) &= f \in L_2(-\infty,\infty) \end{aligned}\right.$$

- Consider:

$$\phi(x,t) \;=\; [T(t)\,f]\,(x) \;=\; f(x \pm ct)$$

In class: $T(t)$ defines a C_0-semigroup on $L_2(-\infty,\infty)$

- The infinitesimal generator of a C_0-semigroup $T(t)$ on $\mathbb{H}$

$$\mathcal{A}\,f \;=\; \lim_{t \to 0^+} \frac{T(t)\,f \,-\, f}{t}$$

$$\mathcal{D}(\mathcal{A}) \;=\; \left\{ f \in \mathbb{H};\; \lim_{t \to 0^+} \frac{T(t)\,f \,-\, f}{t} \text{ exists} \right\}$$

- A couple of additional notes

  ⋆ Change of coordinates:

$$\left.\begin{array}{rcl} \phi_t(x,t) & = & \pm c\,\phi_x(x,t) \\ \phi(x,0) & = & f(x), \quad x \in \mathbb{R} \end{array}\right\} \xrightarrow{z \,=\, x \,\pm\, ct} \left\{\begin{array}{rcl} \phi_t(z,t) & = & 0 \\ \phi(z,0) & = & f(z), \quad z \in \mathbb{R} \end{array}\right.$$

  ⋆ Reaction-convection equation:

$$\left.\begin{array}{rcl} \phi_t(x,t) & = & \pm c\,\phi_x(x,t) \;+\; a\,\phi(x,t) \\ \phi(x,0) & = & f(x), \quad x \in \mathbb{R} \end{array}\right\}$$

C_0-semigroup:

$$\phi(x,t) \;=\; [T(t)\,f]\,(x) \;=\; \mathrm{e}^{a\,t}\,f(x \,\pm\, ct)$$

$a > 0$ exponentially growing traveling wave

$a < 0$ exponentially decaying traveling wave

Infinite number of decoupled scalar states

- Abstract evolution equation on $\ell_2(\mathbb{N})$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\begin{bmatrix}\psi_1(t)\\ \psi_2(t)\\ \vdots\end{bmatrix} = \begin{bmatrix}a_1 & & \\ & a_2 & \\ & & \ddots\end{bmatrix}\begin{bmatrix}\psi_1(t)\\ \psi_2(t)\\ \vdots\end{bmatrix} \;\Leftrightarrow\; \frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} = \mathcal{A}\,\psi(t)$$

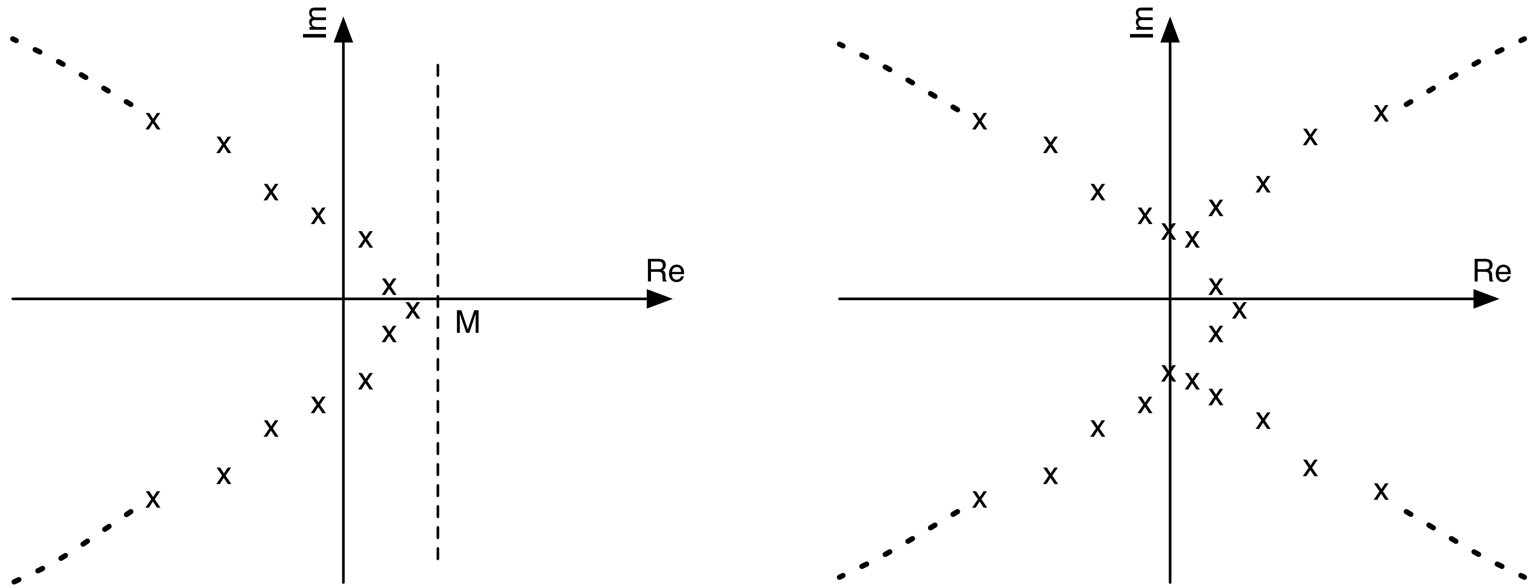
Solution

$$\psi(t) = \begin{bmatrix}\psi_1(t)\\ \psi_2(t)\\ \vdots\end{bmatrix} = \begin{bmatrix}\mathrm{e}^{a_1\,t} & & \\ & \mathrm{e}^{a_2\,t} & \\ & & \ddots\end{bmatrix}\begin{bmatrix}\psi_1(0)\\ \psi_2(0)\\ \vdots\end{bmatrix} = T(t)\,\psi(0)$$

- In class: conditions for well-posedness on $\ell_2(\mathbb{N})$

- Half-plane condition:

$$\sup_n \, Re\,(a_n) \; < \; M \; < \; \infty$$



(a) (b)

Same condition for:

$$T(t)\,f \;=\; \sum_{n\,=\,1}^{\infty} \mathrm{e}^{a_n\,t}\,v_n\,\langle v_n,\,f\rangle$$

Continuum of decoupled scalar states

$$\dot{\psi}(\kappa, t) \;=\; a(\kappa)\, \psi(\kappa, t), \;\; \kappa \in \mathbb{R}$$

Solution

$$\psi(\kappa, t) \;=\; [T(t)\, \psi(\cdot, 0)]\, (\kappa) \;=\; \mathrm{e}^{a(\kappa)\, t}\, \psi(\kappa, 0)$$

- Homework: conditions for well-posedness on $L_2\,(-\infty, \infty)$

Half-plane condition:

$$\sup_{\kappa \,\in\, \mathbb{R}} \; Re\,(a(\kappa)) \;<\; M \;<\; \infty$$

Hille-Yosida Theorem

closed, densely defined operator $\mathcal{A}$ on $\mathbb{H}$:

$\mathcal{A}$ - infinitesimal generator of a C_0-semigroup with $\|T(t)\| \leq M\,\mathrm{e}^{\omega\, t}$

$$\Updownarrow$$

every real $\lambda > \omega$ is in $\rho(\mathcal{A})$ and $\|\,(\lambda I - \mathcal{A})^{-n}\,\| \leq \dfrac{M}{(\lambda - \omega)^n}$ for all $n \geq 1$

- ! Difficult to check
- Important consequence: a method for computing $T(t)$

$$T(t) = \lim_{N \to \infty} \left(I - \frac{t}{N} \mathcal{A} \right)^{-N}$$

Implicit Euler:

$$\frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} = \mathcal{A}\,\psi(t) \quad \Rightarrow \quad \frac{\psi(t + \Delta t) - \psi(t)}{\Delta t} = \mathcal{A}\,\psi(t + \Delta t)$$

Lumer-Phillips Theorem

closed, densely defined operator $\mathcal{A}$ on $\mathbb{H}$:

$$Re\,(\langle \psi, \mathcal{A}\,\psi \rangle) \;\leq\; \omega\,\|\psi\|^2 \quad \text{for all } \psi \,\in\, \mathcal{D}(\mathcal{A})$$

$$Re\,\big(\big\langle \psi, \mathcal{A}^\dagger\,\psi \big\rangle\big) \;\leq\; \omega\,\|\psi\|^2 \quad \text{for all } \psi \,\in\, \mathcal{D}(\mathcal{A}^\dagger)$$

$$\Downarrow$$

$\mathcal{A}$ - infinitesimal generator of a C_0-semigroup with $\|T(t)\| \;\leq\; \mathrm{e}^{\omega\,t}$

- Examples:

$$\left\{ \begin{array}{rcl} [\mathcal{A}\,f]\,(x) & = & \left[\dfrac{\mathrm{d}f}{\mathrm{d}x}\right](x) \\[2ex] \mathcal{D}(\mathcal{A}) & = & \left\{ f \,\in\, L_2\,[-1,\,1],\; \dfrac{\mathrm{d}f}{\mathrm{d}x} \,\in\, L_2\,[-1,\,1],\; f(1) \,=\, 0 \right\} \end{array} \right.$$

$$\left\{ \begin{array}{rcl} [\mathcal{A}\,f]\,(x) & = & \left[\dfrac{\mathrm{d}^2f}{\mathrm{d}x^2}\right](x) \\[2ex] \mathcal{D}(\mathcal{A}) & = & \left\{ f \,\in\, L_2\,[-1,\,1],\; \dfrac{\mathrm{d}^2f}{\mathrm{d}x^2} \,\in\, L_2\,[-1,\,1],\; f(\pm 1) \,=\, 0 \right\} \end{array} \right.$$

Lecture 12: Waves, beams, . . .

- Objective: study dynamics of waves and beams

- Approach: identify commonalities between the two equations
  - ⋆ Inner product that induces energy of wave/beam
  - ⋆ Square-root of a positive self-adjoint operator

Wave equation

$$
\begin{aligned}
\phi_{tt}(x,t) &= \phi_{xx}(x,t) \\
\phi(x,0) &= f(x), \quad \phi_t(x,0) \;=\; g(x) \\
\phi(\pm 1,t) &= 0
\end{aligned}
$$

Define $\psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} = \begin{bmatrix} \phi(\cdot,t) \\ \phi_t(\cdot,t) \end{bmatrix}$ and write an abstract evolution equation:

$$
\begin{aligned}
\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ \mathrm{d}^2/\mathrm{d}x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \\
\phi(t) &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}
\end{aligned}
$$

- Dynamical generator

$$
\begin{aligned}
\mathcal{A} &= \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & 0 \end{bmatrix}, \quad \mathcal{A}_0 \;=\; -\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} \\
\mathcal{D}(\mathcal{A}_0) &= \left\{ f \in L_2\,[-1,\,1],\; \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \in L_2\,[-1,\,1],\; f(\pm 1) \;=\; 0 \right\}
\end{aligned}
$$

Euler-Bernoulli beam

$$\begin{aligned}
\phi_{tt}(x,t) &= -\phi_{xxxx}(x,t) \\
\phi(x,0) &= f(x), \quad \phi_t(x,0) \;=\; g(x) \\
\phi(\pm 1,t) &= 0 \\
\phi_{xx}(\pm 1,t) &= 0
\end{aligned}$$

Define $\psi(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} = \begin{bmatrix} \phi(\cdot,t) \\ \phi_t(\cdot,t) \end{bmatrix}$ and write an abstract evolution equation:

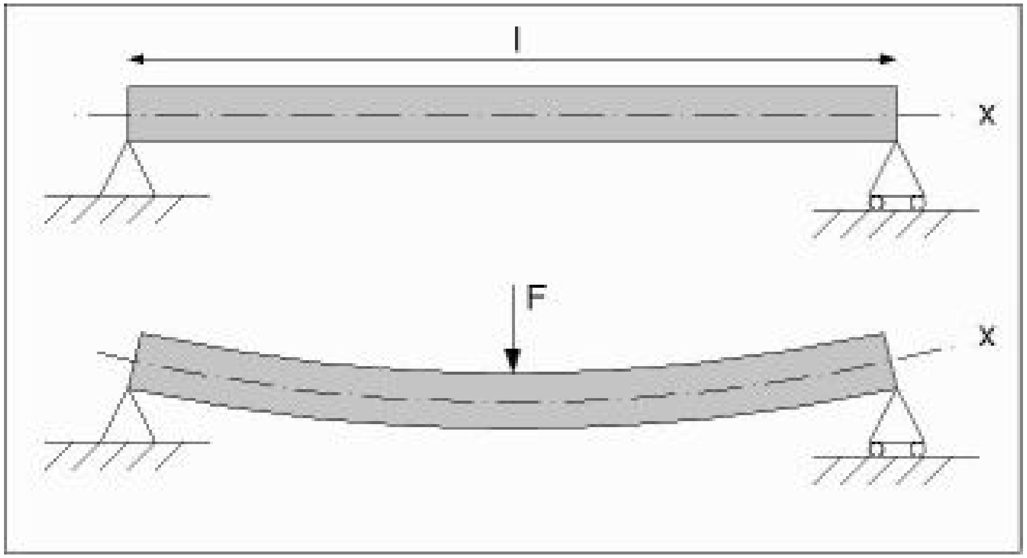
$$\begin{aligned}
\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -\mathrm{d}^4/\mathrm{d}x^4 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \\
\phi(t) &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}
\end{aligned}$$

- Dynamical generator

$$\begin{aligned}
\mathcal{A} &= \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & 0 \end{bmatrix}, \quad \mathcal{A}_0 \;=\; \frac{\mathrm{d}^4}{\mathrm{d}\,x^4} \\
\mathcal{D}(\mathcal{A}_0) &= \left\{ f \in L_2[-1,\,1],\, \frac{\mathrm{d}^4 f}{\mathrm{d}x^4} \in L_2[-1,\,1],\; f(\pm 1) \;=\; f''(\pm 1) \;=\; 0 \right\}
\end{aligned}$$

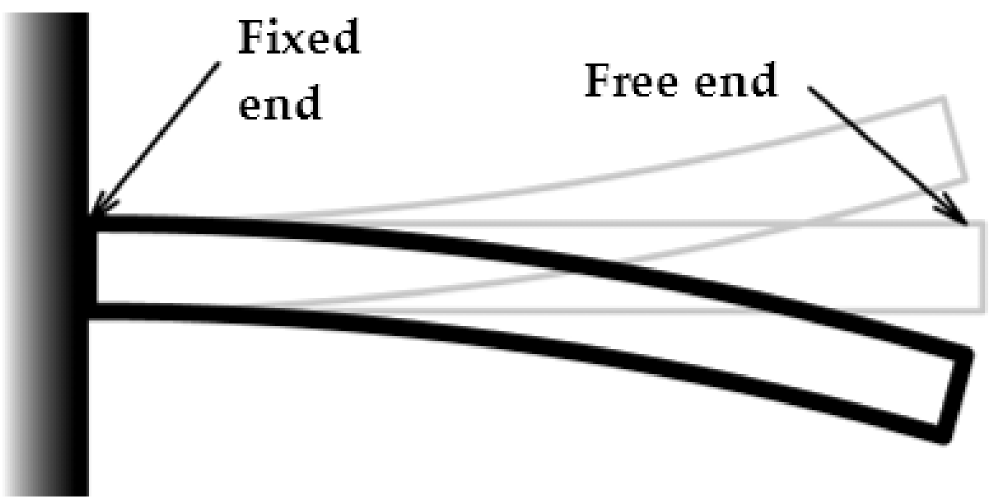
Simply supported and cantilever beams

- Simply supported beams



$$\begin{aligned} \phi(0,t) &= \phi(L,t) \;=\; 0 \\ \phi_{xx}(0,t) &= \phi_{xx}(L,t) \;=\; 0 \end{aligned}$$

- Cantilever beams



$$\begin{aligned} \phi(0,t) &= 0, \;\; \phi_x(0,t) \;=\; 0 \\ \phi_{xx}(L,t) &= 0, \;\; \phi_{xxx}(L,t) \;=\; 0 \end{aligned}$$

Square-root of a positive operator

- Self-adjoint operator $\mathcal{A}$: $\mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H}$ is

  ⋆ positive
  $$\langle \psi, \mathcal{A}\psi \rangle \;>\; 0 \;\text{ for all non-zero } \psi \in \mathcal{D}(\mathcal{A})$$

  ⋆ coercive: if there is $\epsilon > 0$ such that
  $$\langle \psi, \mathcal{A}\psi \rangle \;>\; \epsilon \, \|\psi\|^2 \;\text{ for all } \psi \in \mathcal{D}(\mathcal{A})$$

- Self-adjoint, non-negative $\mathcal{A}$ has a unique non-negative square-root $\mathcal{A}^{\frac{1}{2}}$

$$\begin{cases} \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \supset \mathcal{D}(\mathcal{A}) \\ \mathcal{A}^{\frac{1}{2}}\psi \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) & \text{for all } \psi \in \mathcal{D}(\mathcal{A}) \\ \mathcal{A}^{\frac{1}{2}}\mathcal{A}^{\frac{1}{2}}\psi = \mathcal{A}\psi & \text{for all } \psi \in \mathcal{D}(\mathcal{A}) \end{cases}$$

positive $\mathcal{A}$ $\;\Rightarrow\;$ positive $\mathcal{A}^{\frac{1}{2}}$

- Examples of positive, self-adjoint operators:

$$\mathcal{A}_0 \;=\; -\,\frac{\mathrm{d}^2}{\mathrm{d}\,x^2}, \;\; \mathcal{D}(\mathcal{A}_0) \;=\; \left\{ f \,\in\, L_2\,[-1,\,1],\; \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \in L_2\,[-1,\,1],\; f(\pm 1) \;=\; 0 \right\}$$

$$\mathcal{A}_0 \;=\; \frac{\mathrm{d}^4}{\mathrm{d}\,x^4}, \;\; \mathcal{D}(\mathcal{A}_0) \;=\; \left\{ f \,\in\, L_2\,[-1,\,1],\; \frac{\mathrm{d}^4 f}{\mathrm{d}x^4} \in L_2\,[-1,\,1],\; f(\pm 1) \;=\; f''(\pm 1) \;=\; 0 \right\}$$

$\mathcal{D}(\mathcal{A}_0^{\frac{1}{2}})$ – determined from the following requirement:

$$\left\langle \mathcal{A}_0^{\frac{1}{2}}\, f, \mathcal{A}_0^{\frac{1}{2}}\, g \right\rangle \;=\; \langle f, \mathcal{A}_0\, g\rangle\,, \;\; \text{for all } g \,\in\, \mathcal{D}(\mathcal{A}_0)$$

- For beam (wave left for homework):

$$\mathcal{A}_0^{\frac{1}{2}} \;=\; -\,\frac{\mathrm{d}^2}{\mathrm{d}\,x^2}, \;\; \mathcal{D}(\mathcal{A}_0^{\frac{1}{2}}) \;=\; \left\{ f \,\in\, L_2\,[-1,\,1],\; \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \in L_2\,[-1,\,1],\; f(\pm 1) \;=\; 0 \right\}$$

Abstract evolution equation

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & -a_1 I \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}$$

Hilbert space:

$$\mathbb{H} = \begin{bmatrix} \mathcal{D}(\mathcal{A}_0^{\frac{1}{2}}) \\ L_2[-1, 1] \end{bmatrix}$$

Inner product:

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle_e &= \left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_e \\ &= \left\langle \mathcal{A}_0^{\frac{1}{2}} f_1, \mathcal{A}_0^{\frac{1}{2}} f_2 \right\rangle + \langle g_1, g_2 \rangle \end{aligned}$$

Energy:

$$E(t) = \begin{cases} \dfrac{1}{2} \langle \psi_{1x}, \psi_{1x} \rangle + \dfrac{1}{2} \langle \psi_2, \psi_2 \rangle & \text{wave} \\ \dfrac{1}{2} \langle \psi_{1xx}, \psi_{1xx} \rangle + \dfrac{1}{2} \langle \psi_2, \psi_2 \rangle & \text{beam} \end{cases}$$

- Adjoint of $\mathcal{A}$ (w.r.t. $\langle\cdot,\cdot\rangle_e$):

$$\mathcal{A} \;=\; \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & -a_1 I \end{bmatrix} \;\Rightarrow\; \mathcal{A}^\dagger \;=\; \begin{bmatrix} 0 & -I \\ \mathcal{A}_0 & -a_1 I \end{bmatrix},\; \mathcal{D}(\mathcal{A}^\dagger) \;=\; \mathcal{D}(\mathcal{A}) \;=\; \begin{bmatrix} \mathcal{D}(\mathcal{A}_0) \\ \mathcal{D}(\mathcal{A}_0^{\frac{1}{2}}) \end{bmatrix}$$

- In class:
  - $\star$ well-posedness on $\mathbb{H} = \begin{bmatrix} \mathcal{D}(\mathcal{A}_0^{\frac{1}{2}}) \\ L_2[-1,1] \end{bmatrix}$ using Lumer-Phillips
  - $\star$ spectral decomposition of $\mathcal{A}$ for the undamped wave equation
  - $\star$ solution to the undamped wave equation
  - $\star$ mention different forms of internal damping in beams

Spectral decomposition of the undamped wave equation

$$\begin{bmatrix} 0 & I \\ \partial_{xx} & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \lambda \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \quad \Rightarrow \quad \left\{ \begin{array}{rcl} \psi_2 & = & \lambda\,\psi_1 \\ \psi_1'' & = & \lambda\,\psi_2 \\ 0 & = & \psi_1(\pm 1) \end{array} \right.$$

- Showed:

$$\left. \begin{array}{rcl} \psi_1'' & = & \lambda^2\,\psi_1 \\ 0 & = & \psi_1(\pm 1) \end{array} \right\} \xrightarrow{n\,\in\,\mathbb{N}} \left\{ \begin{array}{rclrcl} \lambda_n & = & +\mathrm{j}\,\dfrac{n\pi}{2}, & v_n(x) & = & \begin{bmatrix} (1/\lambda_n)\,\phi_n(x) \\ \phi_n(x) \end{bmatrix} \\ \lambda_{-n} & = & -\mathrm{j}\,\dfrac{n\pi}{2}, & v_{-n}(x) & = & \begin{bmatrix} (1/\lambda_n)\,\phi_n(x) \\ -\phi_n(x) \end{bmatrix} \\ & & & \phi_n(x) & = & \sin\left(\dfrac{n\pi}{2}\,(x\,+\,1)\right) \end{array} \right.$$

☞ $\{v_n\}_{n\,\in\,\mathbb{Z}\backslash 0}$ – complete orthonormal basis (w.r.t. $\langle\cdot,\cdot\rangle_e$)

Solution of the undamped wave equation

- Represent the solution as

$$
\begin{aligned}
\psi(x,t) &= \sum_{n=1}^{\infty} \alpha_n(t)\, v_n(x) \;+\; \sum_{n=1}^{\infty} \alpha_{-n}(t)\, v_{-n}(x) \\
&= \sum_{n=1}^{\infty} \begin{bmatrix} (\alpha_n(t) + \alpha_{-n}(t))\, \frac{1}{\lambda_n}\, \phi_n(x) \\ (\alpha_n(t) - \alpha_{-n}(t))\, \phi_n(x) \end{bmatrix} \\
&= \sum_{n=1}^{\infty} \begin{bmatrix} a_n(t)\, \frac{1}{\lambda_n}\, \phi_n(x) \\ b_n(t)\, \phi_n(x) \end{bmatrix} \quad \Rightarrow \quad \{a_n(t) \in \mathrm{j}\,\mathbb{R},\; b_n(t) \in \mathbb{R}\}
\end{aligned}
$$

- Substitute into the evolution model

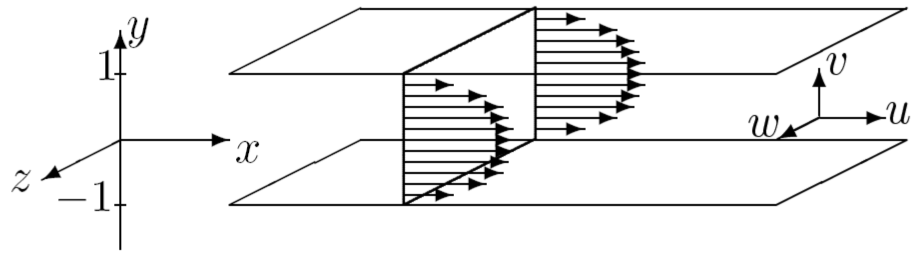
$$
\left.
\begin{aligned}
\dot{\alpha}_n(t) &= +\mathrm{j}\,\frac{n\pi}{2}\,\alpha_n(t) \\
\dot{\alpha}_{-n}(t) &= -\mathrm{j}\,\frac{n\pi}{2}\,\alpha_{-n}(t)
\end{aligned}
\right\} \Rightarrow
\begin{aligned}
\begin{bmatrix} \dot{a}_n(t) \\ \dot{b}_n(t) \end{bmatrix} &= \begin{bmatrix} 0 & \mathrm{j}n\pi/2 \\ \mathrm{j}n\pi/2 & 0 \end{bmatrix} \begin{bmatrix} a_n(t) \\ b_n(t) \end{bmatrix} \\
\begin{bmatrix} a_n(t) \\ b_n(t) \end{bmatrix} &= \begin{bmatrix} \cos\left(\frac{n\pi}{2}\,t\right) & \mathrm{j}\sin\left(\frac{n\pi}{2}\,t\right) \\ \mathrm{j}\sin\left(\frac{n\pi}{2}\,t\right) & \cos\left(\frac{n\pi}{2}\,t\right) \end{bmatrix} \begin{bmatrix} a_n(0) \\ b_n(0) \end{bmatrix}
\end{aligned}
$$

Lectures 13 & 14: . . . and a bit of fluids

- Themes:
  - ⋆ Linearized Navier-Stokes (NS) equations in a channel flow
  - ⋆ Inner product that induces kinetic energy
  - ⋆ Non-normal nature of the dynamical generator
  - ⋆ Riesz spectral basis

- Approach: informal discussion using tools that we've learned so far (more later in the course)

Channel flow



- Steady-state solution: $\begin{bmatrix} U(y) & 0 & 0 \end{bmatrix}^T$

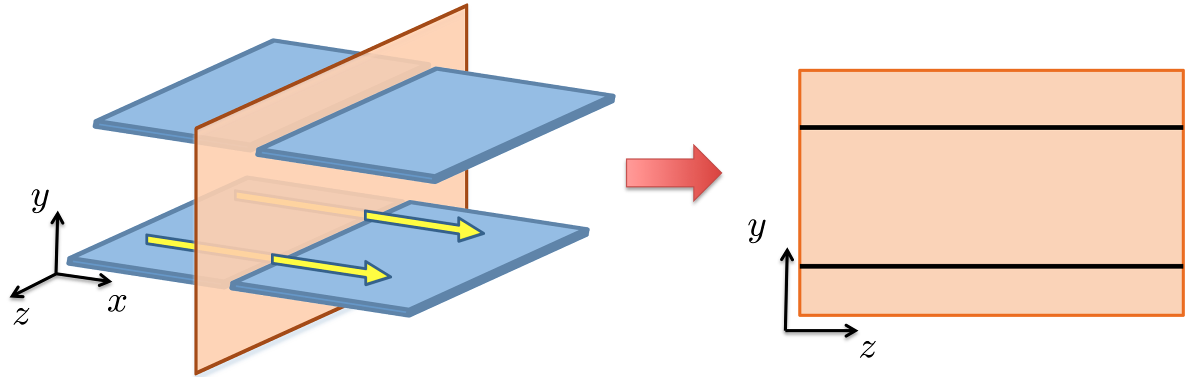
- Linearized NS and continuity equations

$$
\begin{aligned}
u_t \;+\; U(y)\, u_x \;+\; U'(y)\, v \;&=\; -\, p_x \;+\; \frac{1}{Re} \Delta\, u \\
v_t \;+\; U(y)\, v_x \;&=\; -\, p_y \;+\; \frac{1}{Re} \Delta\, v \\
w_t \;+\; U(y)\, w_x \;&=\; -\, p_z \;+\; \frac{1}{Re} \Delta\, w \\
u_x \;+\; v_y \;+\; w_z \;&=\; 0
\end{aligned}
$$

$$
U(y) \;=\; \begin{cases} 1 \,-\, y^2, & \text{pressure driven flow} \\ y, & \text{shear driven flow} \end{cases}
$$

$$
U'(y) \;=\; \frac{\mathrm{d}\, U(y)}{\mathrm{d}\, y} \qquad\qquad \Delta \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
$$

Streamwise constant fluctuations



- Set $(\cdot)_x = 0$

$$
\begin{aligned}
u_t &= -U'(y)\, v + \frac{1}{Re} \Delta u \\
v_t &= -p_y + \frac{1}{Re} \Delta v \\
w_t &= -p_z + \frac{1}{Re} \Delta w \\
0 &= v_y + w_z
\end{aligned}
$$

⋆ Define: stream-function in the (y, z)-plane $\{v = \psi_z, \; w = -\psi_y\}$

⋆ Eliminate pressure from the equations

⋆ Rewrite equations in terms of $\phi = \begin{bmatrix} \psi & u \end{bmatrix}^T$

Evolution model

$$\begin{bmatrix} \psi_t(t) \\ u_t(t) \end{bmatrix} = \begin{bmatrix} (1/Re)\,\mathcal{L} & 0 \\ \mathcal{C}_p & (1/Re)\,\mathcal{S} \end{bmatrix} \begin{bmatrix} \psi(t) \\ u(t) \end{bmatrix}$$

$$\text{Orr-Sommerfeld:} \quad \mathcal{L} = \Delta^{-1}\Delta^2$$

$$\text{Squire:} \quad \mathcal{S} = \Delta$$

$$\text{Coupling:} \quad \mathcal{C}_p = -U'(y)\,\partial_z$$

- After Fourier transform in z

$$\text{Laplacian:} \quad \Delta = \partial_{yy} - k_z^2$$

$$\text{"Square of Laplacian":} \quad \Delta^2 = \partial_{yyyy} - 2\,k_z^2\,\partial_{yy} + k_z^4$$

$$\text{Coupling:} \quad \mathcal{C}_p = -\mathrm{j}k_z\,U'(y)$$

Boundary conditions:

⋆ Dirichlet: $u(y = \pm 1, k_z, t) = 0$

⋆ Dirichlet and Neumann: $\psi(y = \pm 1, k_z, t) = \psi_y(y = \pm 1, k_z, t) = 0$

- Re-scale time: $\tau = t/Re$

$$\begin{bmatrix} \psi_\tau(\tau) \\ u_\tau(\tau) \end{bmatrix} = \begin{bmatrix} \mathcal{L} & 0 \\ Re\,\mathcal{C}_p & \mathcal{S} \end{bmatrix} \begin{bmatrix} \psi(\tau) \\ u(\tau) \end{bmatrix}$$

Inner product:

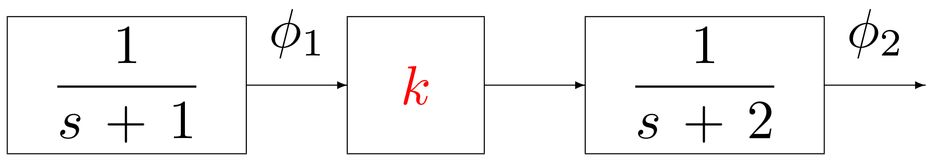
$$\begin{aligned} \langle \phi_1, \phi_2 \rangle_e &= \left\langle \begin{bmatrix} \psi_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} \psi_2 \\ u_2 \end{bmatrix} \right\rangle_e \\ &= \left\langle \begin{bmatrix} \psi_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} -\Delta & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \psi_2 \\ u_2 \end{bmatrix} \right\rangle \\ &= \langle \psi_1, -\Delta\,\psi_2 \rangle \;+\; \langle u_1, u_2 \rangle \end{aligned}$$

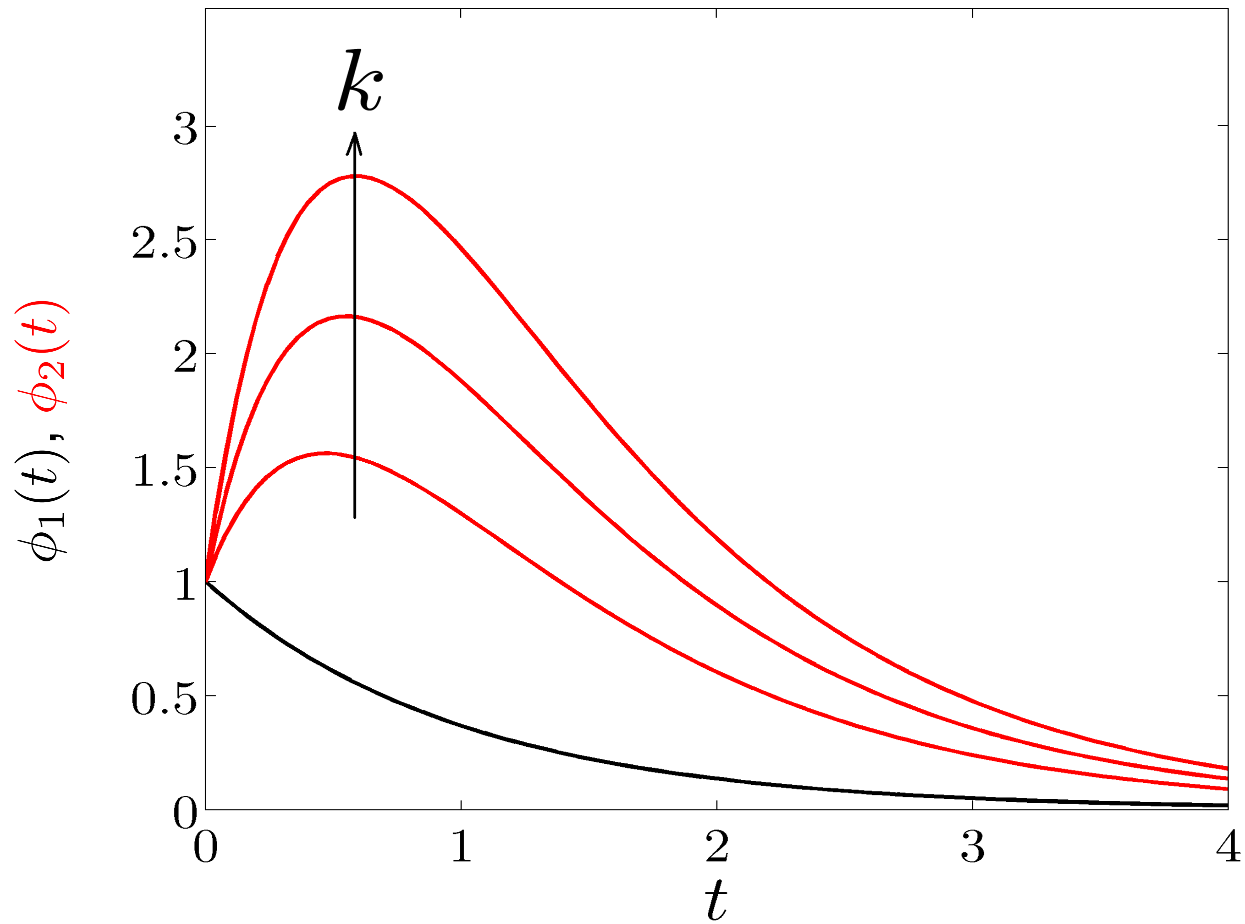
Energy:

$$\begin{aligned} E &= \frac{1}{2} \left(\langle u, u \rangle \;+\; \langle v, v \rangle \;+\; \langle w, w \rangle \right) \\ &= \frac{1}{2} \left(\langle u, u \rangle \;+\; \langle \psi, -\Delta\,\psi \rangle \right) \end{aligned}$$

A finite dimensional example

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$





$$\dot{\phi}(t) \;=\; A\,\phi(t), \qquad A\,A^* \;\neq\; A^*\,A$$

Let A have a full set of linearly independent e-vectors

$$A\,v_i \;=\; \lambda_i\,v_i \quad\Leftrightarrow\quad A\underbrace{\left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]}_{V} \;=\; \underbrace{\left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]}_{V}\underbrace{\left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array}\right]}_{\Lambda}$$

$$A^*\,w_i \;=\; \bar{\lambda}_i\,w_i \quad\Leftrightarrow\quad A^*\underbrace{\left[\begin{array}{ccc} w_1 & \cdots & w_n \end{array}\right]}_{W} \;=\; \underbrace{\left[\begin{array}{ccc} w_1 & \cdots & w_n \end{array}\right]}_{W}\underbrace{\left[\begin{array}{ccc} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{array}\right]}_{\bar{\Lambda}}$$

choose w_i such that $w_i^*\,v_j \;=\; \delta_{ij}$

- A – diagonalizable: $$A \;=\; \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]\left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array}\right]\left[\begin{array}{c} w_1^* \\ \vdots \\ w_n^* \end{array}\right]$$

- Action of A on $f \in \mathbb{C}^n$

$$A\,f \;=\; \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix} f$$

$$= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1\, w_1^*\, f \\ \vdots \\ \lambda_n\, w_n^*\, f \end{bmatrix}$$

$$= \lambda_1\, v_1\, w_1^*\, f \,+\, \cdots \,+\, \lambda_n\, v_n\, w_n^*\, f$$

$$= \sum_{i\,=\,1}^{n} \lambda_i\, v_i\, \langle w_i, f \rangle$$

- Solution to $\dot{\phi}(t) \;=\; A\,\phi(t)$

$$\phi(t) \;=\; \mathrm{e}^{A\,t}\,\phi(0) \;=\; \sum_{i\,=\,1}^{n} \mathrm{e}^{\lambda_i\,t}\, v_i\, \langle w_i, \phi(0) \rangle$$

- E-value decomposition of $A = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}$

$$\{\lambda_1 = -1, \; \lambda_2 = -2\} \quad \begin{array}{l} \left\{ v_1 = \dfrac{1}{\sqrt{1 + k^2}} \begin{bmatrix} 1 \\ k \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \\ \left\{ w_1 = \begin{bmatrix} \sqrt{1 + k^2} \\ 0 \end{bmatrix}, \; w_2 = \begin{bmatrix} -k \\ 1 \end{bmatrix} \right\} \end{array}$$

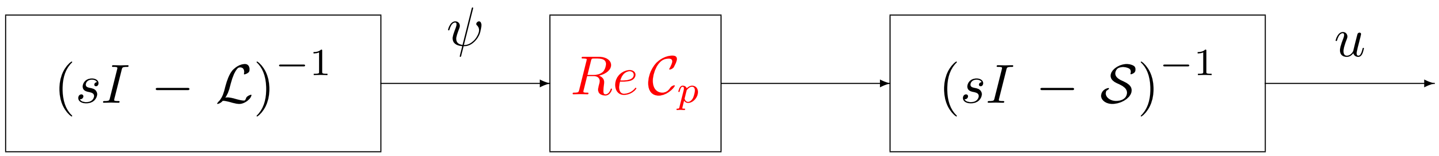
- Solution to $\dot{\phi}(t) = A \, \phi(t)$

$$\begin{aligned} \phi(t) &= \left(\mathrm{e}^{-t} \, v_1 \, w_1^* \; + \; \mathrm{e}^{-2t} \, v_2 \, w_2^* \right) \phi(0) \\ &= \begin{bmatrix} \mathrm{e}^{-t} & 0 \\ k \left(\mathrm{e}^{-t} - \mathrm{e}^{-2t} \right) & \mathrm{e}^{-2t} \end{bmatrix} \begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix} \end{aligned}$$

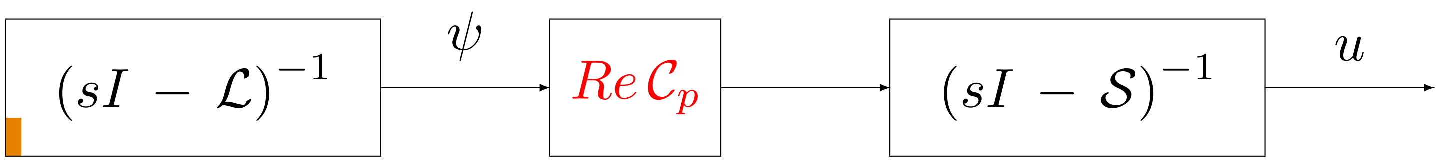
Back to fluids

$$\begin{bmatrix} \psi_\tau(\tau) \\ u_\tau(\tau) \end{bmatrix} = \begin{bmatrix} \mathcal{L} & 0 \\ Re\,\mathcal{C}_p & \mathcal{S} \end{bmatrix} \begin{bmatrix} \psi(\tau) \\ u(\tau) \end{bmatrix}$$

Orr-Sommerfeld **Coupling** **Squire**



'glorified diffusion' **vortex tilting** **viscous dissipation**



- Adjoint of $\mathcal{A}$ (w.r.t. $\langle \cdot, \cdot \rangle_e$):

$$\mathcal{A} = \begin{bmatrix} \mathcal{L} & 0 \\ Re\,\mathcal{C}_p & \mathcal{S} \end{bmatrix} \Rightarrow \left\{ \mathcal{A}^\dagger = \begin{bmatrix} \mathcal{L} & Re\,\mathcal{C}_p^\dagger \\ 0 & \mathcal{S} \end{bmatrix}, \; \mathcal{C}_p^\dagger = -\mathrm{j}k_z\,\Delta^{-1}U'(y) \right\}$$

☞ $\mathcal{A}$: not normal $\Leftrightarrow$ not diagonalizable by a unitary coordinate transformation

Spectral decomposition of $\mathcal{A}$ and $\mathcal{A}^\dagger$

$$\left[\begin{array}{cc} \mathcal{L} & 0 \\ Re\,\mathcal{C}_p & \mathcal{S} \end{array}\right] \left[\begin{array}{c} \psi \\ u \end{array}\right] = \lambda \left[\begin{array}{c} \psi \\ u \end{array}\right] \;\Rightarrow\; \left\{\begin{array}{rcl} \mathcal{L}\,\psi & = & \lambda\,\psi \\ \mathcal{S}\,u & = & \lambda\,u \;-\; Re\,\mathcal{C}_p\,\psi \end{array}\right.$$

- Two sets of eigenvalues

$$(\lambda I \,-\, \mathcal{L}) \text{ not one-to-one} \;\Rightarrow\; \left\{\lambda_{os},\, \left[\begin{array}{c} \psi_{os} \\ u_{os} \end{array}\right]\right\}$$

$$(\lambda I \,-\, \mathcal{S}) \text{ not one-to-one} \;\Rightarrow\; \left\{\lambda_{sq},\, \left[\begin{array}{c} 0 \\ u_{sq} \end{array}\right]\right\}$$

- Homework:
  - ⋆ fill in details for the e-value decomposition of $\mathcal{A}$ and $\mathcal{A}^\dagger$

Orr-Sommerfeld: $$\left\{\begin{array}{rcl} \mathcal{L}\,\psi_{os} & = & \lambda_{os}\,\psi_{os}, \qquad \psi_{os}(\pm 1) \;=\; \psi_{os}'(\pm 1) \;=\; 0 \\ \mathcal{S}\,u_{os} & = & \lambda_{os}\,u_{os} \;-\; Re\,\mathcal{C}_p\,\psi_{os}, \qquad u_{os}(\pm 1) \;=\; 0 \end{array}\right.$$

Squire: $$\left\{\lambda_{sq} \;=\; -\left(\left(\frac{n\pi}{2}\right)^2 \,+\, k_z^2\right),\; \left[\begin{array}{c} 0 \\ u_{sq} \end{array}\right] \;=\; \left[\begin{array}{c} 0 \\ \sin\left(\frac{n\pi}{2}(y \,+\, 1)\right) \end{array}\right]\right\}$$

  - ⋆ show that $\mathcal{A}$ is a Riesz-spectral operator

Riesz-spectral operator

- Action of $\mathcal{A}$ on $f \in \mathbb{H}$

$$[\mathcal{A} f](y) = \sum_{n=1}^{\infty} \lambda_{os,n} v_{os,n}(y) \langle w_{os,n}, f \rangle_e + \sum_{n=1}^{\infty} \lambda_{sq,n} v_{sq,n}(y) \langle w_{sq,n}, f \rangle_e$$

- Solution to $\phi_\tau(\tau) = \mathcal{A} \phi(\tau), \;\; \phi(0) = f$

$$\phi(y,\tau) = \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_{os,n} \tau} v_{os,n}(y) \langle w_{os,n}, f \rangle_e + \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_{sq,n} \tau} v_{sq,n}(y) \langle w_{sq,n}, f \rangle_e$$

- Dependence of $u(y, k_z, \tau)$ on $\psi(y, k_z, 0) = \sum_{n=1}^{\infty} \alpha_n(k_z)\, \psi_{os,n}(y, k_z)$

$$u(y, k_z, \tau) \;=\; Re \sum_{n\,=\,1}^{\infty} \Bigg(\alpha_n \, \mathrm{e}^{\lambda_{os,n}\,\tau}\, u_{os,n}(y, k_z, \tau) \; - \; \sum_{m\,=\,1}^{\infty} \frac{\alpha_m}{\lambda_{os,m} \, - \, \lambda_{sq,n}} \, \mathrm{e}^{\lambda_{sq,n}\,\tau}\, u_{sq,n}(y, k_z, \tau)\, \langle u_{sq,n}, \mathcal{C}_p\, \psi_{os,m} \rangle \Bigg)$$

Orr-Sommerfeld:
$$\begin{cases} \mathcal{L}\,\psi_{os} \;=\; \lambda_{os}\,\psi_{os}, & \psi_{os}(\pm 1) \;=\; \psi_{os}'(\pm 1) \;=\; 0 \\ \mathcal{S}\,u_{os} \;=\; \lambda_{os}\,u_{os} \, - \, \mathcal{C}_p\,\psi_{os}, & u_{os}(\pm 1) \;=\; 0 \end{cases}$$

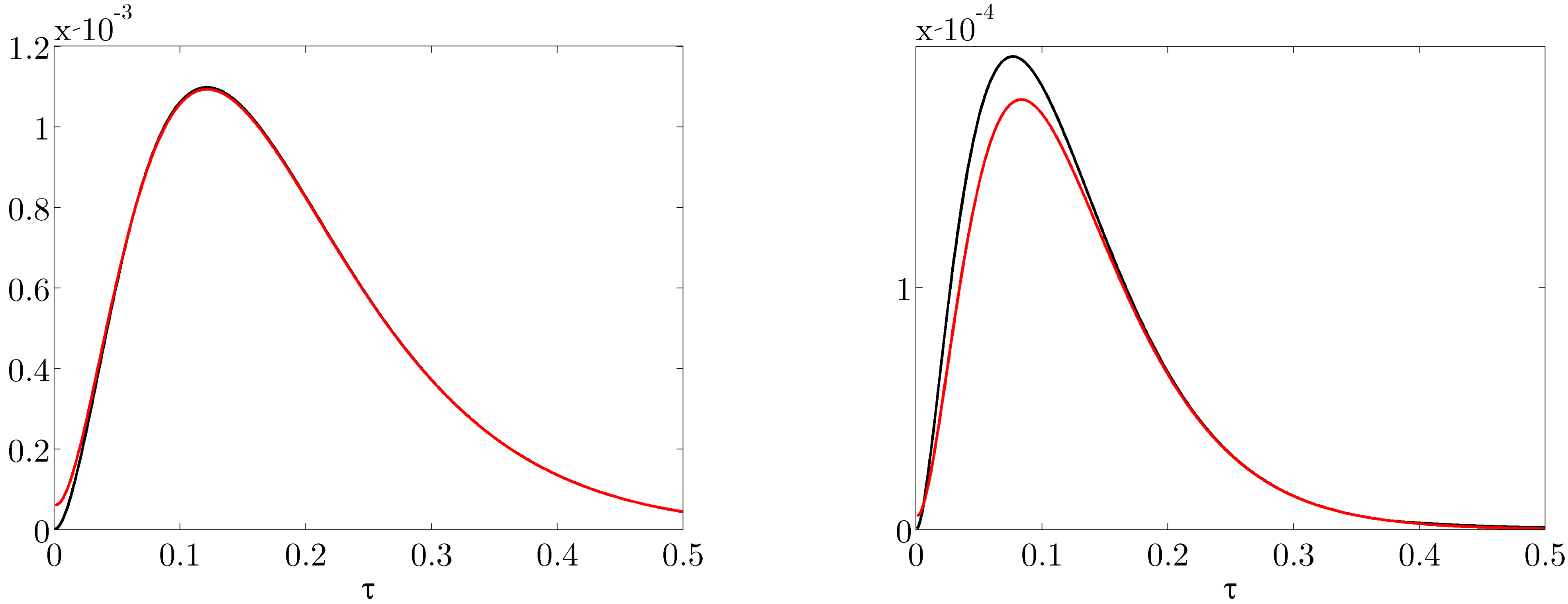
Squire:
$$\left\{ \lambda_{sq} \;=\; -\left(\left(\frac{n\pi}{2} \right)^2 + \, k_z^2 \right), \;\; \begin{bmatrix} 0 \\ u_{sq} \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ \sin\left(\frac{n\pi}{2} (y \, + \, 1) \right) \end{bmatrix} \right.$$

Energy growth

- Worst case energy of u caused by the initial condition in ψ
  - $\star$ $Re = 1$, $k_z = 2$

shear-driven flow
(one OS and one Squire mode)

pressure-driven flow
(one OS and two Squire modes)



Lecture 15: Systems with inputs

- Input types
  - ⋆ Additive inputs
  - ⋆ Boundary inputs
- Input-output mappings
  - ⋆ Transfer function
  - ⋆ Frequency response
  - ⋆ Impulse response
- Abstract evolution equation for boundary control systems
  - ⋆ Objective: bring system into a form that resembles standard formulation
- Two point boundary value problems

Additive inputs

- Example: diffusion equation on $L_2[-1,1]$ with Dirichlet BCs

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) + u(x,t) \\ \phi(x,0) &= \phi_0(x) \\ \phi(\pm 1,t) &= 0 \end{aligned}$$

- Abstract evolution equation

$$\begin{aligned} \psi_t(t) &= \mathcal{A}\,\psi(t) + u(t) \\ \mathcal{A} &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \mathcal{D}(\mathcal{A}) = \{f \in L_2[-1,1],\ f'' \in L_2[-1,1],\ f(\pm 1) = 0\} \end{aligned}$$

- Solution

$$\psi(t) = \mathcal{T}(t)\,\psi(0) + \int_0^t \mathcal{T}(t-\tau)\,u(\tau)\,\mathrm{d}\tau$$

$\mathcal{T}(t)$: C_0-semigroup generated by $\mathcal{A}$

Input-output maps

$$\begin{aligned} \psi_t(t) &= \mathcal{A}\,\psi(t) \,+\, \mathcal{B}\,u(t) \\ \phi(t) &= \mathcal{C}\,\psi(t) \end{aligned}$$

- Underlying operators: $\left\{ \begin{array}{ll} \mathcal{A}: & \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H} \\ \mathcal{B}: & \mathbb{U} \longrightarrow \mathbb{H} \\ \mathcal{C}: & \mathbb{H} \longrightarrow \mathbb{Y} \end{array} \right.$

- Input-output mapping

$$\phi(t) \;=\; [\,\mathcal{H}\,u\,]\,(t) \;=\; \int_0^t \mathcal{C}\,\mathcal{T}(t\,-\,\tau)\,\mathcal{B}\,u(\tau)\,\mathrm{d}\tau$$

  - ⋆ Impulse response

$$\mathcal{H}(t) \;=\; (\mathcal{C}\,\mathcal{T}(t)\,\mathcal{B})\,\mathbb{1}(t)$$

  - ⋆ Transfer function

$$\mathcal{H}(s) \;=\; \mathcal{C}\,(sI\,-\,\mathcal{A})^{-1}\,\mathcal{B}$$

  - ⋆ Frequency response

$$\mathcal{H}(\mathrm{j}\omega) \;=\; \mathcal{C}\,(\mathrm{j}\omega I\,-\,\mathcal{A})^{-1}\,\mathcal{B}$$

An example

$$\left.\begin{array}{rcl}\phi_t(x,t) & = & \phi_{xx}(x,t) \,+\, u(x,t) \\ \phi(\pm 1,t) & = & 0\end{array}\right\} \xrightarrow[\text{transform}]{\text{Laplace}} \left\{\begin{array}{rcl}\phi''(x,s) & = & s\,\phi(x,s) \,-\, u(x,s) \\ \phi(\pm 1,s) & = & 0\end{array}\right.$$

- Spatial realization of $\mathcal{H}(s)$ (with $\psi_1 = \phi$, $\psi_2 = \phi'$)

$$\begin{aligned}
\begin{bmatrix}\psi_1'(x,s)\\ \psi_2'(x,s)\end{bmatrix} &= \begin{bmatrix}0 & 1\\ s & 0\end{bmatrix}\begin{bmatrix}\psi_1(x,s)\\ \psi_2(x,s)\end{bmatrix} + \begin{bmatrix}0\\ -1\end{bmatrix} u(x,s)\\
\phi(x,s) &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\psi_1(x,s)\\ \psi_2(x,s)\end{bmatrix}\\
0 &= \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\psi_1(-1,s)\\ \psi_2(-1,s)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}\psi_1(1,s)\\ \psi_2(1,s)\end{bmatrix}
\end{aligned}$$

- Two point boundary value problem

$$\begin{aligned}
\psi'(x) &= A(x)\,\psi(x) \,+\, B(x)\,u(x)\\
\phi(x) &= C(x)\,\psi(x)\\
0 &= N_a\,\psi(a) \,+\, N_b\,\psi(b)
\end{aligned}$$

Boundary control

- Example: diffusion equation on $L_2\,[-1,1]$

$$\left.\begin{array}{rcl} \phi_t(x,t) & = & \phi_{xx}(x,t) \,+\, d(x,t) \\ \phi(-1,t) & = & u(t) \\ \phi(+1,t) & = & 0 \end{array}\right\} \xrightarrow[\text{transform}]{\text{Laplace}} \left\{\begin{array}{rcl} \phi''(x,s) & = & s\,\phi(x,s) \,-\, d(x,s) \\ \phi(-1,s) & = & u(s) \\ \phi(+1,s) & = & 0 \end{array}\right.$$

- Spatial realization of $\mathcal{H}(s)$ (with $\psi_1 = \phi$, $\psi_2 = \phi'$)

$$\begin{aligned} \begin{bmatrix} \psi_1'(x,s) \\ \psi_2'(x,s) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} d(x,s) \\ \phi(x,s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} \\ \begin{bmatrix} u(s) \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(-1,s) \\ \psi_2(-1,s) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(1,s) \\ \psi_2(1,s) \end{bmatrix} \end{aligned}$$

- Two point boundary value problem

$$\begin{aligned} \psi'(x) &= A(x)\,\psi(x) \,+\, B(x)\,d(x) \\ \phi(x) &= C(x)\,\psi(x) \\ \nu &= N_a\,\psi(a) \,+\, N_b\,\psi(b) \end{aligned}$$

Abstract evolution equation for systems with boundary inputs

$$\begin{aligned}\phi_t(x,t) &= \phi_{xx}(x,t) + d(x,t)\\ \phi(-1,t) &= u(t)\\ \phi(+1,t) &= 0\end{aligned}$$

- Problem: control doesn't enter additively into the equation

- Coordinate transformation

$$\psi(x,t) = \phi(x,t) - f(x)\,u(t)$$

  - ⋆ Choose $f(x)$ to obtain homogeneous boundary conditions $\psi(\pm 1,t) = 0$
  - ⋆ Many possible choices

Conditions for selection of f:

$$\{f(-1) = 1,\ f(1) = 0\} \xrightarrow{\text{simple option}} f(x) = \frac{1 - x}{2}$$

- In new coordinates:

$$\begin{aligned}\phi_t(x,t) &= \phi_{xx}(x,t) + d(x,t)\\ \phi(-1,t) &= u(t)\\ \phi(+1,t) &= 0\end{aligned}$$

$$\Big\downarrow \; \phi(x,t) = \psi(x,t) + f(x)\,u(t)$$

$$\begin{aligned}\psi_t(x,t) + f(x)\,\dot{u}(t) &= \psi_{xx}(x,t) + f''(x)\,u(t) + d(x,t)\\ \psi(\pm 1,t) &= 0\end{aligned}$$

- New input: $v(t) = \dot{u}(t)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\psi(t)\\ u(t)\end{bmatrix} &= \begin{bmatrix}\mathcal{A}_0 & f''\\ 0 & 0\end{bmatrix}\begin{bmatrix}\psi(t)\\ u(t)\end{bmatrix} + \begin{bmatrix}I\\ 0\end{bmatrix} d(t) + \begin{bmatrix}-f\\ I\end{bmatrix} v(t)\\ \phi(t) &= \begin{bmatrix}I & f\end{bmatrix}\begin{bmatrix}\psi(t)\\ u(t)\end{bmatrix}\end{aligned}$$

$$\mathcal{A}_0 = \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \;\; \mathcal{D}(\mathcal{A}_0) = \{f \in L_2[-1,\,1],\; f'' \in L_2[-1,\,1],\; f(\pm 1) = 0\}$$

Two point boundary value problems

$$
\begin{aligned}
\psi'(x) &= A(x)\,\psi(x) \;+\; B(x)\,d(x)\\
\phi(x) &= C(x)\,\psi(x)\\
\nu &= N_a\,\psi(a) \;+\; N_b\,\psi(b)
\end{aligned}
$$

- Solution:

$$
\begin{aligned}
\phi(x) \;=\; & C(x)\,\Phi(x,a)\,(N_a \,+\, N_b\,\Phi(b,a))^{-1}\,\nu \;+\; C(x)\int_a^x \Phi(x,\xi)\,B(\xi)\,d(\xi)\,\mathrm{d}\xi \;-\\
& C(x)\,\Phi(x,a)\,(N_a \,+\, N_b\,\Phi(b,a))^{-1}\,N_b\int_a^b \Phi(b,\xi)\,B(\xi)\,d(\xi)\,\mathrm{d}\xi
\end{aligned}
$$

$\Phi(x,\xi)$: the state transition matrix of $A(x)$

$$
\frac{\mathrm{d}\Phi(x,\xi)}{\mathrm{d}x} \;=\; A(x)\,\Phi(x,\xi), \quad \Phi(\xi,\xi) \;=\; I
$$

For systems with $A \,\neq\, A(x)$:

$$
\Phi(x,\xi) \;=\; \mathrm{e}^{A(x-\xi)}
$$

Examples

- Heat equation with boundary actuation

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) \\ \phi(-1,t) &= u(t) \\ \phi(+1,t) &= 0 \end{aligned}$$

$$\Big\downarrow \text{Laplace trasform}$$

$$\begin{aligned} \begin{bmatrix} \psi_1'(x,s) \\ \psi_2'(x,s) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} \\ \phi(x,s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} \\ \begin{bmatrix} u(s) \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(-1,s) \\ \psi_2(-1,s) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(1,s) \\ \psi_2(1,s) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \phi(x,s) &= C\,\mathrm{e}^{A(s)(x-a)} \left(N_a + N_b\,\mathrm{e}^{A(s)(b-a)}\right)^{-1} \nu(s) \\ &= \frac{\sinh\left(\sqrt{s}\,(1-x)\right)}{\sinh\left(2\sqrt{s}\right)}\, u(s) \\ &= \left(\frac{1-x}{2} - \sum_{n=1}^{\infty} \frac{2}{n\pi} \frac{s}{s + (n\pi/2)^2}\, v_n(x)\right) u(s) \end{aligned}$$

- Eigenvalue problem for streamwise constant linearized NS equations

Orr-Sommerfeld:
$$\left\{ \begin{array}{rclrcl} \mathcal{L}\,\psi_{os} & = & \lambda_{os}\,\psi_{os}, & \psi_{os}(\pm 1) & = \; \psi'_{os}(\pm 1) & = \; 0 \\ \mathcal{S}\,u_{os} & = & \lambda_{os}\,u_{os} \; - \; \mathcal{C}_p\,\psi_{os}, & u_{os}(\pm 1) & = \; 0 & \end{array} \right.$$

$$\downarrow$$

$$\left\{ \begin{array}{rclrcl} \Delta^2\,\psi_{os} & = & \lambda_{os}\,\Delta\,\psi_{os}, & \psi_{os}(\pm 1) & = \; \psi'_{os}(\pm 1) & = \; 0 \\ \Delta\,u_{os} & = & \lambda_{os}\,u_{os} \; - \; \mathrm{j}k_z U'(y)\,\psi_{os}, & u_{os}(\pm 1) & = \; 0 & \end{array} \right.$$

Two point boundary value problem for u_{os}:

$$\begin{bmatrix} x'_1(y,k_z) \\ x'_2(y,k_z) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \lambda_{os} + k_z^2 & 0 \end{bmatrix} \begin{bmatrix} x_1(y,k_z) \\ x_2(y,k_z) \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathrm{j}k_z U'(y) \end{bmatrix} \psi_{os}(y,k_z)$$

$$u_{os}(y,k_z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(y,k_z) \\ x_2(y,k_z) \end{bmatrix}$$

$$0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(-1,k_z) \\ x_2(-1,k_z) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(1,k_z) \\ x_2(1,k_z) \end{bmatrix}$$

Lecture 16: Controllability and observability

- Controllability
  - ⋆ Ability to steer state
- Observability
  - ⋆ Ability to estimate state
- Topics:
  - ⋆ Connections and differences with finite-dimensional case
  - ⋆ Exact vs. approximate controllability/observability
  - ⋆ Conditions for controllability/observability
  - ⋆ Gramians
  - ⋆ Operator Lyapunov equations

An example

- Diffusion equation on $L_2\,[-1, 1]$ with point actuation and sensing

$$
\begin{aligned}
\psi_t(x,t) &= \psi_{xx}(x,t) \; + \; b(x)\,u(t) \\
\phi(t) &= \int_{-1}^{1} c(x)\,\psi(x,t)\,\mathrm{d}x \\
\psi(x,0) &= \psi_0(x) \\
\psi(\pm 1,t) &= 0
\end{aligned}
$$

Control and sensing points x_c and x_s

$$
\begin{aligned}
b(x) &= \frac{1}{2\epsilon}\,\mathbb{1}_{[x_c - \epsilon,\, x_c + \epsilon]}\,(x) \\
c(x) &= \frac{1}{2\delta}\,\mathbb{1}_{[x_s - \delta,\, x_s + \delta]}\,(x) \\
\mathbb{1}_{[a,\,b]}\,(x) &= \begin{cases} 1, & x \in [a,\, b] \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

Controllability operator and Gramian

$$
\begin{aligned}
\psi_t(t) \;&=\; \mathcal{A}\,\psi(t) \,+\, \mathcal{B}\,u(t)\\
\mathcal{A}: &\quad \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \;\longrightarrow\; \mathbb{H}\\
\mathcal{B}: &\quad \mathbb{U} \;\longrightarrow\; \mathbb{H}
\end{aligned}
$$

- Controllability operator

$$\mathcal{R}_t : L_2\left([0,\, t]; \mathbb{U}\right) \;\longrightarrow\; \mathbb{H}$$

$$\psi(t) \;=\; [\,\mathcal{R}_t\, u\,]\,(t) \;=\; \int_0^t \mathcal{T}(t \,-\, \tau)\,\mathcal{B}\,u(\tau)\,\mathrm{d}\tau$$

⋆ Adjoint

$$\left[\mathcal{R}_t^\dagger\, \psi\right](\tau) \;=\; \mathcal{B}^\dagger\, \mathcal{T}^\dagger(t \,-\, \tau), \;\; \tau \,\in\, [0,\, t]$$

- Controllability Gramian

$$\mathcal{P}_t \;=\; \mathcal{R}_t\,\mathcal{R}_t^\dagger \;=\; \int_0^t \mathcal{T}(\tau)\,\mathcal{B}\,\mathcal{B}^\dagger\,\mathcal{T}^\dagger(\tau)\,\mathrm{d}\tau$$

Exact vs. approximate controllability

- Exact controllability on $[0,\, t]$

$$\mathrm{range}\,(\mathcal{R}_t) \;=\; \mathbb{H}$$

  - ⋆ rarely satisfied by infinite-dimensional systems
  - ⋆ never satisfied for systems with finite-dimensional $\mathbb{U}$

- Approximate controllability on $[0,\, t]$

$$\overline{\mathrm{range}\,(\mathcal{R}_t)} \;=\; \mathbb{H}$$

  - ⋆ reasonable notion of controllability for infinite-dimensional systems
  - ⋆ easily checkable conditions for Riesz-spectral systems

approximate controllability on $[0,\, t]$

$$\Updownarrow$$

$$\mathcal{P}_t \;>\; 0 \;\;\Leftrightarrow\;\; \{\langle \psi, \mathcal{P}_t\, \psi \rangle \;>\; 0, \text{ for all } 0 \neq \psi \in \mathbb{H}\}$$

or

$$\mathrm{null}\left(\mathcal{R}_t^{\dagger}\right) \;=\; 0 \;\;\Leftrightarrow\;\; \{\mathcal{B}^{\dagger}\, \mathcal{T}^{\dagger}(\tau)\, \psi \;=\; 0 \text{ on } [0,\, t] \;\Rightarrow\; \psi = 0\}$$

Observability operator and Gramian

$$
\begin{aligned}
\psi_t(t) &= \mathcal{A}\,\psi(t) \\
\phi(t) &= \mathcal{C}\,\psi(t)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}: &\quad \mathbb{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathbb{H} \\
\mathcal{C}: &\quad \mathbb{H} \longrightarrow \mathbb{Y}
\end{aligned}
$$

- Observability operator

$$
\begin{aligned}
\mathcal{O}_t : \mathbb{H} &\longrightarrow L_2\left([0,\, t]; \mathbb{Y}\right) \\
\phi(t) \;=\; \left[\, \mathcal{O}_t\, \psi(0) \,\right](t) &= \mathcal{C}\,\mathcal{T}(t)\,\psi(0)
\end{aligned}
$$

  ⋆ Adjoint

$$
\left[\mathcal{O}_t^{\dagger}\, \phi\right](t) \;=\; \int_0^t \mathcal{T}^{\dagger}(\tau)\,\mathcal{C}^{\dagger}\,\phi(\tau)\,\mathrm{d}\tau
$$

- Observability Gramian

$$
\mathcal{V}_t \;=\; \mathcal{O}_t^{\dagger}\,\mathcal{O}_t \;=\; \int_0^t \mathcal{T}^{\dagger}(\tau)\,\mathcal{C}^{\dagger}\,\mathcal{C}\,\mathcal{T}(\tau)\,\mathrm{d}\tau
$$

Exact vs. approximate observability

- Exact observability on $[0,\, t]$
  - ⋆ $\mathcal{O}_t$ one-to-one and $\mathcal{O}_t^{-1}$ bounded on the range of $\mathcal{O}_t$

- Approximate observability on $[0,\, t]$
  - ⋆ $\operatorname{null}(\mathcal{O}_t) = 0$

- $(\mathcal{A},\, \cdot\,, \mathcal{C})$ approximately obsv on $[0,\, t]$ $\Leftrightarrow$ $(\mathcal{A}^\dagger, \mathcal{C}^\dagger,\, \cdot\,)$ approximately ctrb on $[0,\, t]$

approximate observability on $[0,\, t]$

$$\Updownarrow$$

$$\mathcal{V}_t > 0 \;\;\Leftrightarrow\;\; \{\langle \psi, \mathcal{V}_t\, \psi \rangle > 0, \text{ for all } 0 \neq \psi \in \mathbb{H}\}$$

or

$$\operatorname{null}(\mathcal{O}_t) = 0 \;\;\Leftrightarrow\;\; \{\mathcal{C}\,\mathcal{T}(\tau)\,\psi = 0 \text{ on } [0,\, t] \;\Rightarrow\; \psi = 0\}$$

Infinite horizon Gramians

- Exponentially stable C_0-semigroup $\mathcal{T}(t)$

$$\exists \, M, \, \alpha \, > \, 0 \quad \Rightarrow \quad \|\mathcal{T}(t)\| \, \leq \, M \, \mathrm{e}^{-\alpha t}$$

- Extended (i.e., infinite horizon) Gramians

$$\begin{aligned} \mathcal{P} \; &= \; \mathcal{R}_\infty \, \mathcal{R}_\infty^\dagger \; = \; \int_0^\infty \mathcal{T}(\tau) \, \mathcal{B} \, \mathcal{B}^\dagger \, \mathcal{T}^\dagger(\tau) \, \mathrm{d}\tau \\ \mathcal{V} \; &= \; \mathcal{O}_\infty^\dagger \, \mathcal{O}_\infty \; = \; \int_0^\infty \mathcal{T}^\dagger(\tau) \, \mathcal{C}^\dagger \, \mathcal{C} \, \mathcal{T}(\tau) \, \mathrm{d}\tau \end{aligned}$$

- Approximate controllability

$$\mathcal{P} \, > \, 0 \quad \Leftrightarrow \quad \mathrm{null} \left(\mathcal{R}_\infty^\dagger \right) \; = \; 0$$

- Approximate observability

$$\mathcal{V} \, > \, 0 \quad \Leftrightarrow \quad \mathrm{null} \left(\mathcal{O}_\infty \right) \; = \; 0$$

Lyapunov equations

Controllability Gramian $\mathcal{P}$ – unique self-adjoint solution to:

$$\left\langle \mathcal{A}^\dagger \psi_1, \mathcal{P}\,\psi_2 \right\rangle \;+\; \left\langle \mathcal{P}\,\psi_1, \mathcal{A}^\dagger \psi_2 \right\rangle \;=\; -\left\langle \mathcal{B}^\dagger \psi_1, \mathcal{B}^\dagger \psi_2 \right\rangle \quad \text{for } \psi_1,\, \psi_2 \in \mathcal{D}\left(\mathcal{A}^\dagger\right)$$

$$\Updownarrow$$

$$\mathcal{P}\,\mathcal{D}\left(\mathcal{A}^\dagger\right) \subset \mathcal{D}\left(\mathcal{A}\right) \;\text{ and }\; \mathcal{A}\,\mathcal{P}\,\psi + \mathcal{P}\,\mathcal{A}^\dagger\,\psi \;=\; -\,\mathcal{B}\,\mathcal{B}^\dagger\,\psi \quad \text{for } \psi \in \mathcal{D}\left(\mathcal{A}^\dagger\right)$$

Observability Gramian $\mathcal{V}$ – unique self-adjoint solution to:

$$\langle \mathcal{A}\,\psi_1, \mathcal{V}\,\psi_2 \rangle \;+\; \langle \mathcal{V}\,\psi_1, \mathcal{A}\,\psi_2 \rangle \;=\; -\,\langle \mathcal{C}\,\psi_1, \mathcal{C}\,\psi_2 \rangle \quad \text{for } \psi_1,\, \psi_2 \in \mathcal{D}\left(\mathcal{A}\right)$$

$$\Updownarrow$$

$$\mathcal{V}\,\mathcal{D}\left(\mathcal{A}\right) \subset \mathcal{D}\left(\mathcal{A}^\dagger\right) \;\text{ and }\; \mathcal{A}^\dagger\,\mathcal{V}\,\psi + \mathcal{V}\,\mathcal{A}\,\psi \;=\; -\,\mathcal{C}^\dagger\,\mathcal{C}\,\psi \quad \text{for } \psi \in \mathcal{D}\left(\mathcal{A}\right)$$

Controllability of Riesz-spectral systems

$$\psi_t(x,t) \;=\; [\mathcal{A}\,\psi(\cdot,t)]\,(x) \,+\, \sum_{i\,=\,1}^{m} b_i(x)\,u_i(t)$$

modal controllability $\Leftrightarrow$ approximate controllability

$\mathcal{A}$ – Riesz-spectral operator with e-pair $\{(\lambda_n, v_n)\}_{n\,\in\,\mathbb{N}}$

$\{w_n\}_{n\,\in\,\mathbb{N}}$ – e-functions of $\mathcal{A}^\dagger$ s.t. $\langle w_n, v_m\rangle \;=\; \delta_{nm}$

$$[\mathcal{A}\,f]\,(x) \;=\; \sum_{n\,=\,1}^{\infty} \lambda_n\, v_n(x)\,\langle w_n, f\rangle$$

$$\downarrow$$

approximate controllability $\Leftrightarrow$ $\operatorname{rank}\left(\left[\;\langle w_n, b_1\rangle \;\; \cdots \;\; \langle w_n, b_m\rangle\;\right]\right) \;=\; 1$

- Necessary condition for controllability
  - ⋆ Number of controls $\geq$ maximal multiplicity of e-vectors of $\mathcal{A}$

Example (to be done in class)

- Diffusion equation on $L_2\,[-1,1]$ with Dirichlet BCs

$$\begin{aligned} \psi_t(x,t) &= \psi_{xx}(x,t) \;+\; b(x)\,u(t) \\ \psi(x,0) &= \psi_0(x) \\ \psi(\pm 1,t) &= 0 \end{aligned}$$

Diagonal coordinate form

$$\dot{\alpha}_n(t) \;=\; -\left(\frac{n\pi}{2}\right)^2 \alpha_n(t) \;+\; \underbrace{\langle v_n, b\rangle}_{b_n}\, u(t), \quad n \,\in\, \mathbb{N}$$

approximate/modal controllability $\;\Leftrightarrow\;$ $\{b_n \neq 0$, for all $n \in \mathbb{N}\}$

Lectures 17 & 18: Numerical methods

- Spectral (Galerkin) method
  - ⋆ Basis function expansion
  - ⋆ Compute inner products to determine equation for spectral coefficients
- Pseudo-spectral method
  - ⋆ Satisfy equation at the set of ”collocation” points
  - ⋆ Connection to polynomial interpolation
- Chebyshev polynomials
  - ⋆ Why they should be used
  - ⋆ Basic properties

Online resources

- Freely available books/papers
  - ⋆ Jonh P. Boyd
    Chebyshev and Fourier Spectral Methods
  - ⋆ Lloyd N. Trefethen
    Finite Difference and Spectral Methods for Ordinary and Partial Differential Equations
  - ⋆ Weideman and Reddy
    A Matlab Differentiation Matrix Suite

- Publicly available software
  - ⋆ A Matlab Differentiation Matrix Suite
    http://dip.sun.ac.za/∼weideman/research/differ.html
  - ⋆ Chebfun
    http://www2.maths.ox.ac.uk/chebfun/

Diffusion equation on $L_2\,[-1,1]$

$$\begin{aligned}\psi_t(x,t) &= \psi_{xx}(x,t)\\ \psi(x,0) &= \psi_0(x)\\ \psi(\pm 1,t) &= 0\end{aligned}$$

Basis function expansion

$$\psi(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\,\phi_n(x)$$

$\alpha_n(t)$ – (unknown) spectral coefficients

$\phi_n(x)$ – (known) basis functions

Galerkin method

- Approximate solution by

$$\psi(x,t) \;\approx\; \sum_{n\,=\,1}^{N} \alpha_n(t)\,\phi_n(x) \;=\; \begin{bmatrix} \phi_1(x) & \cdots & \phi_N(x) \end{bmatrix} \begin{bmatrix} \alpha_1(t) \\ \vdots \\ \alpha_N(t) \end{bmatrix}$$

substitute into the equation and take an inner product with $\{\phi_m\}$

$$\begin{bmatrix} \langle \phi_1, \phi_1 \rangle & \cdots & \langle \phi_1, \phi_N \rangle \\ \vdots & & \vdots \\ \langle \phi_N, \phi_1 \rangle & \cdots & \langle \phi_N, \phi_N \rangle \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1(t) \\ \vdots \\ \dot{\alpha}_N(t) \end{bmatrix} = \begin{bmatrix} \langle \phi_1, \phi_1'' \rangle & \cdots & \langle \phi_1, \phi_N'' \rangle \\ \vdots & & \vdots \\ \langle \phi_N, \phi_1'' \rangle & \cdots & \langle \phi_N, \phi_N'' \rangle \end{bmatrix} \begin{bmatrix} \alpha_1(t) \\ \vdots \\ \alpha_N(t) \end{bmatrix}$$

- Done if basis functions satisfy BCs

  Otherwise, need additional conditions on spectral coefficients

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \phi_1(-1) & \cdots & \phi_N(-1) \\ \phi_1(+1) & \cdots & \phi_N(+1) \end{bmatrix} \begin{bmatrix} \alpha_1(t) \\ \vdots \\ \alpha_N(t) \end{bmatrix}$$

Pros and cons

- Advantage: superior convergence
  (if basis functions selected properly)

- Problem: requires integration
  - ⋆ Cumbersome in spatially-varying and nonlinear systems

Example: Orr-Sommerfeld equation in fluid mechanics

$$\Delta \, \psi_t \;=\; \left(\mathrm{j} k_x \, (U''(y) \;-\; U(y) \, \Delta) \;+\; \frac{1}{R} \Delta^2 \right) \psi$$

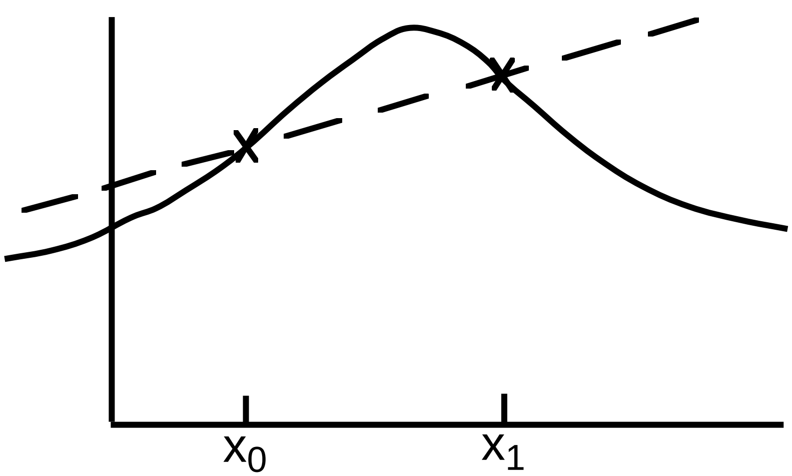
Polynomial interpolation

- Approximate $f(x)$ by a polynomial that matches $f(x)$ at interpolation points

$$p_{N-1}(x_i) \;=\; f(x_i), \quad i \;=\; \{1, \ldots, N\}$$

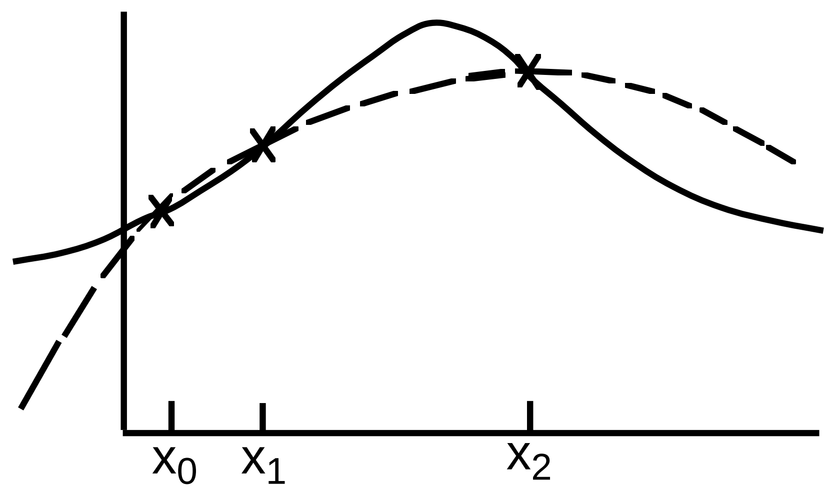
- Examples:

$N = 2 \;\Rightarrow\;$ Linear Interpolation



$$f(x) \;\approx\; \frac{x - x_1}{x_0 - x_1} f(x_0) \;+\; \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$N = 3 \;\Rightarrow\;$ Quadratic Interpolation



$$\begin{aligned} f(x) \;\approx\; & \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) \;+ \\ & \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \;+ \\ & \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \end{aligned}$$

Lagrange interpolation formula

$$p_N(x) = \sum_{i=0}^{N} f(x_i)\, C_i(x)$$

$$C_i(x) = \prod_{j=0,\, j \neq i}^{N} \frac{x - x_j}{x_i - x_j}$$

- Cardinal functions $C_i(x_j) = \delta_{ij}$
  - ⋆ Not efficient for computations
  - ⋆ Suitable for theoretical arguments

- Runge Phenomenon

$$f(x) = \frac{1}{1 + x^2}, \quad x \in [-5, 5]$$

  - ⋆ Evenly spaced points $\Rightarrow$ convergence for $|x| \leq 3.63$

    Interactive Demo

Choice of grid points

- Cauchy interpolation error theorem

$$\left.\begin{array}{lll} f & - & \text{has } N+1 \text{ derivatives} \\ p_N & - & \text{interpolant of degree } N \end{array}\right\} \;\Rightarrow\; f(x) - p_N(x) \;=\; \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{i\,=\,0}^{N} (x \,-\, x_i)$$

- Chebyshev minimal amplitude theorem
  - ⋆ Among all polynomials $q_N(x)$ of degree N, with leading coefficient 1,

$$\frac{T_N(x)}{2^{N-1}} \;=\; \frac{N\text{th Chebyshev polynomial}}{2^{N-1}}$$

has the smallest $L_\infty[-1,\,1]$ norm

$$\sup_{x\,\in\,[-1,\,1]} |q_N(x)| \;\geq\; \sup_{x\,\in\,[-1,\,1]} \left|\frac{T_N(x)}{2^{N-1}}\right| \;=\; \frac{1}{2^{N-1}}, \quad \text{for all } q_N(x)$$

Optimal interpolation points

- Select polynomial part of $f(x) - p_N(x)$ as

$$\prod_{i=0}^{N} (x - x_i) = \frac{T_{N+1}(x)}{2^N}$$

- Optimal interpolation points: roots of $T_{N+1}(x)$

$$x_i = \cos\left(\frac{(2i - 1)\,\pi}{2\,(N + 1)}\right), \quad i = \{1, \ldots, N+1\}$$

Chebyshev polynomials

- Solutions to Sturm-Liouville Problem

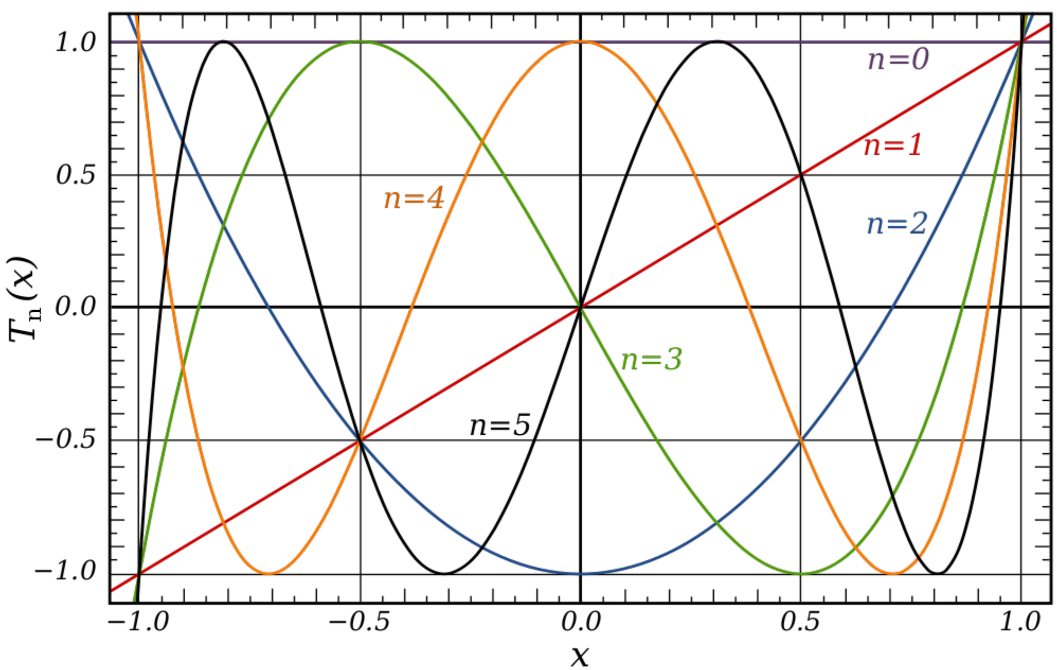
$$(1 - x^2)\, T_n''(x) \; - \; x\, T_n'(x) \; + \; n^2\, T_n(x) \; = \; 0, \quad x \in [-1,\, 1], \quad n \; = \; 0, 1, \ldots$$

- Three-term recurrence

$$\{T_0 \; = \; 1; \;\; T_1(x) \; = \; x; \;\; T_{n+1}(x) \; = \; 2x\, T_n(x) \; - \; T_{n-1}(x), \; n \; \geq \; 1\}$$

- Alternative definition

$$T_n\,(\cos\,(t)) \; = \; \cos\,(n\,t) \quad \Rightarrow \quad |T_n(x)| \; \leq \; 1, \;\; \text{for all } x \; \in \; [-1,\, 1], \; n \; = \; 0, 1, \ldots$$



- Inner product

$$\langle T_m, T_n \rangle_w \;=\; \int_{-1}^{1} \frac{T_m(x)\, T_n(x)}{\sqrt{1 \,-\, x^2}}\, \mathrm{d}x \;=\; \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \dfrac{\pi}{2} & m = n \neq 0 \end{cases}$$

- Collocation points

$$\text{Gauss-Chebyshev:} \quad x_i \;=\; \cos\left(\frac{(2i \,-\, 1)\,\pi}{2\,N}\right), \quad i \;=\; \{1, \ldots, N\}$$

$$\text{Gauss-Lobatto:} \quad x_i \;=\; \cos\left(\frac{\pi\, i}{N \,-\, 1}\right), \quad i \;=\; \{0, \ldots, N - 1\}$$

- Integration

$$\int_{-1}^{x} T_n(\xi)\, \mathrm{d}\xi \;=\; \frac{T_{n+1}(x)}{2\,(n \,+\, 1)} \;+\; \frac{T_{n-1}(x)}{2\,(n \,-\, 1)}, \quad n \;\geq\; 2$$

Gaussian integration

- Approximate $f(x)$ by a polynomial that matches $f(x)$ at interpolation points

$$p_N(x_i) \;=\; f(x_i), \quad i \;=\; \{0, \ldots, N\}$$

$$f(x) \;\approx\; p_N(x) \;=\; \sum_{i\,=\,0}^{N} f(x_i)\, C_i(x)$$

- Evaluate integral of $f(x)$ by integrating $p_N(x)$

$$\int_a^b f(x)\, \mathrm{d}x \;\approx\; \sum_{i\,=\,0}^{N} w_i\, f(x_i)$$

Quadrature weights:

$$w_i \;=\; \int_a^b C_i(x)\, \mathrm{d}x$$

- Gaussian integration: exact if integrand is a polynomial of degree N

- Can be made exact for polynomials of degree $2N+1$ by optimal selection of
  - ⋆ interpolation points $\{x_i\}$
  - ⋆ weights $\{w_i\}$

- Gauss-Jacobi integration
  - ⋆ orthogonal polynomials w.r.t. the inner product with weight function $\rho(x)$
  - ⋆ interpolation points: zeros of $p_{N+1}(x)$
  - ⋆ quadrature formula: exact for polynomials of degree $2N+1$ or smaller

$$\int_a^b f(x)\,\rho(x)\,\mathrm{d}x \;=\; \sum_{i\,=\,0}^{N} w_i\,f(x_i)$$

- Good candidates for quadrature points:

$$\text{Gauss-Lobatto:} \quad x_i \;=\; \cos\left(\frac{\pi\, i}{N}\right), \quad i \;=\; \{0,\ldots,N\}$$

Interpolation by quadrature

- Orthogonality w.r.t. discrete inner product

$$\langle \phi_i, \phi_j \rangle \;=\; \delta_{ij} \quad \Rightarrow \quad \langle \phi_i, \phi_j \rangle_G \;=\; \sum_{m\,=\,0}^{N} w_m \, \phi_i(x_m) \, \phi_j(x_m) \;=\; \delta_{ij}$$

- Basis function expansion

$$f(x) \;=\; \sum_{n\,=\,0}^{\infty} \alpha_n \, \phi_n(x) \;=\; \sum_{n\,=\,0}^{N} \alpha_n \, \phi_n(x) \;+\; E_N(x)$$

- Discrete vs. exact spectral coefficients

$$\begin{aligned} \alpha_{m,G} \;&=\; \langle \phi_m, f \rangle_G \\ &=\; \left\langle \phi_m, \sum_{n\,=\,0}^{N} \alpha_n \, \phi_n \;+\; E_N \right\rangle_G \\ &=\; \sum_{n\,=\,0}^{N} \alpha_n \, \langle \phi_m, \phi_n \rangle_G \;+\; \langle \phi_m, E_N \rangle_G \\ &=\; \alpha_m \;+\; \langle \phi_m, E_N \rangle_G \end{aligned}$$

Error bound for Chebyshev interpolation

- Error between Galerkin and Pseudo-spectral
  twice the sum of absolute values of neglected spectral coefficients

  ⋆ $f(x) \;=\; \sum_{n=0}^{\infty} \alpha_n T_n(x)$

  ⋆ $p_N(x)$ – polynomial that interpolates $f(x)$ at Gauss-Lobatto points

$$|f(x) \;-\; p_N(x)| \;\leq\; 2 \sum_{n=N+1}^{\infty} |\alpha_n|, \quad \text{for all } N \text{ and all } x \in [-1,\, 1]$$

Back to cardinal functions

- Lagrange interpolation

$$p_N(x) = \sum_{i=0}^{N} f(x_i)\, C_i(x)$$

$$C_i(x) = \prod_{j=0,\, j \neq i}^{N} \frac{x - x_j}{x_i - x_j}$$

Cardinal functions $C_i(x_j) = \delta_{ij}$ ▮

- Sinc functions

$$C_k(x;h) = \frac{\sin\left(\dfrac{(x - kh)\,\pi}{h}\right)}{\dfrac{(x - kh)\,\pi}{h}} = \operatorname{sinc}\left(\frac{x - kh}{h}\right)$$

$$\{x_j = j\,h;\ j \in \mathbb{Z}\} \Rightarrow C_k(x_j;h) = \delta_{jk}$$

Approximate f by

$$f(x) = \sum_{j=-\infty}^{\infty} f(x_j)\, C_j(x;h)$$

Cardinal functions for Chebyshev polynomials

- Gauss-Chebyshev points: zeros of $T_{N+1}(x)$
  - ⋆ Taylor series expansion around x_j

$$T_{N+1}(x) \;=\; \underbrace{T_{N+1}(x_j)}_{0} + T'_{N+1}(x_j)\,(x \,-\, x_j) \,+\, \frac{1}{2}\,T''_{N+1}(x_j)\,(x \,-\, x_j)^2 + O\left(|x \,-\, x_j|^3\right)$$

Cardinal functions

$$C_j(x) \;=\; \frac{T_{N+1}(x)}{T'_{N+1}(x_j)\,(x \,-\, x_j)} \;=\; 1 \;+\; \frac{T''_{N+1}(x_j)\,(x \,-\, x_j)}{2T'_{N+1}(x_j)} \;+\; O\left(|x \,-\, x_j|^2\right)$$

- Gauss-Lobatto points: zeros of $(1 - x^2)\,T'_N(x)$

Cardinal functions: $$C_j(x) \;=\; \frac{\left(1 \,-\, x^2\right)T'_N(x)}{\left((1 \,-\, x^2)\,T'_N(x)\right)'\Big|_{x\,=\,x_j}(x \,-\, x_j)}$$

Matlab Differentiation Matrix Suite: A Demo

```
%% number of grid points without boundaries (no \pm 1)
N = 50

%% 1st & 2nd order differentiation matrices
[yT,DM] = chebdif(N+2,2);
y = yT(2:end-1);

%% 1st & 2nd derivatives wrt y on a total grid (no BCs)
DT1 = DM(:,:,1); DT2 = DM(:,:,2);

%% implement homogeneous Dirichlet BCs
%% ammounts to deleting 1st rows and columns of DT1 & DT2
D1 = DT1(2:N+1,2:N+1); D2 = DT2(2:N+1,2:N+1);

%% 4th derivative with Dirichlet & Neumann BCs at both ends
%% D4 - obtained on a grid without \pm 1
[y1,D4] = cheb4c(N+2);

%% e-value decomposition of D2 with Dirichlet BCs
[Vh,Dh] = eig(D2);  % compare with analytical results
```

Lecture 19: Introduction to Chebfun

- Freely available
  - ⋆ Chebfun project: download and enjoy!
    http://www2.maths.ox.ac.uk/chebfun/

- Online resources
  - ⋆ Tutorial by Nick Trefethen
    Introduction to Chebfun
  - ⋆ Book under preparation by Nick Trefethen
    Approximation Theory and Approximation Practice
  - ⋆ Papers
    Publications about Chebfun

- In-class demonstration

Lecture 20: Input-output norms; Pseudospectra

- Singular Value Decomposition of the frequency response operator

- Measures of input-output amplification (across frequency)
  - ⋆ Largest singular value
  - ⋆ Hilbert-Schmidt norm (power spectral density)

- Systems with one spatial variable
  - ⋆ Two point boundary value problems

- Input-output norms
  - ⋆ H_∞ norm: $\begin{cases} \text{worst-case amplification of deterministic disturbances} \\ \text{measure of robustness} \end{cases}$
  - ⋆ H_2 norm: $\begin{cases} \text{energy of the impulse response} \\ \text{variance amplification} \end{cases}$

- Pseudospectra of linear operators

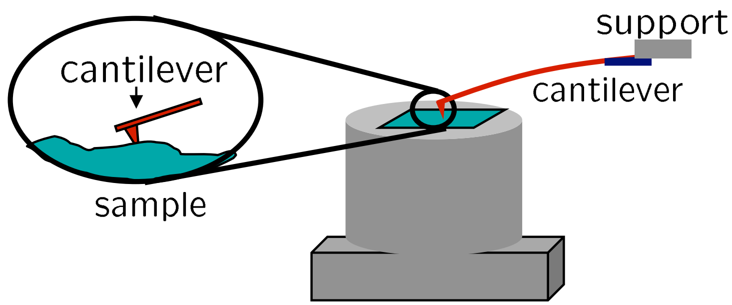
Example: cantilever beam

$$
\left\{
\begin{array}{rclrcl}
\mu\,\psi_{tt} + \alpha EI\,\psi_{txxxx} + EI\,\psi_{xxxx} & = & 0 & & & \\
\psi(0,t) & = & 0, & \psi_x(0,t) & = & 0 \\
\alpha EI\,\psi_{txxx}(l,t) + EI\psi_{xxx}(l,t) & = & u(t), & \psi_{xx}(l,t) & = & 0 \\
\psi(l,t) & = & y(t) & & &
\end{array}
\right.
$$

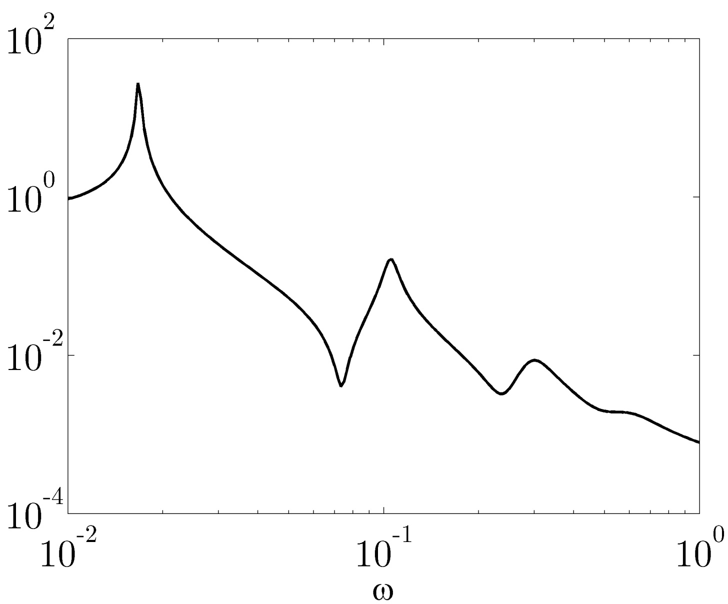
---

input: $u(t)$

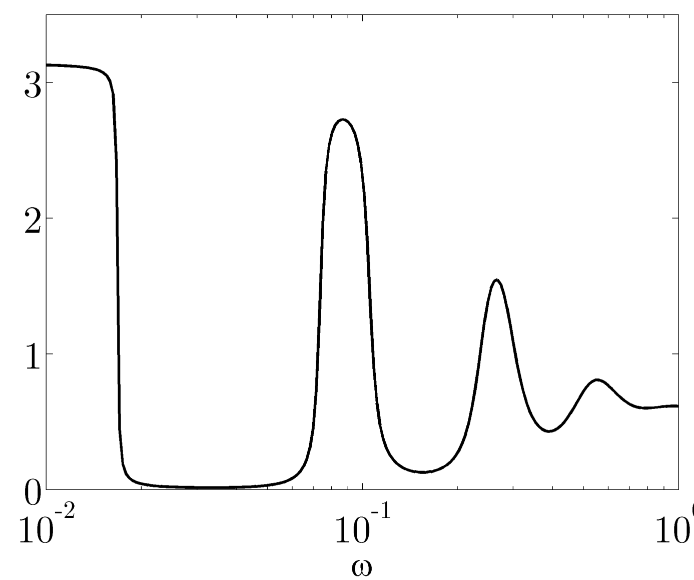
output: $y(t)$



magnitude:



phase:



Example: diffusion equation on $L_2[-1,1]$

- Distributed input and output fields

$$\begin{aligned} \phi_t(y,t) &= \phi_{yy}(y,t) \; + \; d(y,t) \\ \phi(y,0) &= 0 \\ \phi(\pm 1,t) &= 0 \end{aligned}$$

Frequency response operator

$$\begin{aligned} \phi(y,\omega) &= \left[\,\mathcal{T}(\omega)\, d(\,\cdot\,,\omega)\,\right](y) \\ &= \left[\,(\mathrm{j}\omega I \;-\; \partial_{yy})^{-1}\, d(\,\cdot\,,\omega)\,\right](y) \\ &= \int_{-1}^{1} T_{\mathrm{ker}}(y,\eta;\omega)\, d(\eta,\omega)\,\mathrm{d}\eta \end{aligned}$$

Two point boundary value realizations of $\mathcal{T}(\omega)$

- Input-output differential equation

$$\mathcal{T}(\omega): \left\{ \begin{array}{rcl} \phi''(y,\omega) \, - \, \mathrm{j}\omega \, \phi(y,\omega) & = & -d(y,\omega) \\ \phi(\pm 1,\omega) & = & 0 \end{array} \right.$$

- Spatial state-space realization

$$\mathcal{T}(\omega): \left\{ \begin{array}{rcl} \begin{bmatrix} x_1'(y,\omega) \\ x_2'(y,\omega) \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ \mathrm{j}\omega & 0 \end{bmatrix} \begin{bmatrix} x_1(y,\omega) \\ x_2(y,\omega) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} d(y,\omega) \\ \phi(y,\omega) & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(y,\omega) \\ x_2(y,\omega) \end{bmatrix} \\ 0 & = & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(-1,\omega) \\ x_2(-1,\omega) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(1,\omega) \\ x_2(1,\omega) \end{bmatrix} \end{array} \right.$$

Frequency response operator

- Evolution equation

$$\begin{aligned} \mathcal{E}\,\boldsymbol{\phi}_t(y,t) &= \mathcal{F}\,\boldsymbol{\phi}(y,t) \;+\; \mathcal{G}\,\mathbf{d}(y,t) \\ \boldsymbol{\varphi}(y,t) &= \mathcal{C}\,\boldsymbol{\phi}(y,t) \end{aligned}$$

  ⋆ Spatial differential operators

$$\mathcal{F} \;=\; [\,\mathcal{F}_{ij}\,] \;=\; \left[\,\sum_{k\,=\,0}^{n_{ij}} f_{ij,k}(y)\,\frac{\mathrm{d}^k}{\mathrm{d}y^k}\,\right]$$

- Frequency response operator

$$\mathcal{T}(\omega) \;=\; \mathcal{C}\,(\mathrm{j}\omega\mathcal{E} \;-\; \mathcal{F})^{-1}\,\mathcal{G}$$

Singular Value Decomposition of $\mathcal{T}(\omega)$

- compact operator $\mathcal{T}(\omega)$: $\mathbb{H}_{\rm in} \longrightarrow \mathbb{H}_{\rm out}$

$$\boldsymbol{\varphi}(y,\omega) \;=\; [\,\mathcal{T}(\omega)\,\mathbf{d}(\,\cdot\,,\omega)\,]\,(y) \;=\; \sum_{n\,=\,1}^{\infty} \sigma_n(\omega)\,\mathbf{u}_n(y,\omega)\,\langle \mathbf{v}_n, \mathbf{d}\rangle$$

$$\left[\,\mathcal{T}(\omega)\,\mathcal{T}^{\dagger}(\omega)\,\mathbf{u}_n(\,\cdot\,,\omega)\right](y) \;=\; \sigma_n^2(\omega)\,\mathbf{u}_n(y,\omega) \;\Rightarrow\; \{\mathbf{u}_n\} \text{ orthonormal basis of } \mathbb{H}_{\rm out}$$

$$\left[\,\mathcal{T}^{\dagger}(\omega)\,\mathcal{T}(\omega)\,\mathbf{v}_n(\,\cdot\,,\omega)\right](y) \;=\; \sigma_n^2(\omega)\,\mathbf{v}_n(y,\omega) \;\Rightarrow\; \{\mathbf{v}_n\} \text{ orthonormal basis of } \mathbb{H}_{\rm in}$$

$$\sigma_1(\omega) \;\geq\; \sigma_2(\omega) \;\geq\; \cdots \;>\; 0$$

$$\mathbf{d}(y,\omega) \;=\; \mathbf{v}_m(y,\omega) \;\;\Rightarrow\;\; \boldsymbol{\varphi}(y,\omega) \;=\; \sigma_m(\omega)\,\mathbf{u}_m(y,\omega)$$

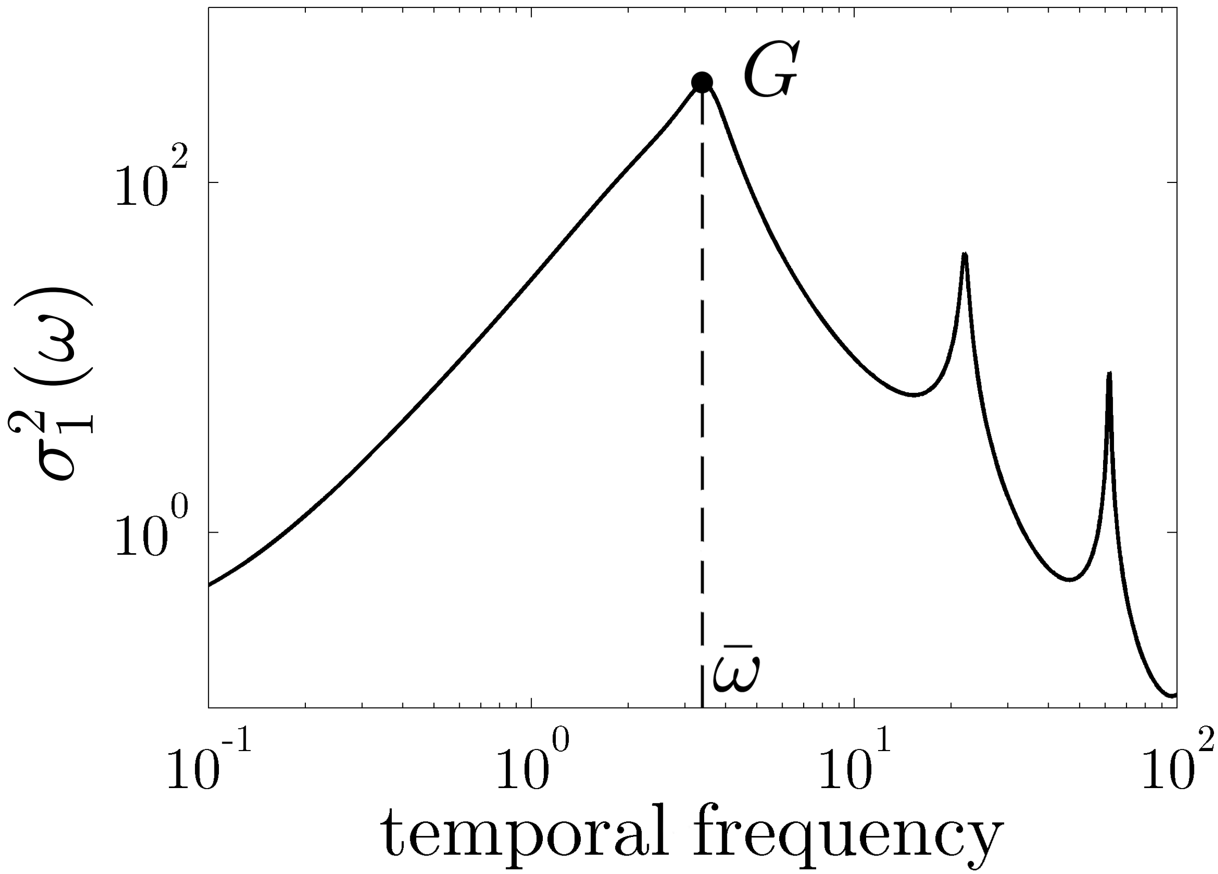
$\sigma_1(\omega)$: the largest amplification at any frequency

Input-output gains

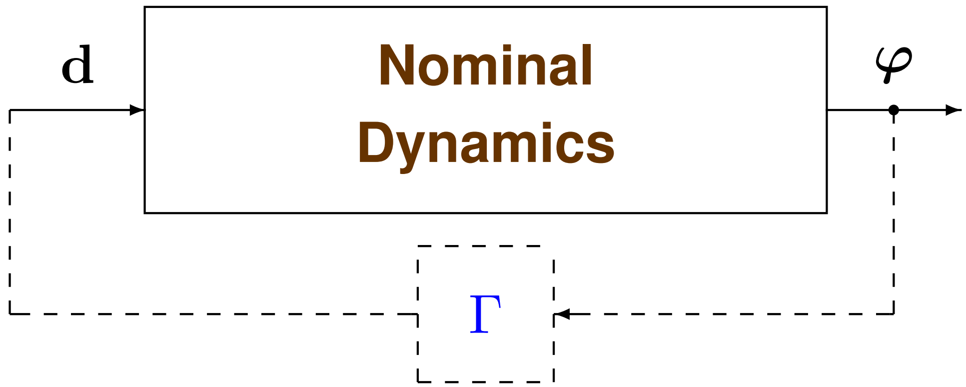
- **Determined by singular values of $\mathcal{T}(\omega)$**
  - $\star$ H_∞ norm: an induced L_2 gain (of a system)

worst case amplification:

$$\|\mathcal{T}\|_\infty^2 \;=\; \sup \frac{\text{output energy}}{\text{input energy}} \;=\; \sup_{\omega} \sigma_1^2(\omega)$$



- Robustness interpretation



modeling uncertainty
(can be nonlinear or time-varying)

small-gain theorem:

stability for all Γ with $\|\Gamma\|_\infty \le \gamma$	$\Leftrightarrow$	$\gamma < \dfrac{1}{\|\mathcal{T}\|_\infty}$

LARGE
worst case amplification $\Rightarrow$ **small** **stability margins**

- Hilbert-Schmidt norm of $\mathcal{T}(\omega)$

power spectral density:

$$\|\mathcal{T}(\omega)\|_{\mathrm{HS}}^2 \;=\; \mathrm{trace}\left(\mathcal{T}(\omega)\,\mathcal{T}^\dagger(\omega)\right) \;=\; \sum_{n\,=\,1}^{\infty} \sigma_n^2\,(\omega)$$

☞ Both $\sigma_1(\omega)$ and $\|\mathcal{T}(\omega)\|_{\mathrm{HS}}^2$ can be computed efficiently using Chebfun

- ⋆ Enabling tool: TPBVRs of $\mathcal{T}(\omega)$ and $\mathcal{T}^\dagger(\omega)$

$\|\mathcal{T}(\omega)\|_{\mathrm{HS}}^2$: *Jovanović & Bamieh, Syst. Control Lett. '06*

$\sigma_1(\omega)$: *Lieu & Jovanović, J. Comput. Phys. '11* (submitted; also: arXiv:1112.0579v1)

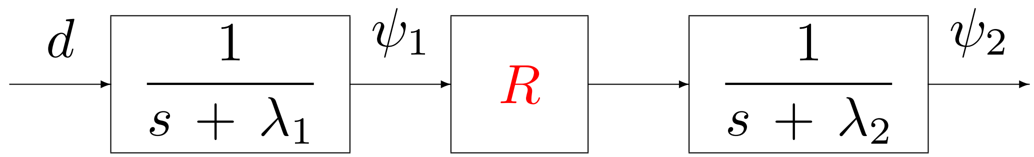
software: Frequency Responses of PDEs in Chebfun

- H_2 norm: variance amplification

$$\|\mathcal{T}\|_2^2 \;=\; \frac{1}{2\,\pi}\int_{-\infty}^{\infty} \|\mathcal{T}(\omega)\|_{\mathrm{HS}}^2\,\mathrm{d}\omega$$

A toy example

$$\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 \\ R & -\lambda_2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d$$



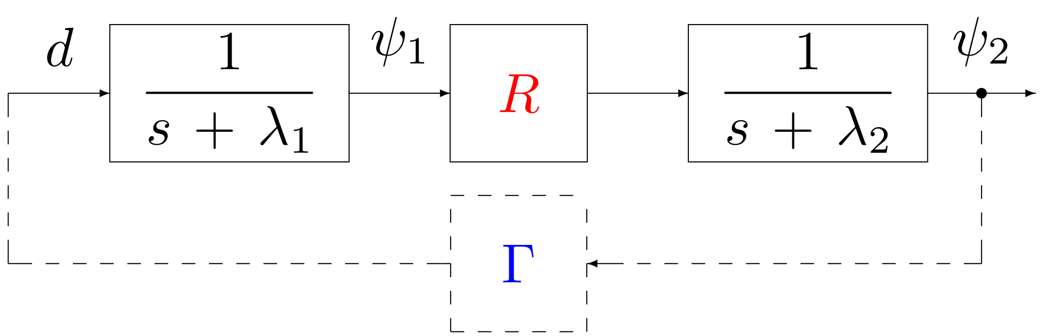
- **Worst case amplification**

$$\sup \frac{\text{energy of } \psi_2}{\text{energy of } d} = \sup_{\omega} |T(\mathrm{j}\omega)|^2 = \frac{R^2}{(\lambda_1 \lambda_2)^2}$$

Variance amplification

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |T(\mathrm{j}\omega)|^2 \, \mathrm{d}\omega = \frac{R^2}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)}$$

- **Robustness**



modeling uncertainty
(can be nonlinear or time-varying)

small-gain theorem:

stability for all Γ with

$$\|\Gamma\|_\infty \le \gamma$$

$$\Updownarrow$$

$$\gamma < \lambda_1 \lambda_2 / R$$

A note on computation of H_2 and H_∞ norms

$$
\begin{aligned}
\boldsymbol{\phi}_t(y,t) &= \mathcal{A}\,\boldsymbol{\phi}(y,t) \,+\, \mathcal{B}\,\mathbf{d}(y,t) \\
\boldsymbol{\varphi}(y,t) &= \mathcal{C}\,\boldsymbol{\phi}(y,t)
\end{aligned}
$$

- H_2 norm
  - ⋆ Operator Lyapunov equation

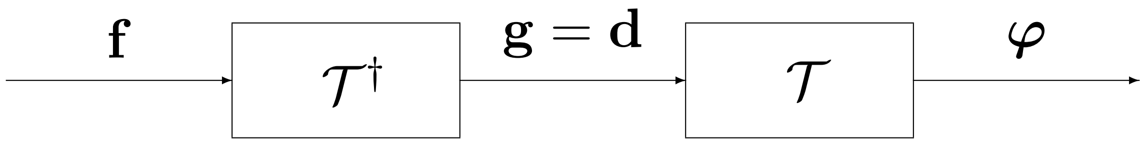
$$
\begin{aligned}
\|\mathcal{T}\|_2^2 &= \mathrm{trace}\left(\mathcal{C}\,\mathcal{X}\,\mathcal{C}^\dagger\right) \\
\mathcal{A}\,\mathcal{X} \,+\, \mathcal{X}\,\mathcal{A}^\dagger &= -\,\mathcal{B}\,\mathcal{B}^\dagger
\end{aligned}
$$

- H_∞ norm
  - ⋆ E-value decomposition of Hamiltonian in conjunction with bisection

$$
\|\mathcal{T}\|_\infty \geq \gamma \;\;\Leftrightarrow\;\; \left[\begin{array}{cc} \mathcal{A} & \frac{1}{\gamma}\,\mathcal{B}\,\mathcal{B}^\dagger \\ -\frac{1}{\gamma}\,\mathcal{C}^\dagger\,\mathcal{C} & -\mathcal{A}^\dagger \end{array}\right] \text{ has at least one imaginary e-value}
$$

Spatial state-space realization of $\mathcal{T}(\omega)$

- Cascade connection of $\mathcal{T}^\dagger$ and $\mathcal{T}$



- Realization of $\mathcal{T}$

$$\mathcal{T}: \left\{ \begin{array}{rcl} \mathbf{x}'(y) & = & \mathbf{A}_0(y)\,\mathbf{x}(y) \; + \; \mathbf{B}_0(y)\,\mathbf{d}(y) \\ \boldsymbol{\varphi}(y) & = & \mathbf{C}_0(y)\,\mathbf{x}(y) \\ 0 & = & \mathbf{N}_a\,\mathbf{x}(a) \; + \; \mathbf{N}_b\,\mathbf{x}(b) \end{array} \right.$$

- Realization of $\mathcal{T}^\dagger$

$$\mathcal{T}^\dagger: \left\{ \begin{array}{rcl} \mathbf{z}'(y) & = & -\mathbf{A}_0^*(y)\,\mathbf{z}(y) \; - \; \mathbf{C}_0^*(y)\,\mathbf{f}(y) \\ \mathbf{g}(y) & = & \mathbf{B}_0^*(y)\,\mathbf{z}(y) \\ 0 & = & \mathbf{M}_a\,\mathbf{z}(a) \; + \; \mathbf{M}_b\,\mathbf{z}(b) \end{array} \right.$$

$$\left[\begin{array}{cc} \mathbf{M}_a & \mathbf{M}_b \end{array} \right] \left[\begin{array}{r} \mathbf{N}_a^* \\ -\mathbf{N}_b^* \end{array} \right] \; = \; 0$$

Integral form of a differential equation

- 1D diffusion equation: differential form

$$\left(D^{(2)} \,-\, \mathrm{j}\omega I\right)\phi(y) \;=\; -\,d(y)$$

$$\left(\left[\begin{array}{c} 1 \\ 0 \end{array}\right] E_{-1} \,+\, \left[\begin{array}{c} 0 \\ 1 \end{array}\right] E_{1}\right)\phi(y) \;=\; \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Auxiliary variable: $\nu(y) \;=\; \left[D^{(2)}\,\phi\right](y)$

Integrate twice

$$\begin{aligned}
\phi'(y) &= \int_{-1}^{y} \nu(\eta_1)\,\mathrm{d}\eta_1 \,+\, k_1 \;=\; \left[J^{(1)}\,\nu\right](y) \,+\, k_1 \\
\phi(y) &= \int_{-1}^{y} \left(\int_{-1}^{\eta_2} \nu(\eta_1)\,\mathrm{d}\eta_1\right)\mathrm{d}\eta_2 \,+\, \left[\begin{array}{cc} 1 & (y \,+\, 1) \end{array}\right] \left[\begin{array}{c} k_2 \\ k_1 \end{array}\right] \\
&= \left[J^{(2)}\,\nu\right](y) \,+\, K^{(2)}\,\mathbf{k}
\end{aligned}$$

- 1D diffusion equation: integral form

$$\left(I \,-\, \mathrm{j}\omega J^{(2)}\right)\nu(y) \,-\, \mathrm{j}\omega\, K^{(2)}\,\mathbf{k} \;=\; -\,d(y)$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} k_2 \\ k_1 \end{bmatrix} \;=\; -\,\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} E_{-1} \,+\, \begin{bmatrix} 0 \\ 1 \end{bmatrix} E_1\right) J^{(2)}\nu(y)$$

Eliminate $\mathbf{k}$ from the equations to obtain

$$\left(I \,-\, \mathrm{j}\omega J^{(2)} \,+\, \frac{1}{2}\,\mathrm{j}\omega\,(y\,+\,1)\,E_1 J^{(2)}\right)\nu(y) \;=\; -d(y)$$

☞ More suitable for numerical computations than differential form
integral operators and point evaluation functionals are well-conditioned

Pseudospectra

- Book
  - ⋆ Trefethen and Embree: Spectra and Pseudospectra

- Online resources
  - ⋆ Talk by Nick Trefethen: Pseudospectra and EigTool
  - ⋆ Software: { Pseudospectra Gateway; EigTool }

perturbed system: $\psi_t = (\mathcal{A} + \Gamma)\,\psi$

ϵ-pseudospectrum:

$$\begin{aligned}\sigma_\epsilon(\mathcal{A}) &= \{s \in \mathbb{C};\ \|(sI - \mathcal{A})^{-1}\| > 1/\epsilon\} \\ &= \{s \in \mathbb{C};\ s \in \sigma(\mathcal{A} + \Gamma),\ \|\Gamma\| < \epsilon\}\end{aligned}$$

can be converted to an input-output problem

Lecture 21: Input-output analysis in fluid mechanics

- Linear analyses: Input-output vs. Stability

AMPLIFICATION:

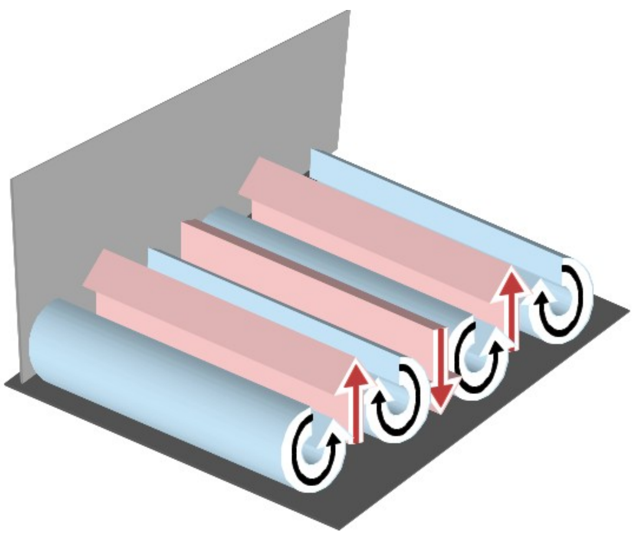
$$\mathbf{v} = \mathcal{T}\,\mathbf{d}$$

singular values of $\mathcal{T}$

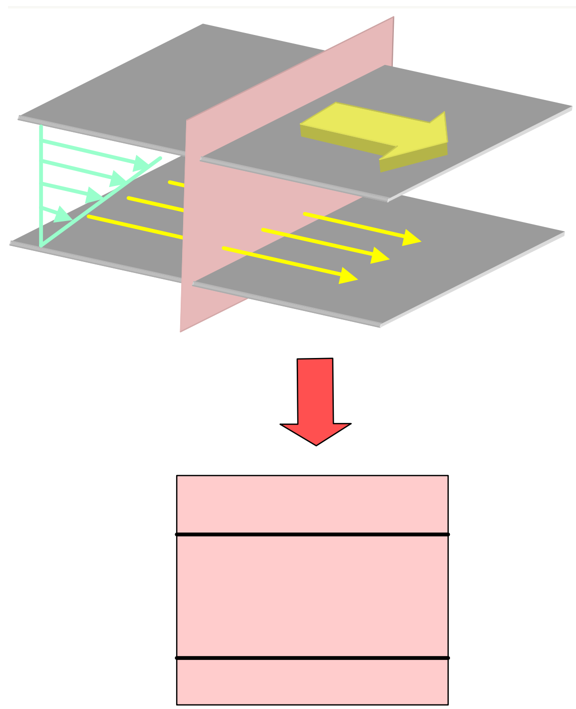
STABILITY:

$$\psi_t = \mathcal{A}\,\psi$$

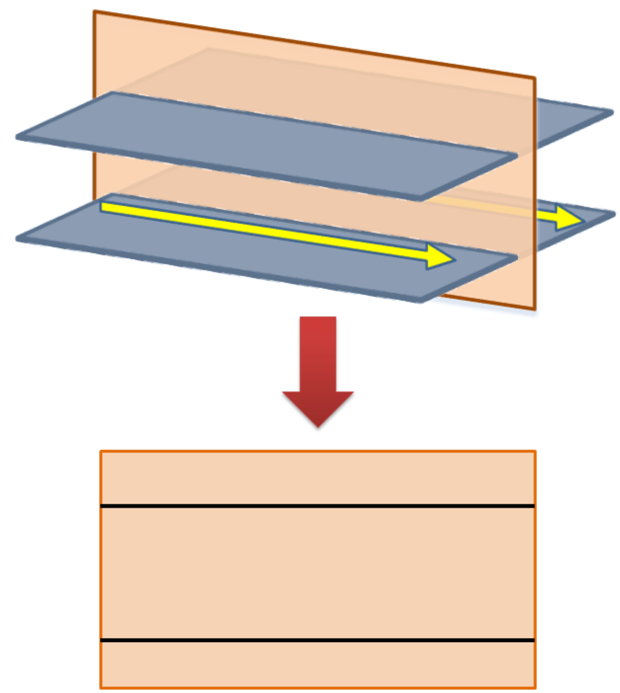
e-values of $\mathcal{A}$



typical structures



cross-sectional dynamics



2D models

Transition in Newtonian fluids

- LINEAR HYDRODYNAMIC STABILITY: **unstable normal modes**
  - ⋆ **successful in:** Benard Convection, Taylor-Couette flow, etc.
  - ⋆ **fails in:** wall-bounded shear flows (channels, pipes, boundary layers)

- DIFFICULTY #1

  Inability to predict: Reynolds number for the onset of turbulence (Re_c)

  Experimental onset of turbulence: $\begin{cases} \textbf{much before instability} \\ \textbf{no sharp value for } Re_c \end{cases}$

- DIFFICULTY #2

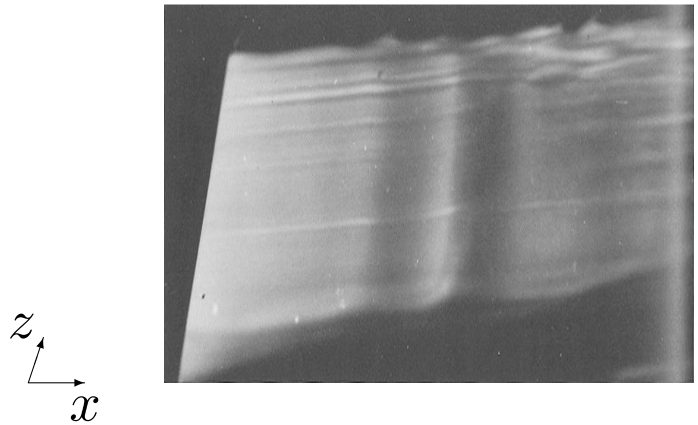
  Inability to predict: flow structures observed at transition

  (except in carefully controlled experiments)

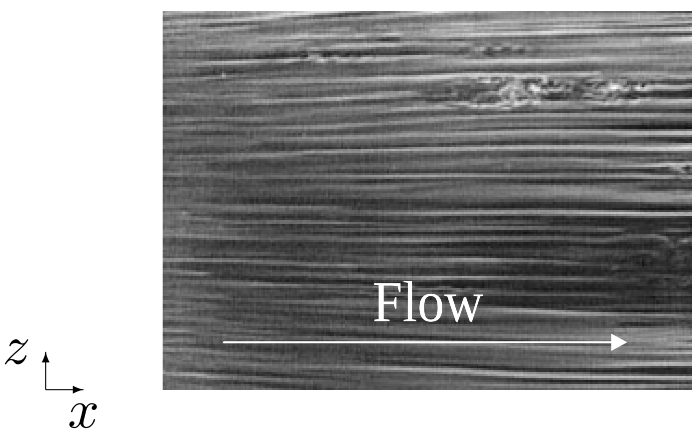
LINEAR STABILITY:

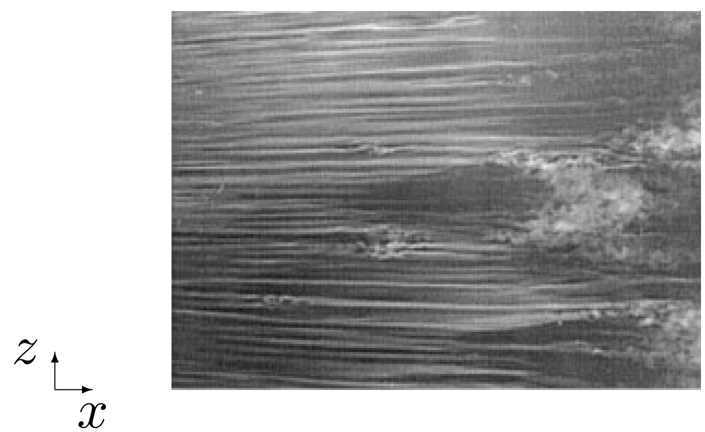
⋆ For $Re \geq Re_c \;\; \Rightarrow$ exp. growing normal modes

corresponding e-functions (TS-waves) $\Big\}$:= exp. growing flow structures



EXPERIMENTS: **streaky boundary layers and turbulent spots**





Matsubara & Alfredsson, *J. Fluid Mech. ’01*

- FAILURE OF LINEAR HYDRODYNAMIC STABILITY
  caused by high flow sensitivity
  - ⋆ large transient responses
  - ⋆ large noise amplification
  - ⋆ small stability margins

TO COUNTER THIS SENSITIVITY: **must account for modeling imperfections**

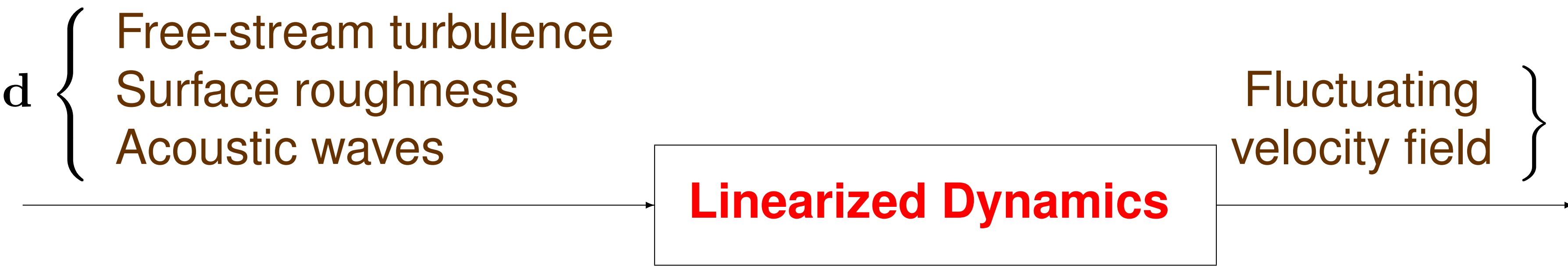
TRANSITION ≈ STABILITY + RECEPTIVITY + ROBUSTNESS

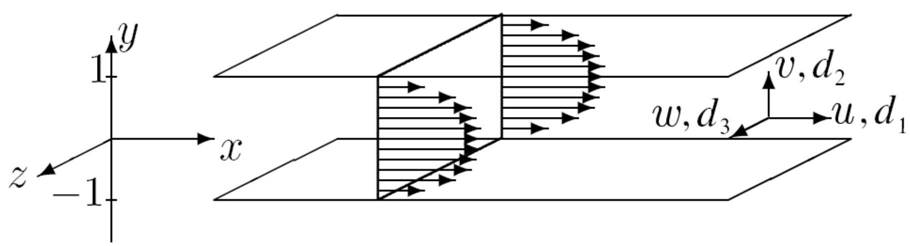
RECEPTIVITY → **flow disturbances**

ROBUSTNESS → **unmodeled dynamics**

Tools for quantifying sensitivity

- INPUT-OUTPUT ANALYSIS: **spatio-temporal frequency responses**





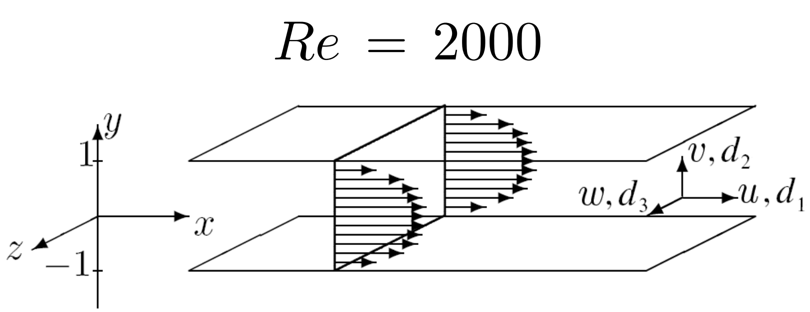
$$\underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{\mathbf{d}} \xrightarrow{\textbf{amplification}} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\mathbf{v}}$$

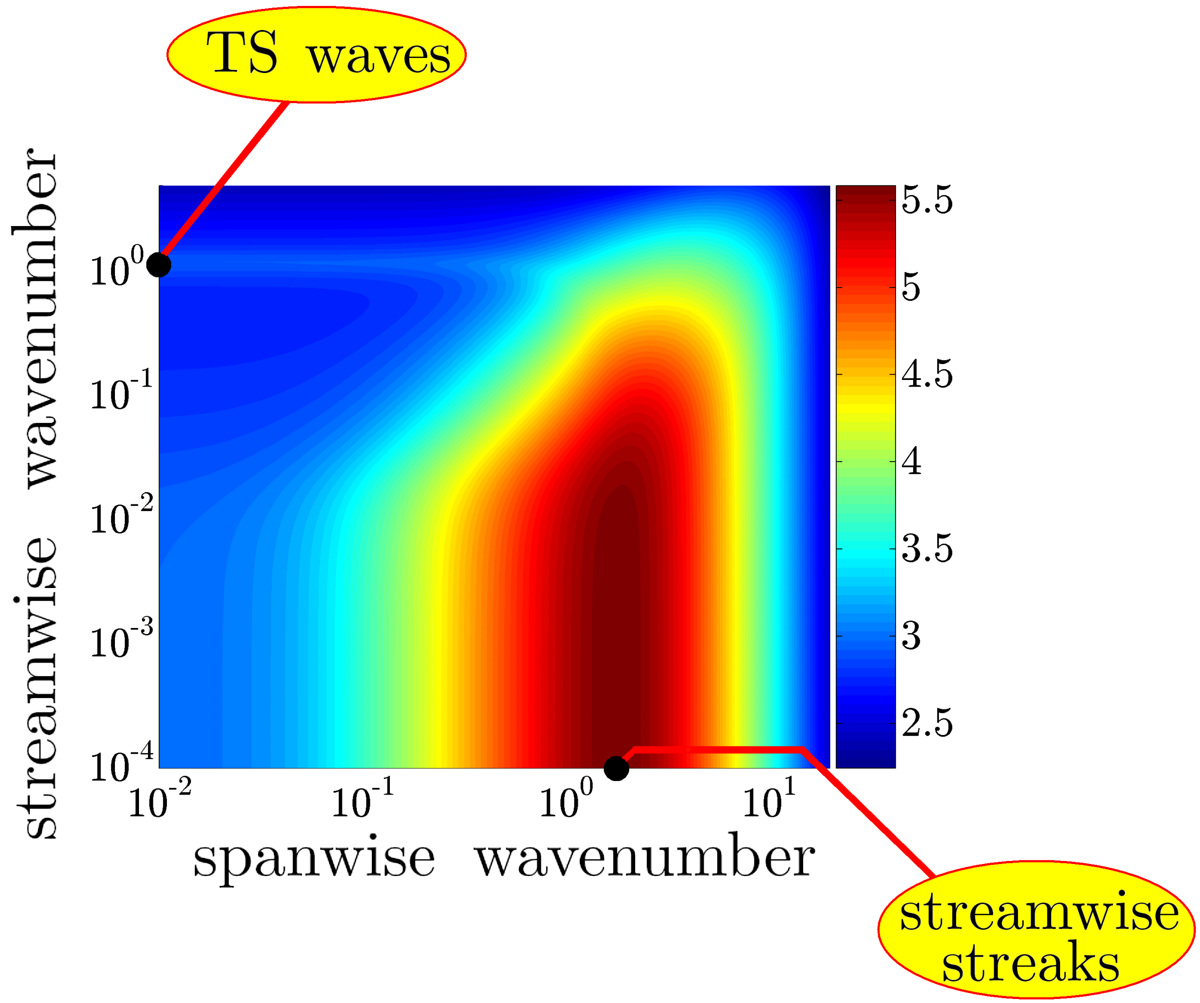
IMPLICATIONS FOR:

transition: insight into mechanisms

control: control-oriented modeling

Ensemble average energy density





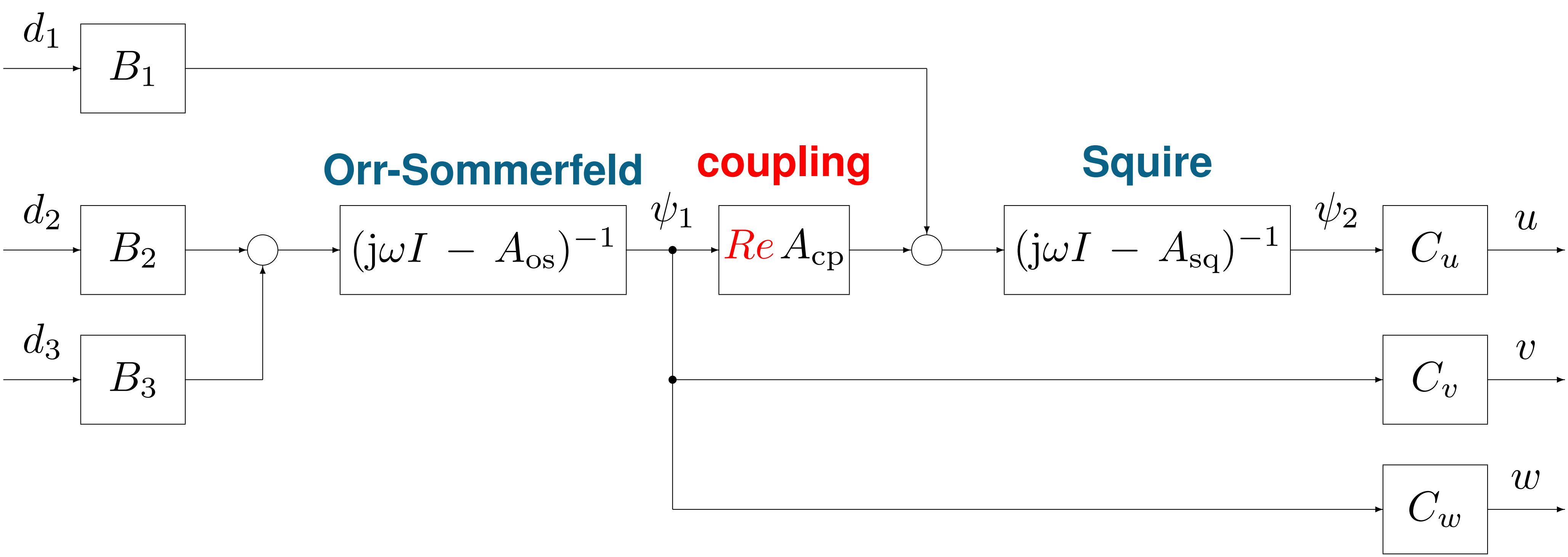
- **Dominance of streamwise elongated structures**

  streamwise streaks!

Influence of Re: streamwise-constant model

$$\begin{bmatrix} \psi_{1t} \\ \psi_{2t} \end{bmatrix} = \begin{bmatrix} A_{\rm os} & 0 \\ Re\, A_{\rm cp} & A_{\rm sq} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 & B_2 & B_3 \\ B_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

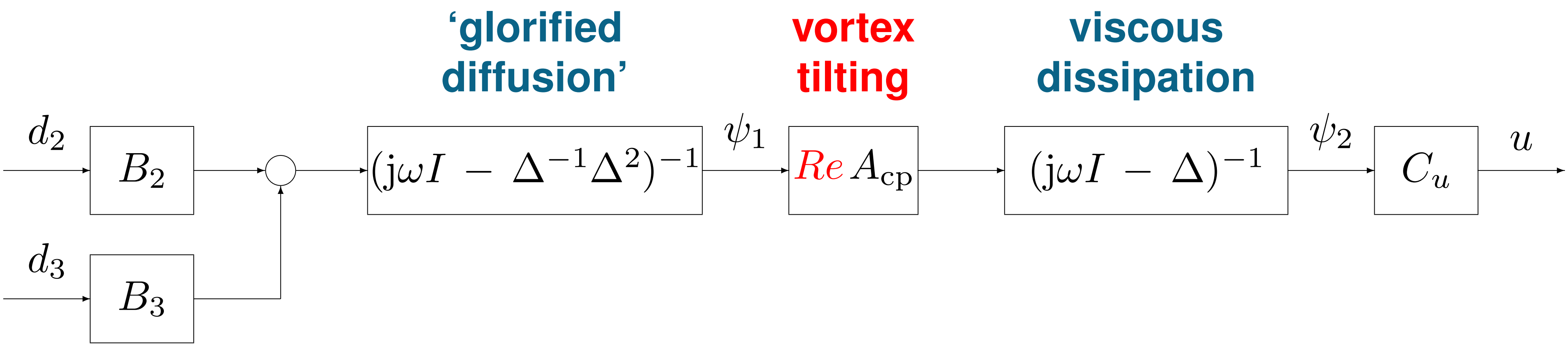
$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & C_u \\ C_v & 0 \\ C_w & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$



Jovanović & Bamieh, *J. Fluid Mech.* ’05

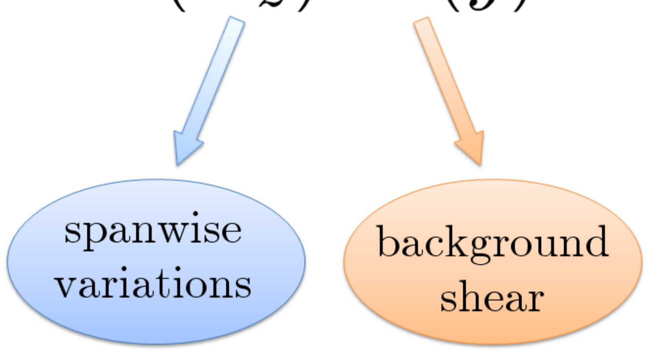
Amplification mechanism in flows with high Re

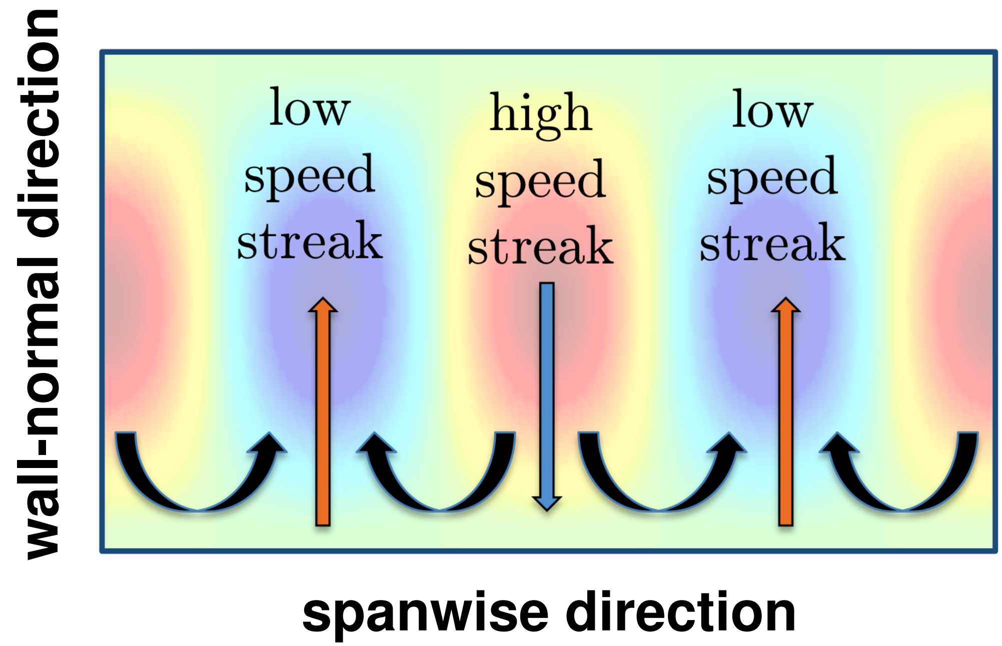
- HIGHEST AMPLIFICATION: $(d_2, d_3) \to u$



☞ AMPLIFICATION MECHANISM: **vortex tilting** or **lift-up**

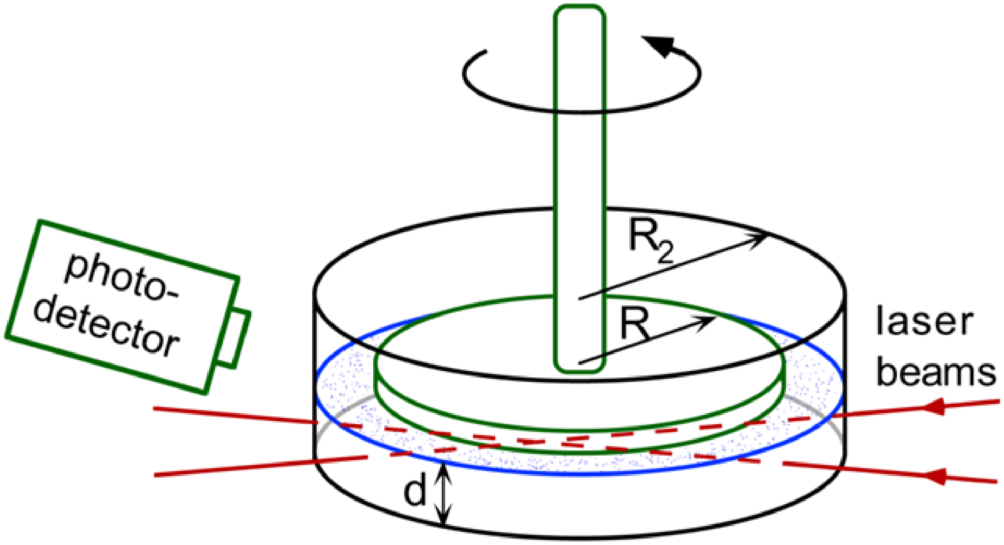
$$A_{cp} = -(\mathrm{i}k_z)\; U'(y)$$





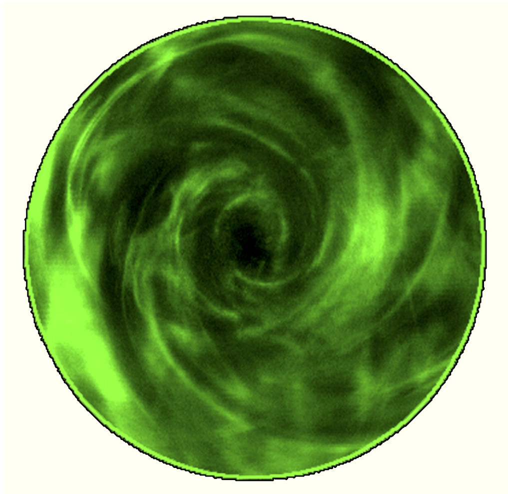
Turbulence without inertia

NEWTONIAN: **inertial turbulence** VISCOELASTIC: **elastic turbulence**



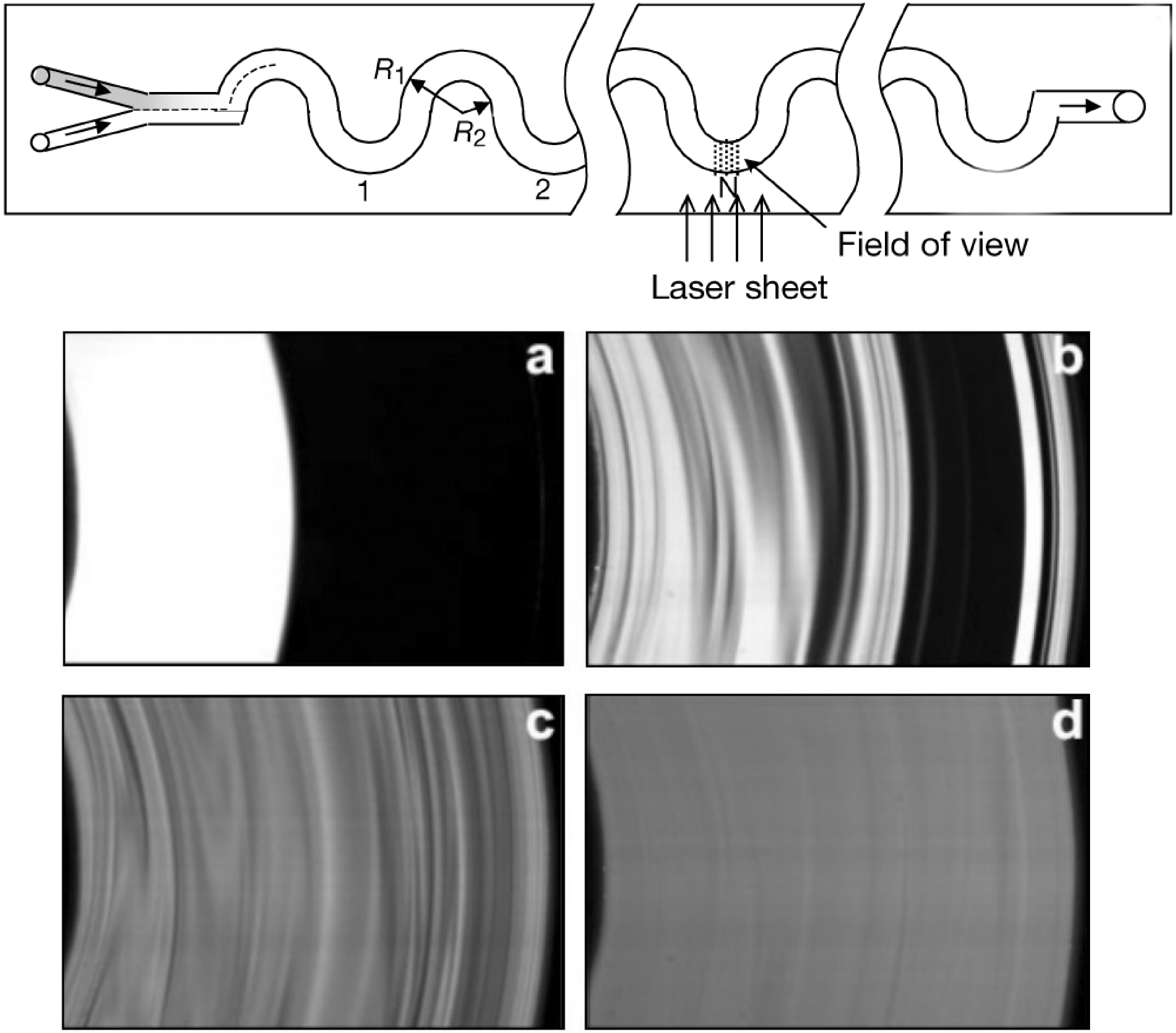
Groisman & Steinberg, *Nature '00*

NEWTONIAN: VISCOELASTIC:



☞ FLOW RESISTANCE: **increased 20 times!**

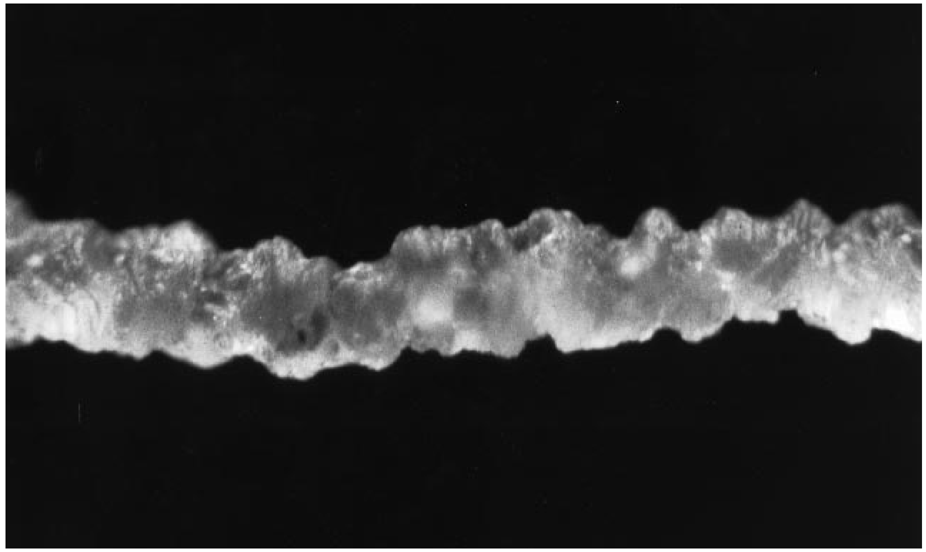
Turbulence: good for mixing . . .



Groisman & Steinberg, *Nature '01*

. . . bad for processing

POLYMER MELT EMERGING FROM A CAPILLARY TUBE

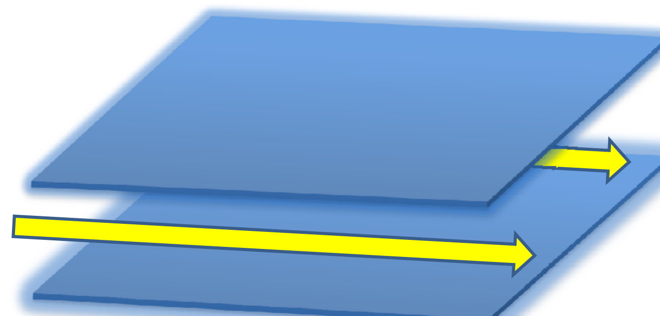


Kalika & Denn, *J. Rheol. '87*

CURVILINEAR FLOWS: **purely elastic instabilities**

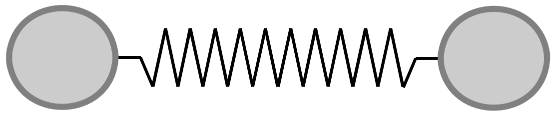
Larson, Shaqfeh, Muller, *J. Fluid Mech. '90*

RECTILINEAR FLOWS: **no modal instabilities**



Oldroyd-B fluids

HOOKEAN SPRING:



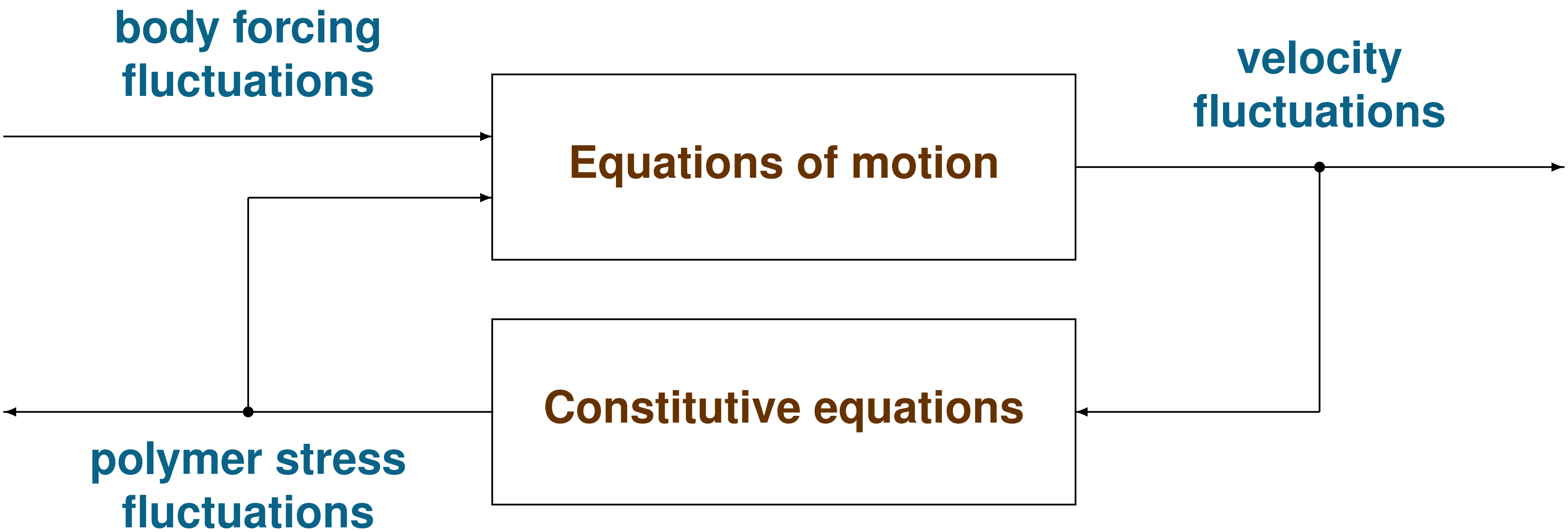
$$
\begin{aligned}
(Re/We)\,\mathbf{v}_t &= -Re\,(\mathbf{v}\cdot\nabla)\,\mathbf{v} \;-\; \nabla p \;+\; \beta\,\Delta\mathbf{v} + (1-\beta)\,\nabla\cdot\boldsymbol{\tau} \;+\; \mathbf{d} \\
0 &= \nabla\cdot\mathbf{v} \\
\boldsymbol{\tau}_t &= -\boldsymbol{\tau} \;+\; \nabla\mathbf{v} \;+\; (\nabla\mathbf{v})^T \;+\; We\left(\boldsymbol{\tau}\cdot\nabla\mathbf{v} + (\nabla\mathbf{v})^T\cdot\boldsymbol{\tau} \;-\; (\mathbf{v}\cdot\nabla)\boldsymbol{\tau}\right)
\end{aligned}
$$

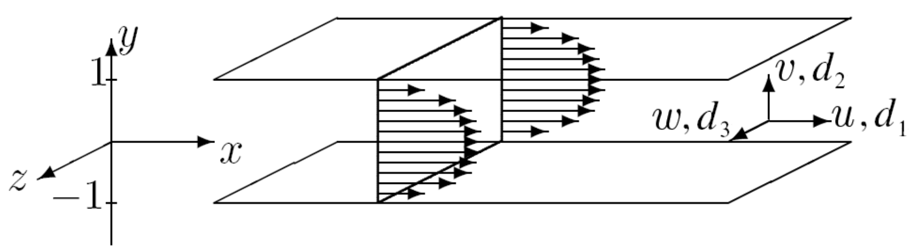
VISCOSITY RATIO: $\beta \;:=\; \dfrac{\text{solvent viscosity}}{\text{total viscosity}}$

WEISSENBERG NUMBER: $We \;:=\; \dfrac{\text{fluid relaxation time}}{\text{characteristic flow time}}$

REYNOLDS NUMBER: $Re \;:=\; \dfrac{\text{inertial forces}}{\text{viscous forces}}$

Input-output analysis





$$\underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{\mathbf{d}} \xrightarrow{\text{amplification}} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\mathbf{v}}$$

- INSIGHT INTO AMPLIFICATION MECHANISMS

  importance of streamwise elongated structures

Hoda, Jovanović, Kumar, *J. Fluid Mech. '08, '09*

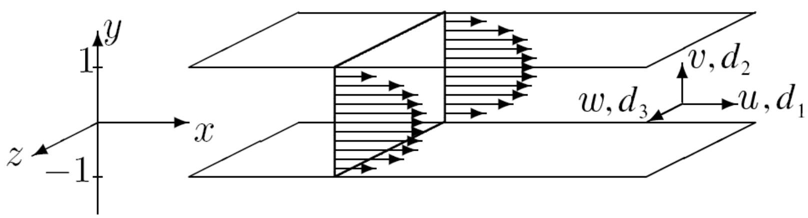
Jovanović & Kumar, *JNNFM '11*

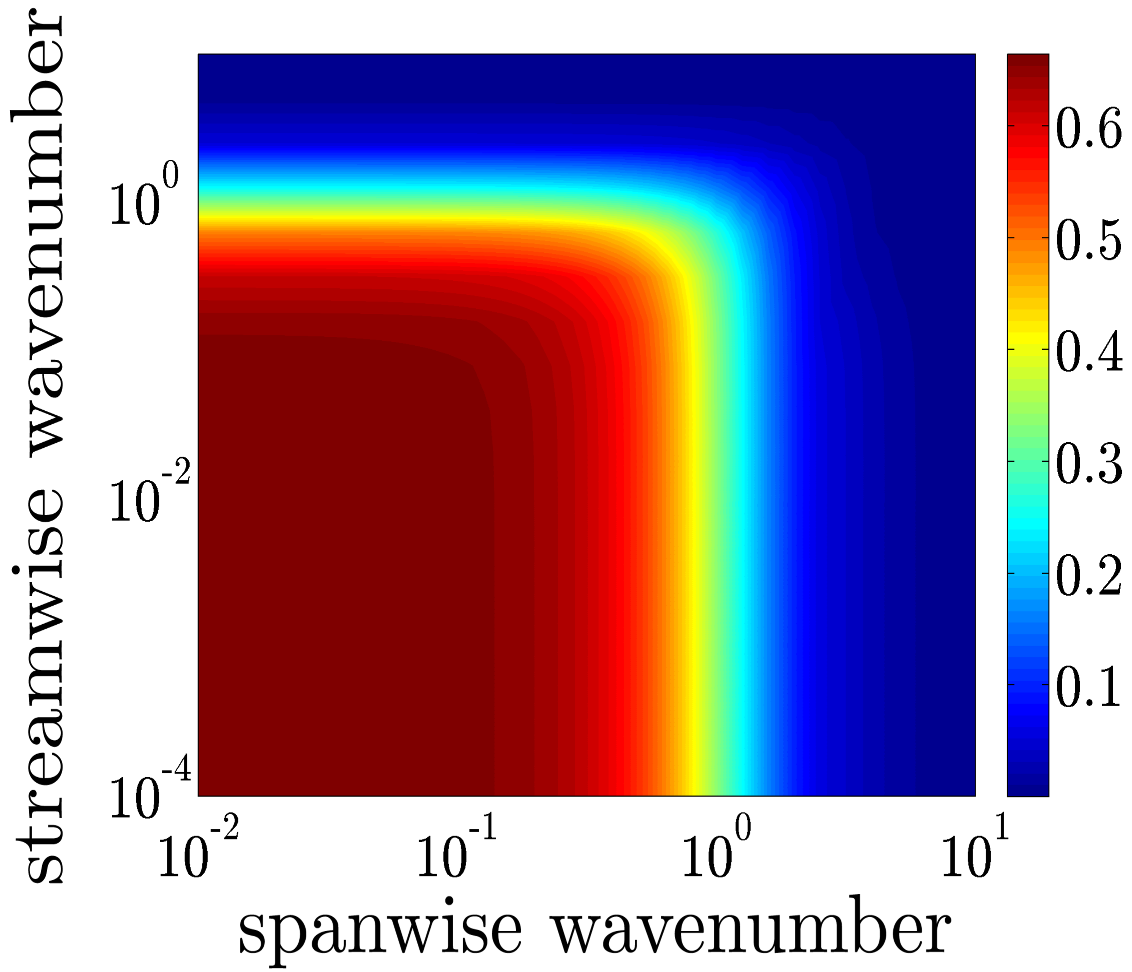
Inertialess channel flow: worst case amplification

- **No single constitutive equation can describe the range of phenomena**
  - ⋆ **important to quantify influence of modeling imperfections on dynamics**

$$G(k_x, k_z) \;=\; \sup_{\omega}\, \sigma_{\max}^2 \left(\mathcal{T}(k_x, k_z, \omega) \right):$$

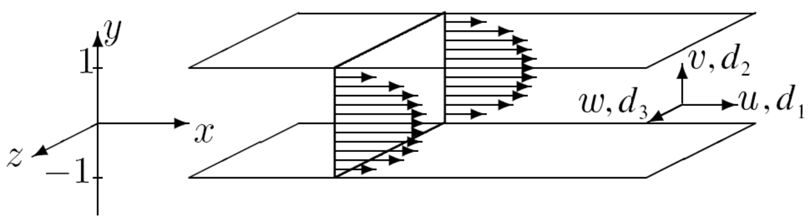
$We = 10$, $\beta = 0.5$, $Re = 0$

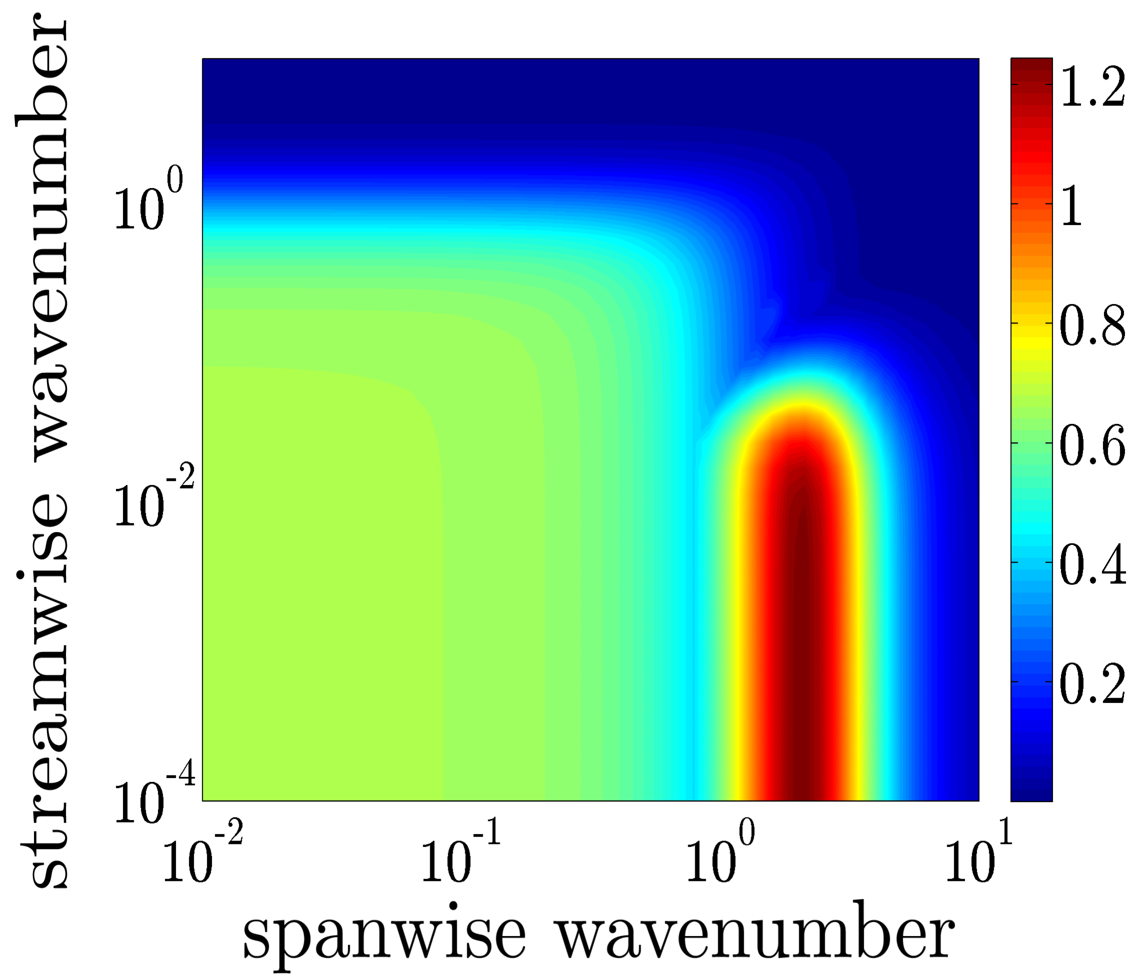




$G(k_x, k_z)$:

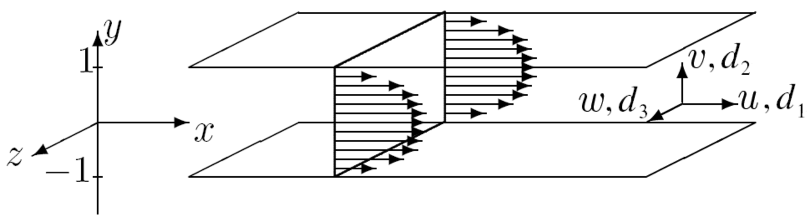
$We = 50$, $\beta = 0.5$, $Re = 0$

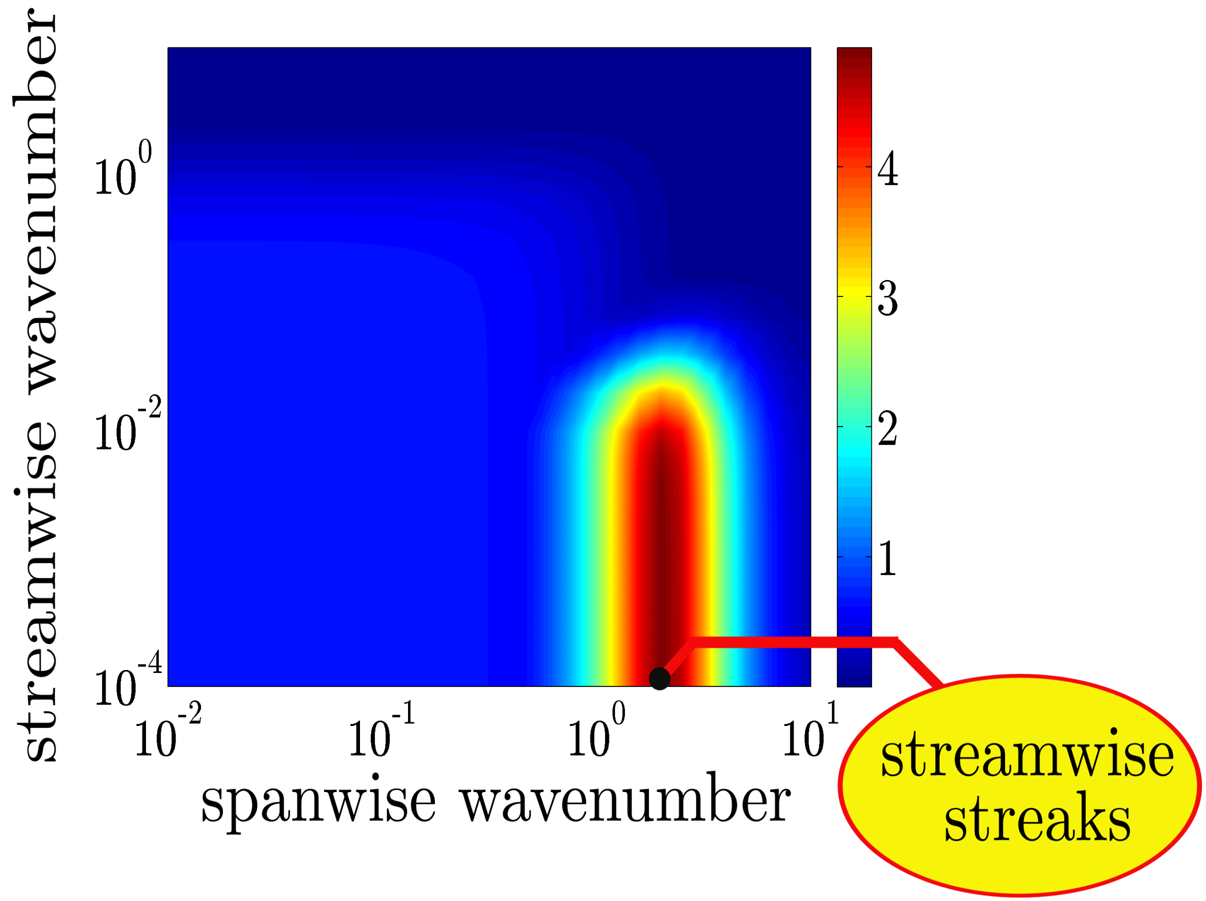




$G(k_x, k_z)$:

$We = 100$, $\beta = 0.5$, $Re = 0$



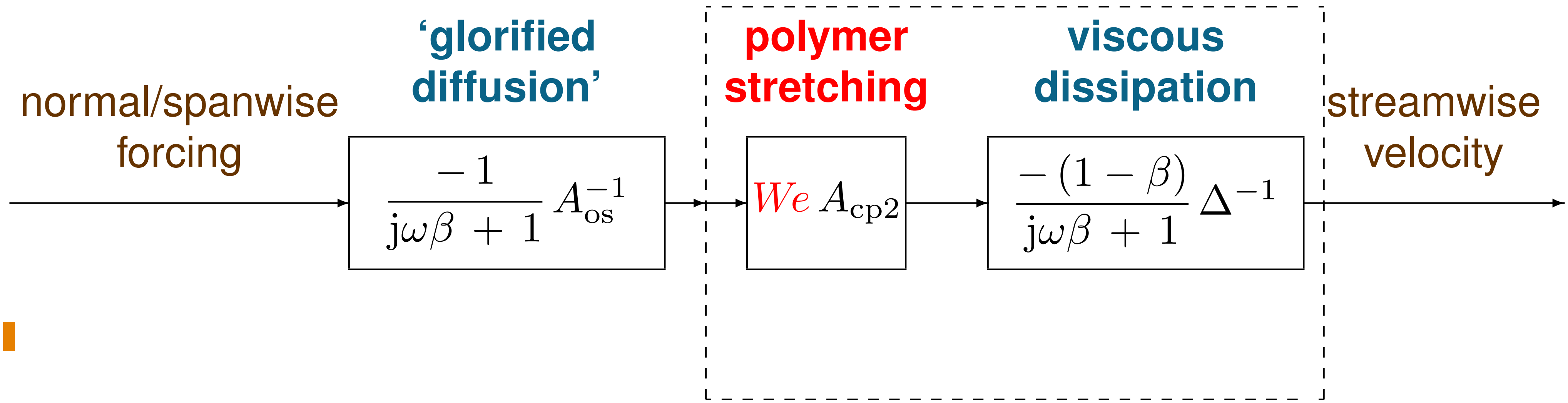


- **Dominance of streamwise elongated structures**
  streamwise streaks!

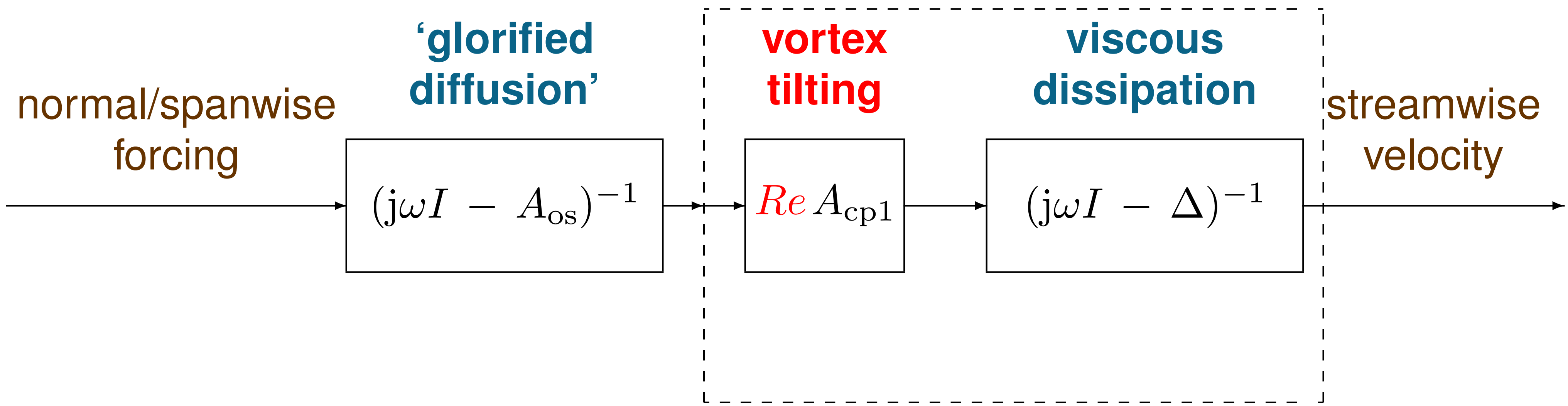
Amplification mechanism

- **Highest amplification:** $(d_2, d_3) \to u$

INERTIALESS VISCOELASTIC:

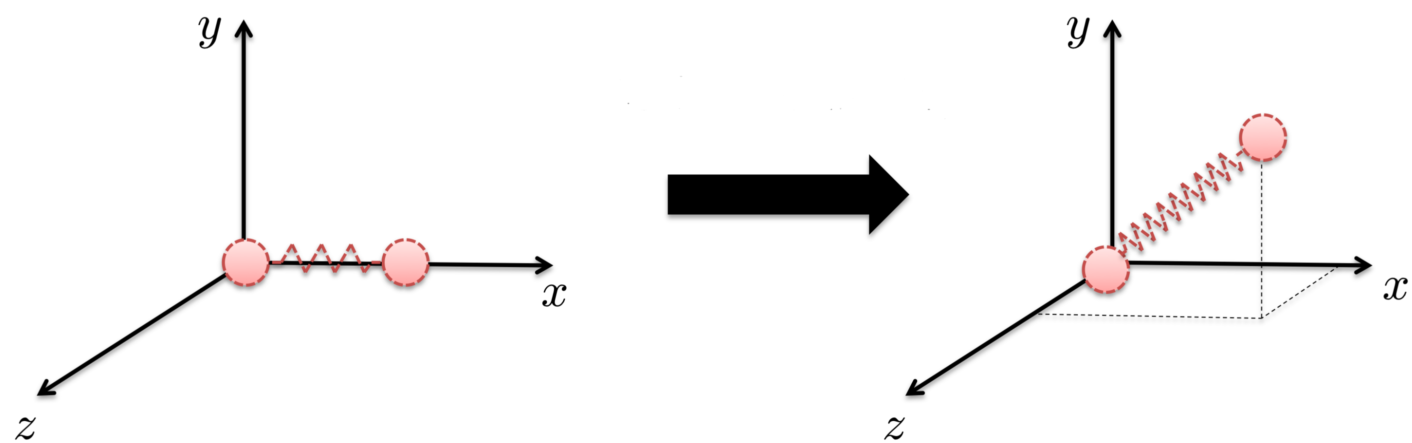


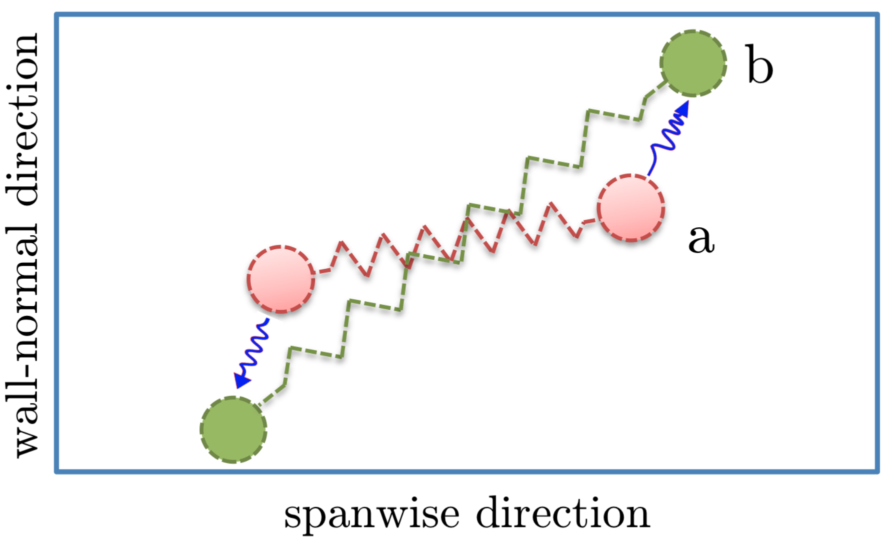
INERTIAL NEWTONIAN:

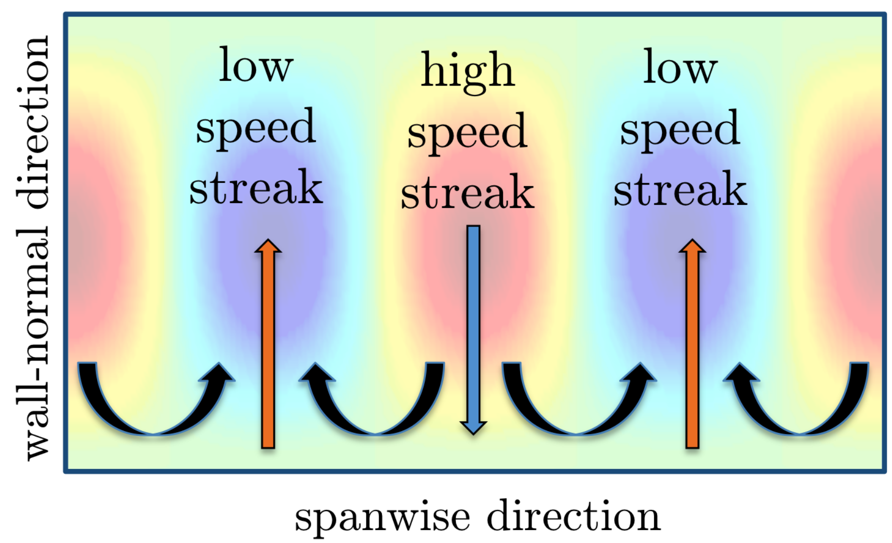


Inertialess lift-up mechanism

$$\begin{aligned}\Delta\,\eta_t &= -(1/\beta)\Delta\eta \;+\; \color{red}{We}\,(1 - 1/\beta)\,A_{\mathrm{cp2}}\,\vartheta \\ &= -(1/\beta)\Delta\eta \;+\; \color{red}{We}\,(1 - 1/\beta)\left(\partial_{yz}\,(U'(y)\,\tau_{22}) + \partial_{zz}\,(U'(y)\,\tau_{23})\right)\end{aligned}$$





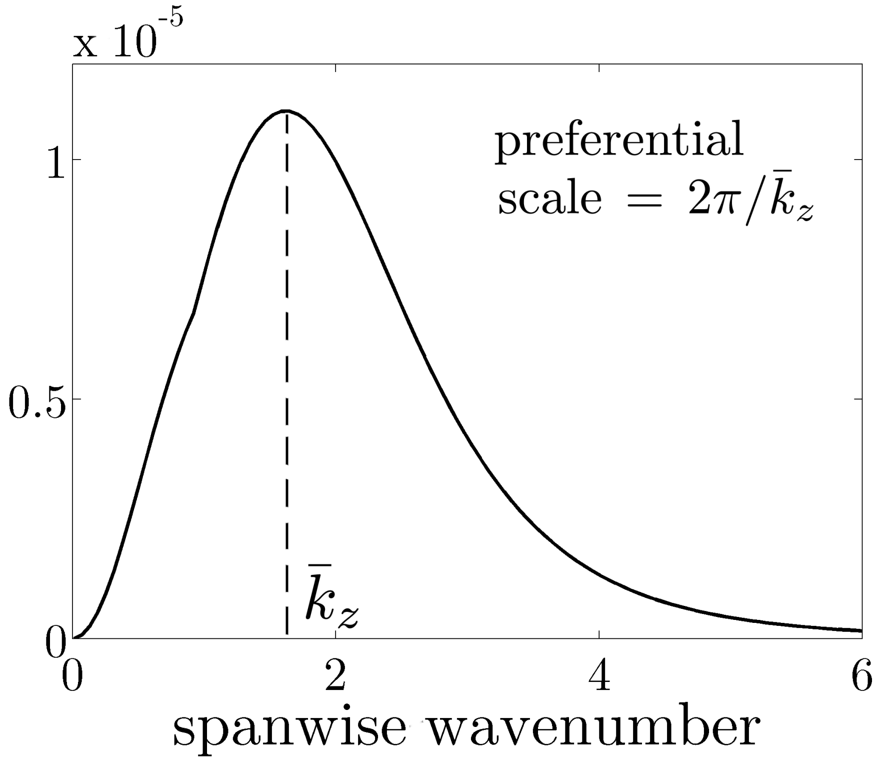


Spatial frequency responses

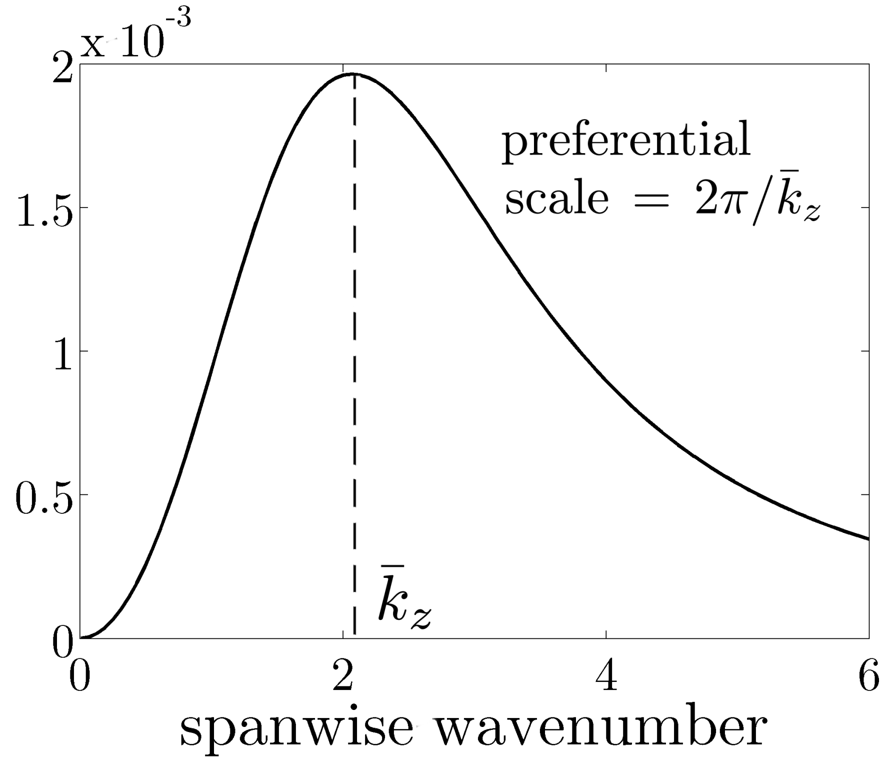
$$(d_2, d_3) \xrightarrow{\text{amplification}} u$$

INERTIAL NEWTONIAN:	$G(k_z; Re) \;=\; Re^2\, f(k_z)$
INERTIALESS VISCOELASTIC:	$G(k_z; We, \beta) \;=\; We^2\, g(k_z)\,(1-\beta)^2$

vortex tilting: $f(k_z)$



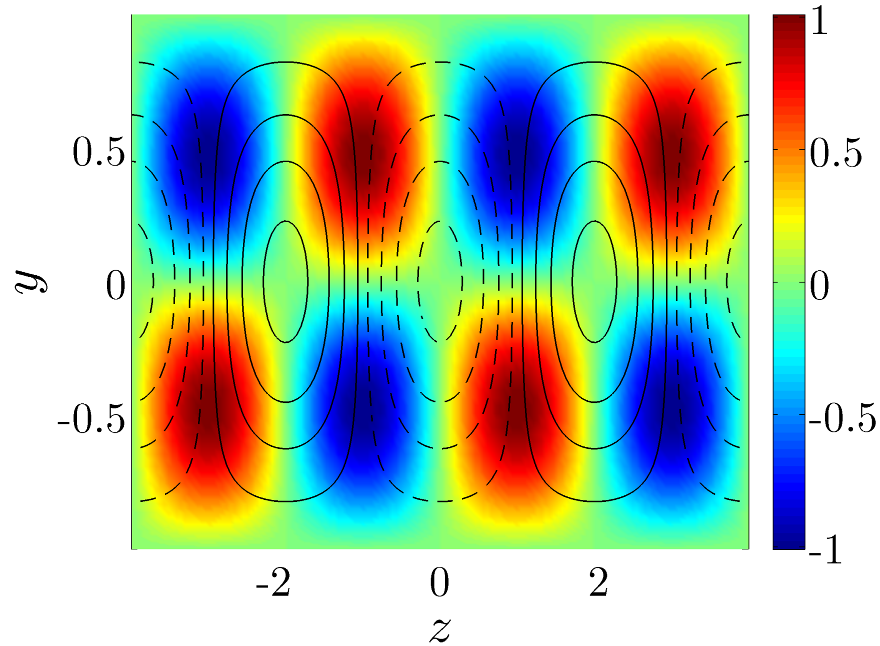
polymer stretching: $g(k_z)$



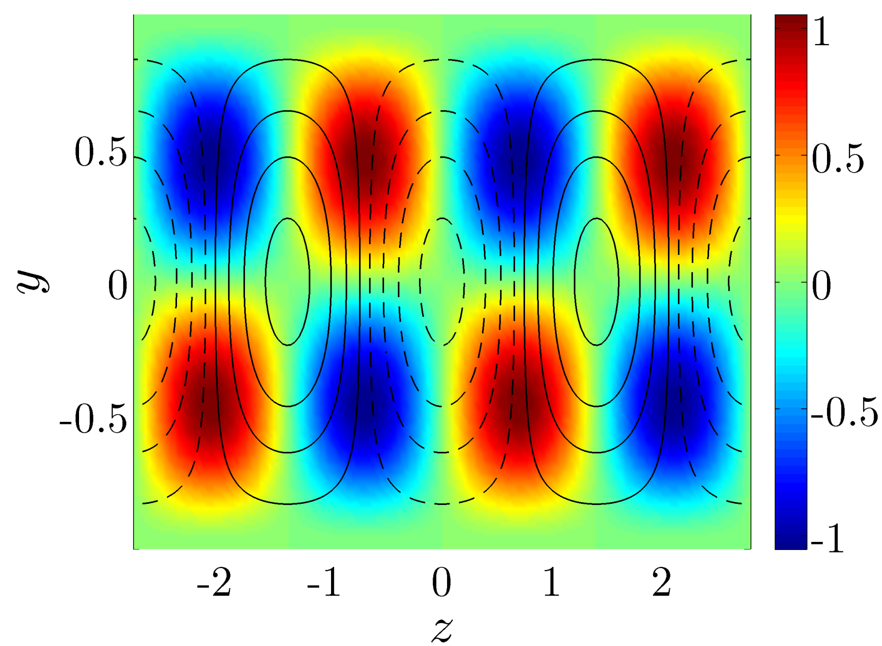
Dominant flow patterns

- FREQUENCY RESPONSE PEAKS
  - ☞ **streamwise vortices and streaks**

Inertial Newtonian:



Inertialess viscoelastic:

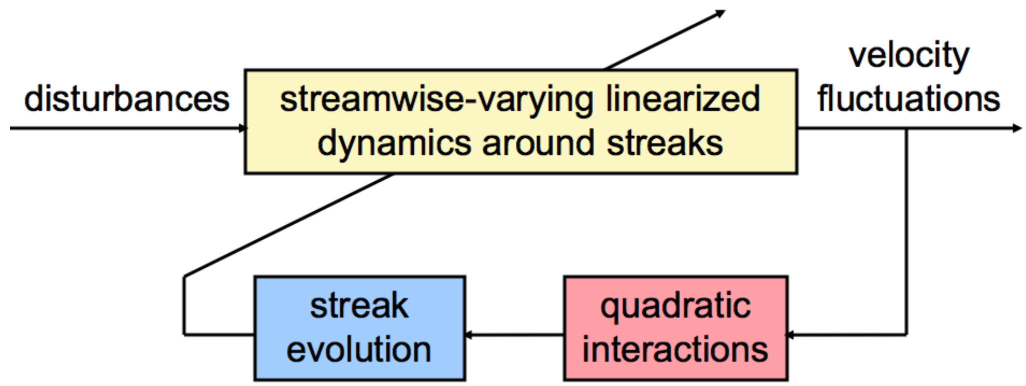


- CHANNEL CROSS-SECTION VIEW: { color plots: streamwise velocity; contour lines: stream-function }

color plots:	streamwise velocity
contour lines:	stream-function

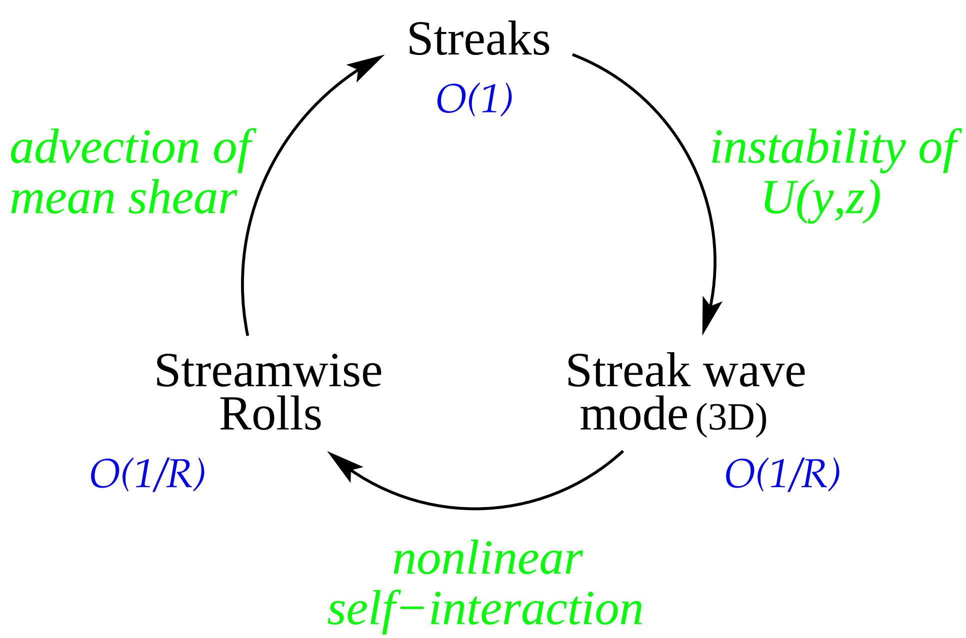
Flow sensitivity vs. nonlinearity

- **Challenge: relative roles of flow sensitivity and nonlinearity**



- **Newtonian fluids: self-sustaining process for transition to turbulence**

  Waleffe, *Phys. Fluids '97*



Lecture 22: Stability of infinite dimensional systems

- Exponential stability
  - ⋆ Definition
  - ⋆ Conditions
  - ⋆ Lyapunov-based characterization
  - ⋆ Examples

Exponential stability

$$\psi_t(t) \;=\; \mathcal{A}\,\psi(t), \;\; \psi(0) \;=\; \psi_0 \in \mathbb{H}$$

- Exponential stability of a C_0-semigroup $\mathcal{T}(t)$ generated by $\mathcal{A}$

  there exist $M > 0$, $\alpha > 0$ s.t. $\|\mathcal{T}(t)\| \leq M \mathrm{e}^{-\alpha\, t}$ for all $t \geq 0$

- Consequence

  ⋆ exponential convergence to zero of solutions to $\psi_t(t) \;=\; \mathcal{A}\,\psi(t)$

$$\|\psi(t)\| \;\leq\; M\,\|\psi_0\|\,\mathrm{e}^{-\alpha\, t}$$

Conditions for exponential stability

DATKO'S LEMMA:

Exponential stability of $\mathcal{T}(t)$ on $\mathbb{H}$

$$\Updownarrow$$

for every $\psi_0 \in \mathbb{H}$ there exists positive $\gamma_\psi < \infty$ s.t.

$$\int_0^\infty \|\mathcal{T}(t)\,\psi_0\|^2\,\mathrm{d}t \;\leq\; \gamma_\psi$$

Lyapunov-based characterization

Exponential stability of $\mathcal{T}(t)$ on $\mathbb{H}$

$$\Updownarrow$$

there exists a bounded positive operator $\mathcal{P}$ s.t.

$$\langle \mathcal{A}\,\psi, \mathcal{P}\,\psi\rangle \;+\; \langle \mathcal{P}\,\psi, \mathcal{A}\,\psi\rangle \;=\; -\,\langle \psi, \psi\rangle \quad \text{for all } \psi \in \mathcal{D}\,(\mathcal{A})$$

- $\mathcal{P}$ – infinite horizon observability Gramian of system with $\mathcal{C} = I$

$$\mathcal{P}\,\psi_0 \;=\; \int_0^\infty \mathcal{T}^\dagger(t)\,\mathcal{T}(t)\,\psi_0\,\mathrm{d}t, \quad \psi_0 \in \mathbb{H}$$

- Lyapunov functional

$$V(\psi(t)) \;=\; \langle \psi(t), \mathcal{P}\,\psi(t)\rangle \;=\; \langle \mathcal{T}(t)\,\psi(0), \mathcal{P}\,\mathcal{T}(t)\,\psi(0)\rangle$$

Example: diffusion equation on $L_2\,[-1, 1]$

$$\begin{aligned} \psi_t(x,t) &= \psi_{xx}(x,t) \\ \psi(x,0) &= \psi_0(x) \\ \psi(\pm 1, t) &= 0 \end{aligned}$$

- Lyapunov equation

$$\mathcal{A}^\dagger \mathcal{P} + \mathcal{P}\mathcal{A} = -I \quad \text{on } \mathcal{D}(\mathcal{A})$$

$$\mathcal{A}^\dagger = \mathcal{A} \;\Rightarrow\; \phi = \mathcal{P}\psi = -\frac{1}{2}\mathcal{A}^{-1}\psi$$

- Lyapunov functional

$$\left.\begin{aligned} V(\psi) &= \langle \psi, \mathcal{P}\psi \rangle = \langle \psi, \phi \rangle \\ \phi''(x) &= -\frac{1}{2}\psi(x), \;\; \phi(\pm 1) = 0 \end{aligned}\right\}$$

$$\Downarrow$$

$$V(\psi(t)) = \int_{-1}^{1}\int_{-1}^{1} \psi^*(x,t)\, P_{\mathrm{ker}}(x,\xi)\, \psi(\xi,t)\, \mathrm{d}\xi\, \mathrm{d}x$$

- Alternative approach

$$V(\psi) \;=\; \frac{1}{2}\,\langle \psi, \psi \rangle \quad \Rightarrow \quad \left\{ \begin{array}{l} \dfrac{\mathrm{d}\,V(\psi(t))}{\mathrm{d}\,t} \;=\; \langle \psi(t), \partial_{xx}\,\psi(t) \rangle \;\le\; -\,\epsilon_Q\,\|\psi(t)\|^2 \\[2ex] \dfrac{\mathrm{d}\,\|\psi(t)\|^2}{\mathrm{d}\,t} \;\le\; -2\,\epsilon_Q\,\|\psi(t)\|^2 \end{array} \right.$$

- In class:

  ⋆ Use $V(\psi) \;=\; \frac{1}{2}\,\langle \psi, \psi \rangle$ to show exponential stability of

$$\begin{aligned} \psi_t(x,t) \;&=\; \psi_{xx}(x,t) \;-\; \mathrm{j}\kappa\,U(x)\,\psi(x,t) \\ \psi(x,0) \;&=\; \psi_0(x) \\ \psi(\pm 1,t) \;&=\; 0 \end{aligned}$$

Lecture 23: Optimal control of distributed systems

- Linear Quadratic Regulator (LQR)
  - ⋆ Linear: plant
  - ⋆ Quadratic: performance index
  - ⋆ Infinite horizon problem
  - ⋆ Algebraic Riccati Equation (ARE)

- Spatially invariant systems
  - ⋆ LQR: also spatially invariant
  - ⋆ Feedback gains decay exponentially with spatial distance

- Examples
  - ⋆ Distributed control
  - ⋆ Boundary control

Linear Quadratic Regulator

$$\text{minimize} \quad J \;=\; \int_0^\infty \Big(\langle \psi(t), \mathcal{Q}\,\psi(t) \rangle \;+\; \langle u(t), \mathcal{R}\,u(t) \rangle \Big)\,\mathrm{d}t$$

$$\text{subject to} \quad \psi_t(t) \;=\; \mathcal{A}\,\psi(t) \;+\; \mathcal{B}\,u(t), \quad \psi(0) \in \mathbb{H}$$

- Finite dimensional problems

  ⋆ Optimal controller determined by

$$\begin{aligned} u(t) &= -K\,\psi(t) \\ K &= R^{-1}B^T P \end{aligned}$$

  ⋆ $P = P^*$ – non-negative solution to ARE

$$A^*P \;+\; P\,A \;+\; Q \;-\; P\,B\,R^{-1}B^*P \;=\; 0$$

  ⋆ ARE – quadratic equation in the elements of P

- Infinite dimensional problems
  - ⋆ Optimal controller determined by

$$
\begin{aligned}
u(t) &= -\mathcal{K}\,\psi(t) \\
\mathcal{K} &= \mathcal{R}^{-1}\mathcal{B}^{\dagger}\,\mathcal{P}
\end{aligned}
$$

  - ⋆ $\mathcal{P} = \mathcal{P}^{\dagger}$ – bounded non-negative operator that solves ARE

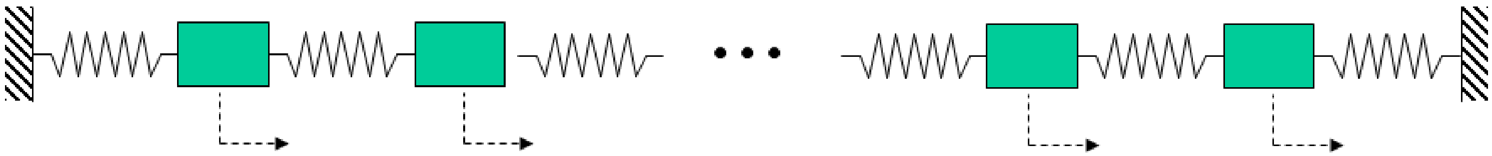
$$
\langle \mathcal{A}\,\psi_1, \mathcal{P}\,\psi_2 \rangle + \langle \mathcal{P}\,\psi_1, \mathcal{A}\,\psi_2 \rangle + \left\langle \mathcal{Q}^{\frac{1}{2}}\,\psi_1, \mathcal{Q}^{\frac{1}{2}}\,\psi_2 \right\rangle - \langle \mathcal{B}^{\dagger}\,\mathcal{P}\,\psi_1, \mathcal{R}^{-1}\mathcal{B}^{\dagger}\,\mathcal{P}\,\psi_2 \rangle = 0
$$

$$
\psi_1, \psi_2 \in \mathcal{D}(\mathcal{A})
$$

  - ⋆ ARE – operator-valued equation in the unknown $\mathcal{P}$

An example

- Mass-spring system on a line



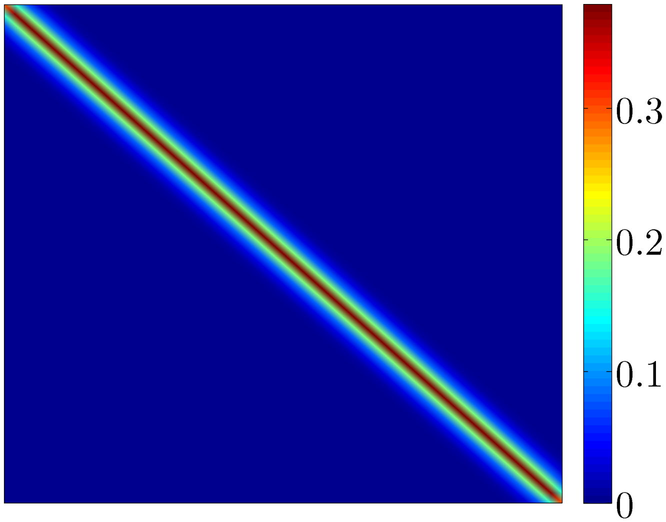
$$\begin{bmatrix} \dot{p} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ T & 0 \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

$$T \sim \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

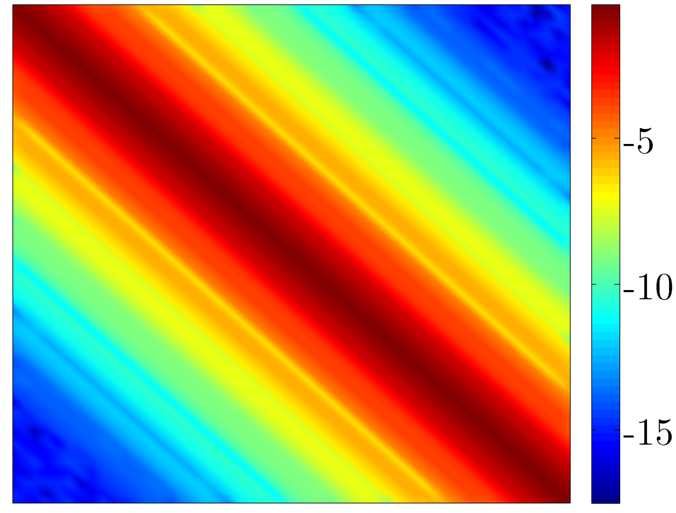
In class: use Matlab to illustrate structure of optimal feedback gains

Structure of optimal solution

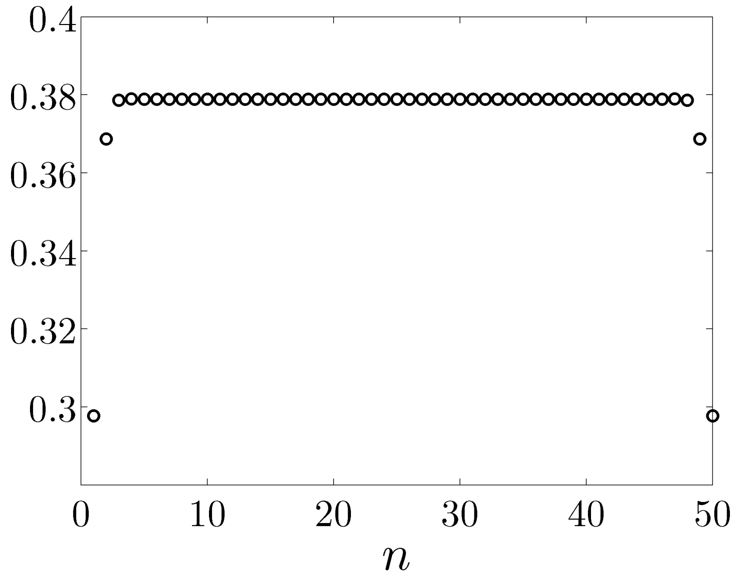
K_p:



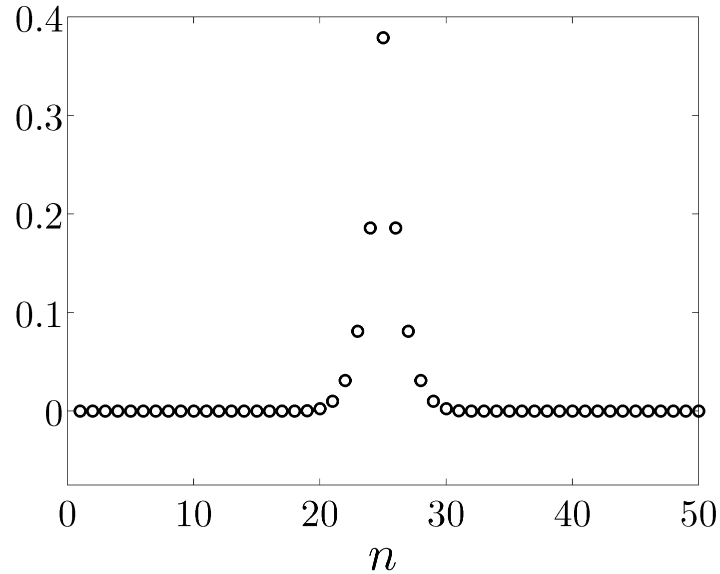
$\log_{10}(|K_p|)$:

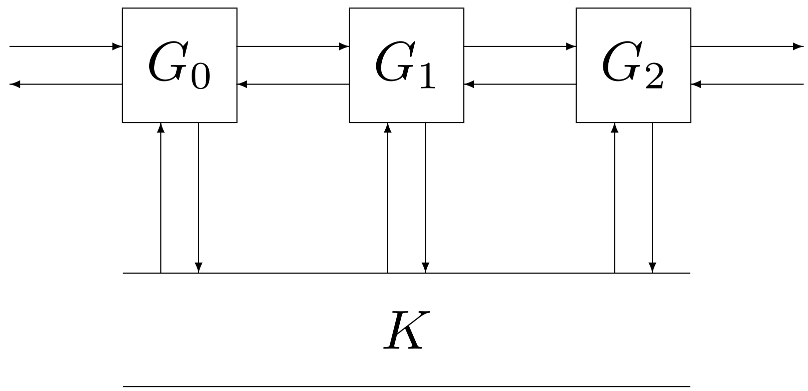


$\mathrm{diag}(K_p)$:



$K_p(25,:)$:





$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = -\underbrace{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}}_{K_p} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix} - \underbrace{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}}_{K_v} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \end{bmatrix}$$

- Observations:
  - ⋆ LQR – centralized controller
  - ⋆ Diagonals almost constant (modulo edges)
  - ⋆ Off-diagonal decay of centralized gain

Spatially invariant systems

$$\psi_t(x,t) \;=\; [\,\mathcal{A}\,\psi(\cdot,t)\,]\,(x) \;+\; [\,\mathcal{B}\,u(\cdot,t)\,]\,(x)$$

spatial coordinate: $x \in \mathbb{G}$

translation invariant operators: $\mathcal{A}$, $\mathcal{B}$

SPATIAL FOURIER TRANSFORM

$$\dot{\hat{\psi}}(\kappa,t) \;=\; \hat{\mathcal{A}}(\kappa)\,\hat{\psi}(\kappa,t) \;+\; \hat{\mathcal{B}}(\kappa)\,\hat{u}(\kappa,t)$$

spatial frequency: $\kappa \in \hat{\mathbb{G}}$

multiplication operators: $\hat{\mathcal{A}}(\kappa)$, $\hat{\mathcal{B}}(\kappa)$

$\mathbb{G}$	$\mathbb{R}$	$\mathbb{S}$	$\mathbb{Z}$	$\mathbb{Z}_N$
$\hat{\mathbb{G}}$	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{S}$	$\mathbb{Z}_N$

- $\mathbb{R}$ reals
- $\mathbb{Z}$ integers
- $\mathbb{S}$ unit circle
- $\mathbb{Z}_N$ integers modulo N

- Partial Differential Equations

  ⋆ Constant coefficients + Infinite spatial extent

$$\psi_t(x,t) \;=\; \psi_{xx}(x,t) \;+\; u(x,t), \quad x \in \mathbb{R}$$

$$\Big\downarrow \textbf{Fourier transform}$$

$$\dot{\hat{\psi}}(\kappa,t) \;=\; -\,\kappa^2\,\hat{\psi}(\kappa,t) \;+\; \hat{u}(\kappa,t), \quad \kappa \in \mathbb{R}$$

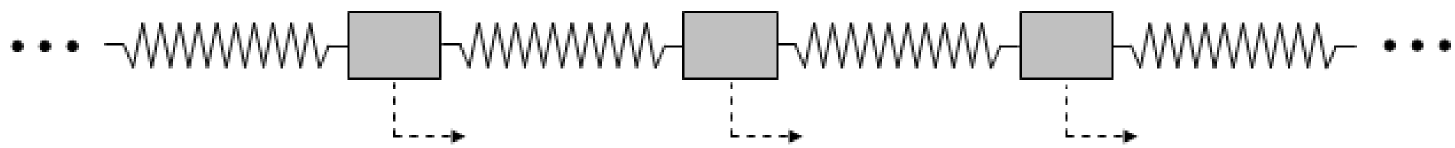
  ⋆ Constant coefficients + Periodic domain

$$\psi_t(x,t) \;=\; \psi_{xx}(x,t) \;+\; u(x,t), \quad x \in \mathbb{S}$$

$$\Big\downarrow \textbf{Fourier series}$$

$$\dot{\hat{\psi}}(\kappa,t) \;=\; -\,\kappa^2\,\hat{\psi}(\kappa,t) \;+\; \hat{u}(\kappa,t), \quad \kappa \in \mathbb{Z}$$

- Spatially discrete systems (Interconnected ODEs)
  - ⋆ Constant coefficients $+$ Infinite lattices

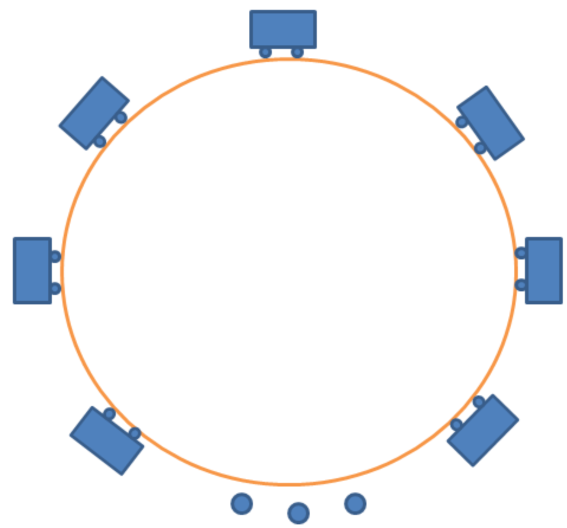


$$\dot{\psi}(x,t) \;=\; \begin{bmatrix} 0 & 1 \\ S_{-1} \,-\, 2 \,+\, S_1 & 0 \end{bmatrix} \psi(x,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(x,t), \quad x \in \mathbb{Z}$$

$\downarrow$ **$\mathcal{Z}$-transform evaluated at $z = \mathrm{e}^{\mathrm{j}\kappa}$**

$$\dot{\hat{\psi}}(\kappa,t) \;=\; \begin{bmatrix} 0 & 1 \\ 2\,(\cos\,\kappa \,-\, 1) & 0 \end{bmatrix} \hat{\psi}(\kappa,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(\kappa,t), \quad \kappa \in \mathbb{S}$$

⋆ Constant coefficients $+$ Circular lattices



Example: Mass-spring system on a circle

$$\dot{\psi}(x,t) \;=\; \begin{bmatrix} 0 & 1 \\ S_{-1} \,-\, 2 \,+\, S_1 & 0 \end{bmatrix} \psi(x,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(x,t), \quad x \,\in\, \mathbb{Z}_N$$

$\downarrow$ **discrete Fourier transform**

$$\dot{\hat{\psi}}(\kappa,t) \;=\; \begin{bmatrix} 0 & 1 \\ 2\left(\cos \frac{2\,\pi\,\kappa}{N} \,-\, 1\right) & 0 \end{bmatrix} \hat{\psi}(\kappa,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(\kappa,t), \quad \kappa \,\in\, \mathbb{Z}_N$$

LQR for spatially invariant system over $\mathbb{Z}_N$

$$\text{minimize} \quad J \;=\; \int_0^\infty \Big(\psi^*(t)\, Q\, \psi(t) \,+\, u^*(t)\, R\, u(t) \Big)\, \mathrm{d}t$$

$$\text{subject to} \quad \dot{\psi}(t) \;=\; A\, \psi(t) \,+\, B\, u(t)$$

- Circulant matrices: A, B, Q, R

  ⋆ Jointly unitarily diagonalizable by DFT Matrix V

$$\begin{aligned} \dot{\hat{\psi}}(t) &= A_d\, \hat{\psi}(t) \,+\, B_d\, \hat{u}(t) \\ A_d &= \mathrm{diag}\Big(\hat{A}(\kappa)\Big) \;=\; V A V^* \\ \psi^*\, Q\, \psi &= \hat{\psi}^*\, Q_d\, \hat{\psi} \end{aligned}$$

  ⋆ Entries into ARE – diagonal matrices

$$A_d^* P_d \,+\, P_d\, A_d \,+\, Q_d \,-\, P_d\, B_d\, R_d^{-1} B_d^* P_d \;=\; 0$$

$$\Updownarrow$$

$$\hat{A}^*(\kappa)\, \hat{P}(\kappa) \,+\, \hat{P}(\kappa)\, \hat{A}(\kappa) \,+\, \hat{Q}(\kappa) \,-\, \hat{P}(\kappa)\, \hat{B}(\kappa)\, \hat{R}^{-1}(\kappa)\, \hat{B}^*(\kappa)\, \hat{P}(\kappa) \;=\; 0, \quad \kappa \in \mathbb{Z}_N$$

Lecture 24: LQR for spatially invariant systems

- Structure of optimal distributed controllers
  - ⋆ Also spatially invariant
  - ⋆ Feedback gains decay exponentially with spatial distance
  - ⋆ Obtained from solving parameterized family of AREs

- Examples
  - ⋆ Systems on lattices
  - ⋆ PDEs
  - ⋆ Vehicular formations

Spatially invariant systems

$$\psi_t(x,t) \;=\; [\,\mathcal{A}\,\psi(\cdot,t)\,]\,(x) \;+\; [\,\mathcal{B}\,u(\cdot,t)\,]\,(x)$$

spatial coordinate: $x \in \mathbb{G}$

translation invariant operators: $\mathcal{A}$, $\mathcal{B}$

Spatial Fourier transform

$$\dot{\hat{\psi}}(\kappa,t) \;=\; \hat{\mathcal{A}}(\kappa)\,\hat{\psi}(\kappa,t) \;+\; \hat{\mathcal{B}}(\kappa)\,\hat{u}(\kappa,t)$$

spatial frequency: $\kappa \in \hat{\mathbb{G}}$

multiplication operators: $\hat{\mathcal{A}}(\kappa)$, $\hat{\mathcal{B}}(\kappa)$

$\mathbb{G}$	$\mathbb{R}$	$\mathbb{S}$	$\mathbb{Z}$	$\mathbb{Z}_N$
$\hat{\mathbb{G}}$	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{S}$	$\mathbb{Z}_N$

$$\begin{cases} \mathbb{R} & \text{reals} \\ \mathbb{Z} & \text{integers} \\ \mathbb{S} & \text{unit circle} \\ \mathbb{Z}_N & \text{integers modulo } N \end{cases}$$

LQR for spatially invariant systems over $\mathbb{Z}_N$

$$\text{minimize} \quad J \;=\; \int_0^\infty \Big(\psi^*(t)\, Q\, \psi(t) \;+\; u^*(t)\, R\, u(t)\Big)\, \mathrm{d}t$$

$$\text{subject to} \quad \dot{\psi}(t) \;=\; A\,\psi(t) \;+\; B\,u(t)$$

- Circulant matrices: A, B, Q, R

⋆ Jointly unitarily diagonalizable by DFT Matrix V

$$\begin{aligned} \dot{\hat{\psi}}(t) &= A_d\,\hat{\psi}(t) \;+\; B_d\,\hat{u}(t) \\ A_d &= \mathrm{diag}\left(\hat{A}(\kappa)\right) \;=\; V A V^* \\ \psi^*\, Q\, \psi &= \hat{\psi}^*\, Q_d\, \hat{\psi} \end{aligned}$$

⋆ Entries into ARE – diagonal matrices

$$A_d^* P_d \;+\; P_d A_d \;+\; Q_d \;-\; P_d B_d R_d^{-1} B_d^* P_d \;=\; 0$$

$$\Updownarrow$$

$$\hat{A}^*(\kappa)\,\hat{P}(\kappa) \;+\; \hat{P}(\kappa)\,\hat{A}(\kappa) \;+\; \hat{Q}(\kappa) \;-\; \hat{P}(\kappa)\,\hat{B}(\kappa)\,\hat{R}^{-1}(\kappa)\,\hat{B}^*(\kappa)\,\hat{P}(\kappa) \;=\; 0, \quad \kappa \in \mathbb{Z}_N$$

Example: mass-spring system on a circle

$$\dot{\psi}(x,t) \;=\; \begin{bmatrix} 0 & 1 \\ S_{-1} \,-\, 2 \,+\, S_1 & 0 \end{bmatrix} \psi(x,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(x,t), \quad x \in \mathbb{Z}_N$$

$\downarrow$ **discrete Fourier transform**

block-diagonal family of 2nd order systems:

$$\dot{\hat{\psi}}(\kappa,t) \;=\; \begin{bmatrix} 0 & 1 \\ \hat{a}_{21}(\kappa) & 0 \end{bmatrix} \hat{\psi}(\kappa,t) \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(\kappa,t), \quad \kappa \in \mathbb{Z}_N$$

$$\hat{a}_{21}(\kappa) \;=\; -\,2\left(1 \,-\, \cos\tfrac{2\,\pi\,\kappa}{N}\right)$$

- State and control weights

$$\left\{ Q \;=\; \begin{bmatrix} Q_p & 0 \\ 0 & Q_v \end{bmatrix};\; R \right\} \;\Rightarrow\; \left\{ \hat{Q}(\kappa) \;=\; \begin{bmatrix} \hat{q}_p(\kappa) & 0 \\ 0 & \hat{q}_v(\kappa) \end{bmatrix};\; \hat{R}(\kappa) \;=\; \hat{r}(\kappa) \right\}$$

- Solution to ARE

$$\hat{P}(\kappa) \;=\; \begin{bmatrix} \hat{p}_1(\kappa) & \hat{p}_0^*(\kappa) \\ \hat{p}_0(\kappa) & \hat{p}_2(\kappa) \end{bmatrix} \;\Rightarrow\; \left\{ \begin{array}{rcl} \hat{a}_{21}\,(\hat{p}_0 \,+\, \hat{p}_0^*) \,+\, \hat{q}_p \,-\, \dfrac{\hat{p}_0\,\hat{p}_0^*}{\hat{r}} & = & 0 \\ \hat{p}_0 \,+\, \hat{p}_0^* \,+\, \hat{q}_v \,-\, \dfrac{\hat{p}_2^2}{\hat{r}} & = & 0 \\ \hat{a}_{21}\,\hat{p}_2 \,+\, \hat{p}_1 \,-\, \dfrac{\hat{p}_2\,\hat{p}_0^*}{\hat{r}} & = & 0 \\ \hat{a}_{21}\,\hat{p}_2 \,+\, \hat{p}_1 \,-\, \dfrac{\hat{p}_2\,\hat{p}_0}{\hat{r}} & = & 0 \end{array} \right.$$

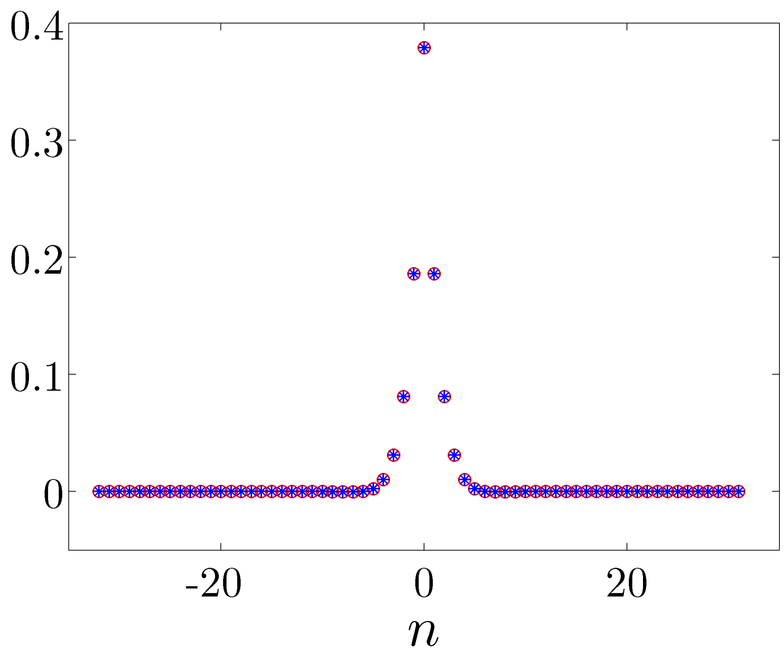
A bit of algebra yields

$$\begin{aligned} \hat{p}_0(\kappa) &= \hat{r}(\kappa)\left(\hat{a}_{21}(\kappa) \,+\, \sqrt{\hat{a}_{21}^2(\kappa) \,+\, \hat{q}_p(\kappa)/\hat{r}(\kappa)}\right) \\ \hat{p}_2(\kappa) &= \sqrt{\hat{r}(\kappa)\,(\hat{q}_v(\kappa) \,+\, 2\,\hat{p}_0(\kappa))} \\ \hat{p}_1(\kappa) &= \hat{p}_2(\kappa)\,(\hat{p}_0(\kappa)/\hat{r}(\kappa) \,-\, \hat{a}_{21}(\kappa)) \end{aligned}$$

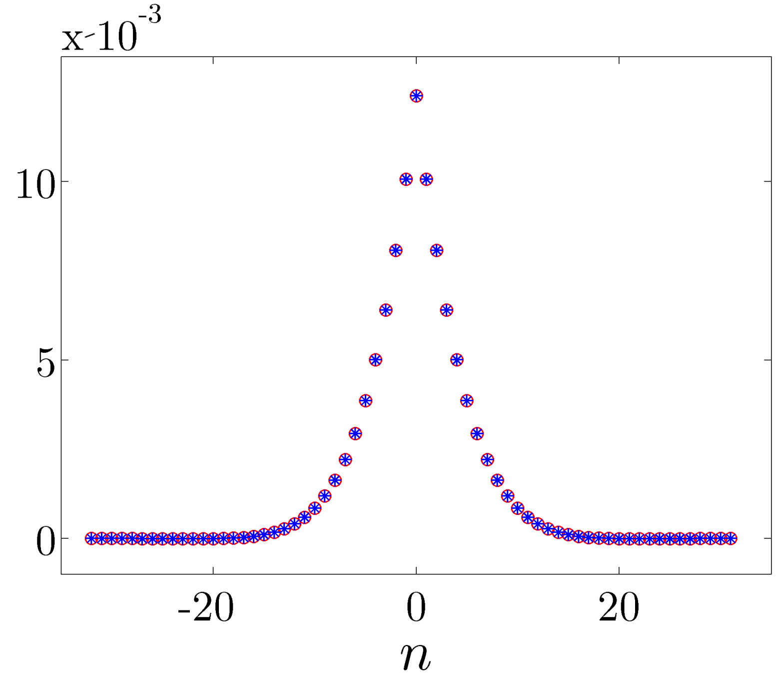
Structure of optimal solution

- Optimal position gain

$Q = I, R = I$:



$Q = I, R = 100\, I$:



- General trends
  - ⋆ High actuation authority $\xrightarrow{\text{cheap control}}$ Less communication
  - ⋆ Low actuation authority $\xrightarrow{\text{expensive control}}$ More communication

Bamieh, Paganini, Dahleh, IEEE TAC '02

LQR for systems with standard L_2 (or l_2) inner product

- Optimal controller determined by

$$
\begin{aligned}
u(x,t) &= -\left[\, \mathcal{K}\, \psi(\,\cdot\,,t) \,\right](x), \quad x \in \mathbb{G} \\
\mathcal{K} &= \mathcal{R}^{-1} \mathcal{B}^{\dagger}\, \mathcal{P}
\end{aligned}
$$

  $\star$ $\mathcal{P} = \mathcal{P}^{\dagger}$ – bounded non-negative operator that solves ARE

$$
\langle \mathcal{A}\,\psi_1, \mathcal{P}\,\psi_2 \rangle + \langle \mathcal{P}\,\psi_1, \mathcal{A}\,\psi_2 \rangle + \left\langle \mathcal{Q}^{\frac{1}{2}}\,\psi_1, \mathcal{Q}^{\frac{1}{2}}\,\psi_2 \right\rangle - \langle \mathcal{B}^{\dagger}\,\mathcal{P}\,\psi_1, \mathcal{R}^{-1}\mathcal{B}^{\dagger}\,\mathcal{P}\,\psi_2 \rangle = 0
$$

$$
\psi_1, \psi_2 \in \mathcal{D}(\mathcal{A})
$$

- For standard L_2 (or l_2) inner product $\langle\,\cdot\,,\,\cdot\,\rangle$

$$
\begin{aligned}
\hat{u}(\kappa,t) &= -\hat{K}(\kappa)\,\hat{\psi}(\kappa,t), \quad \kappa \in \hat{\mathbb{G}} \\
\hat{K}(\kappa) &= \hat{R}^{-1}(\kappa)\,\hat{B}^{*}(\kappa)\,\hat{P}(\kappa) \\
0 &= \hat{A}^{*}(\kappa)\,\hat{P}(\kappa) + \hat{P}(\kappa)\,\hat{A}(\kappa) + \hat{Q}(\kappa) - \hat{P}(\kappa)\,\hat{B}(\kappa)\,\hat{R}^{-1}(\kappa)\,\hat{B}^{*}(\kappa)\,\hat{P}(\kappa)
\end{aligned}
$$

In class: diffusion equation on $L_2(-\infty,\infty)$

Lectures 25 & 26: Consensus and vehicular formation problems

- Consensus
  - ⋆ Make subsystems (agents, nodes) reach agreement
  - ⋆ Distributed decision making

- Vehicular formations
  - ⋆ How does performance scale with size?
  - ⋆ Are there any fundamental limitations?
  - ⋆ Is it enough to only look at neighbors?
  - ⋆ Should information be broadcast to all?

Collective behavior in nature

SNOW GEESE STRING FORMATION

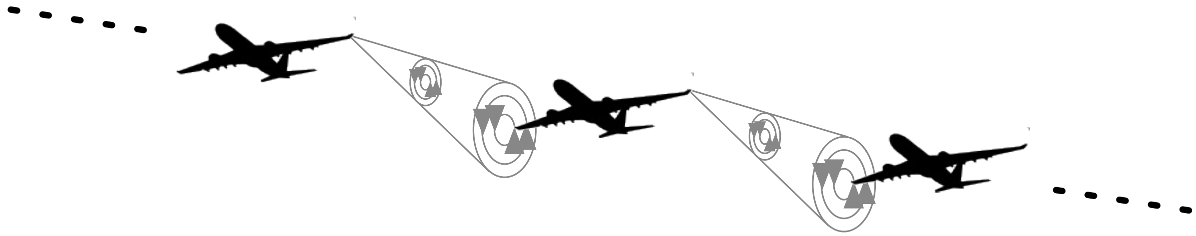
WILDEBEEST HERD MIGRATION

COLLECTIVE MOTION IN 3D

Coordinated control of formations

FORMATION FLIGHT FOR AERODYNAMIC ADVANTAGE

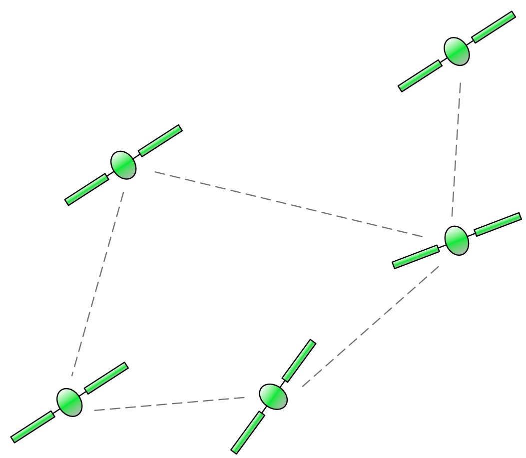
e.g. additional lift in V-formations



precise control needed

MICRO-SATELLITE FORMATIONS

e.g. for synthetic aperture



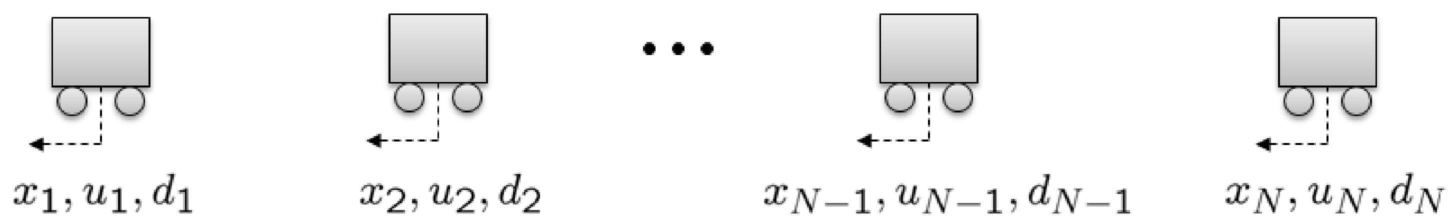
MAKE VEHICLES SMALLER AND CHEAPER $\Rightarrow$ USE MANY

cooperative control becomes a major issue

Vehicular strings

AUTOMATED CONTROL OF EACH VEHICLE

tight spacing at highway speeds



KEY ISSUES (**also in: control of swarms, flocks, formation flight**)

⋆ Is it enough to only look at neighbors?

⋆ How does performance scale with size?

⋆ Are there any fundamental limitations?

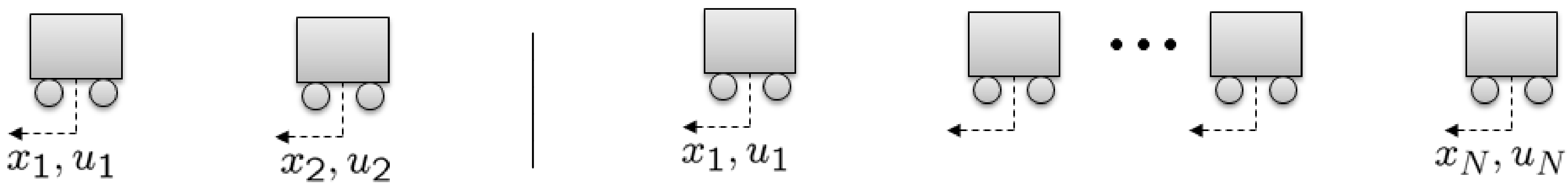
FUNDAMENTALLY DIFFICULT PROBLEM (**scales poorly**)

⋆ *Jovanović & Bamieh, IEEE TAC '05*

⋆ *Bamieh, Jovanović, Mitra, Patterson, IEEE TAC '11 (to appear)*

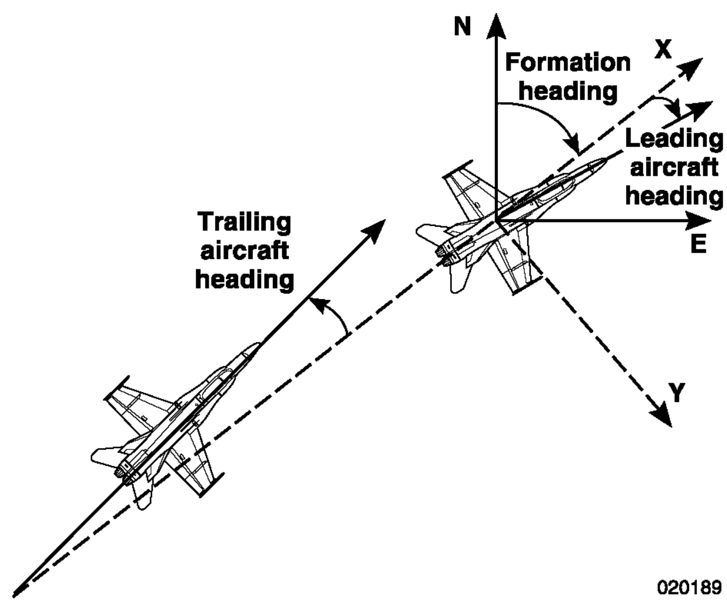
String instability

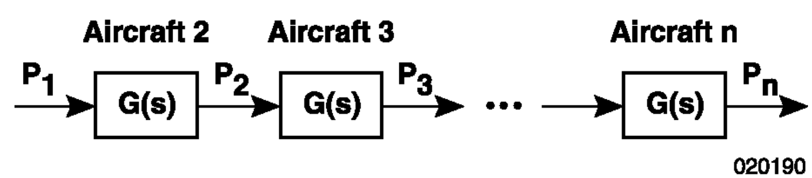
ONE APPROACH: design a follower cruise control $\Rightarrow$ chain into a formation



PROBLEM: STRING INSTABILITY

FLIGHT FORMATION EXAMPLE (Allen et al., 2002)

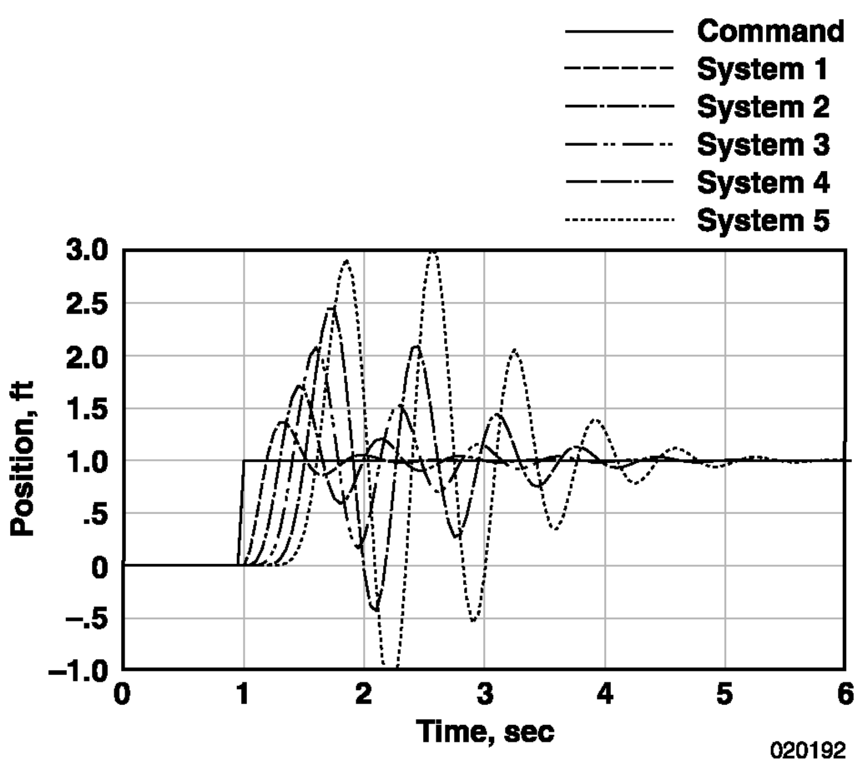




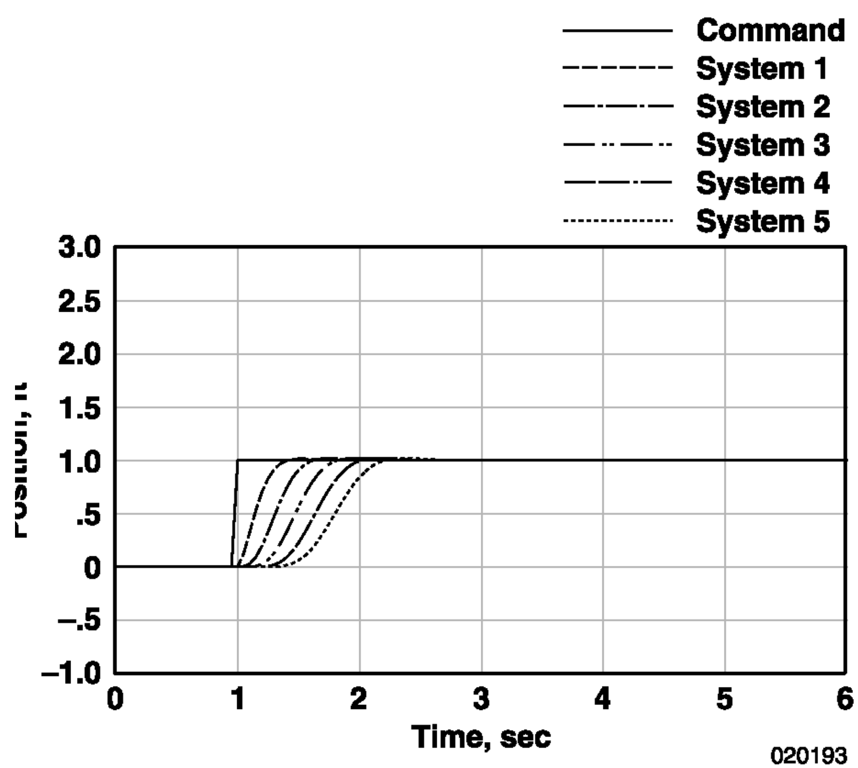
CHAINING OF A FOLLOWER CONTROLLER $\Rightarrow$ STRING INSTABILITY

Allen et al., 2002

STRING INSTABILITY:



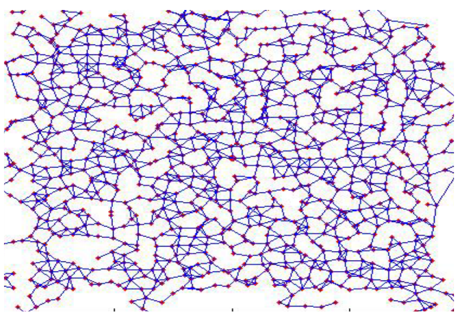
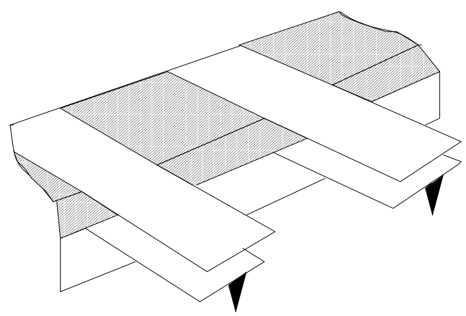
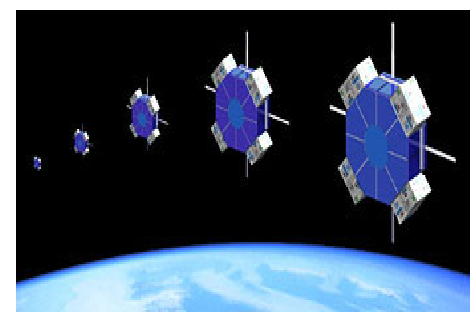
BETTER DESIGN:



Control of vehicular platoons

- ACTIVE RESEARCH AREA FOR $\approx$ 40 YEARS
  (Levine & Athans, Melzer & Kuo, Chu, Ioannou, Varaiya, Hedrick, Swaroop, ...)

- SPATIO-TEMPORAL SYSTEMS
  signals depend on time & discrete spatial variable n

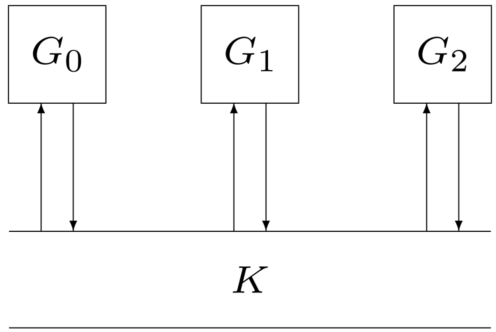
sensor networks	arrays of micro-cantilevers arrays of micro-mirrors	UAV formations satellite constellations
		

- INTERACTIONS CAUSE COMPLEX BEHAVIOR
  'string instability' in vehicular platoons

- SPECIAL STRUCTURE
  every unit has sensors and actuators

Controller architectures: platoons

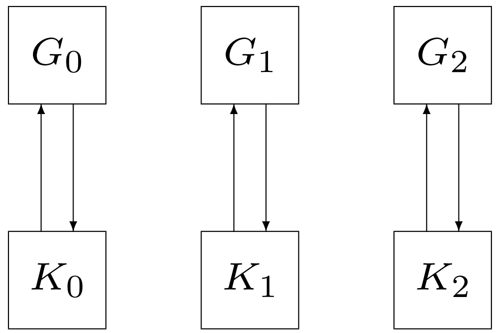
CENTRALIZED:



best performance

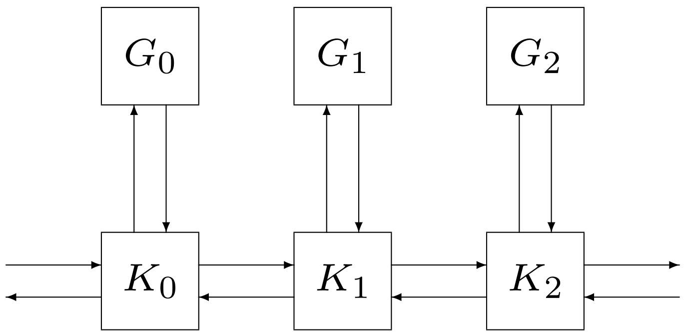
excessive communication

FULLY DECENTRALIZED:



not safe!

LOCALIZED:

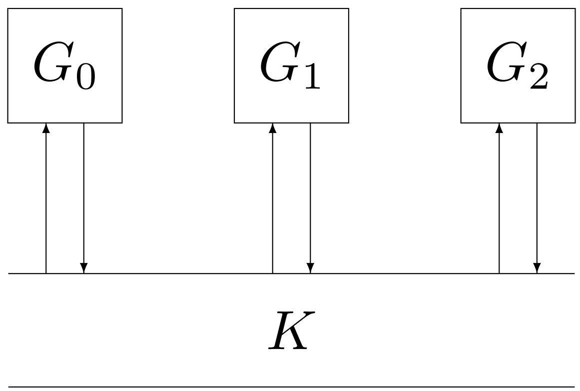


many possible architectures

• FUNDAMENTAL LIMITATIONS

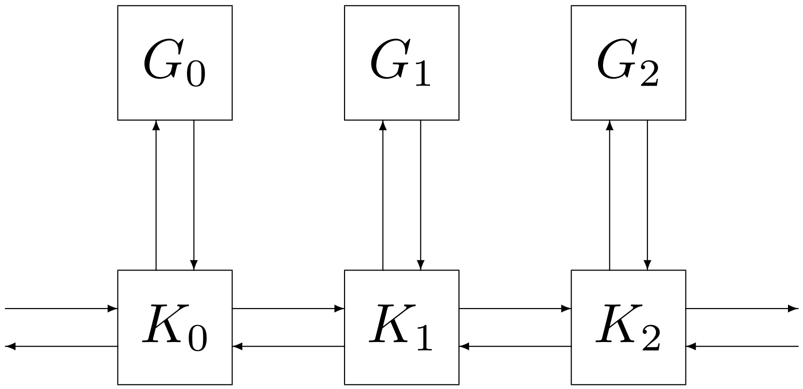
⋆ spatially invariant theory

CENTRALIZED:



performance vs. size

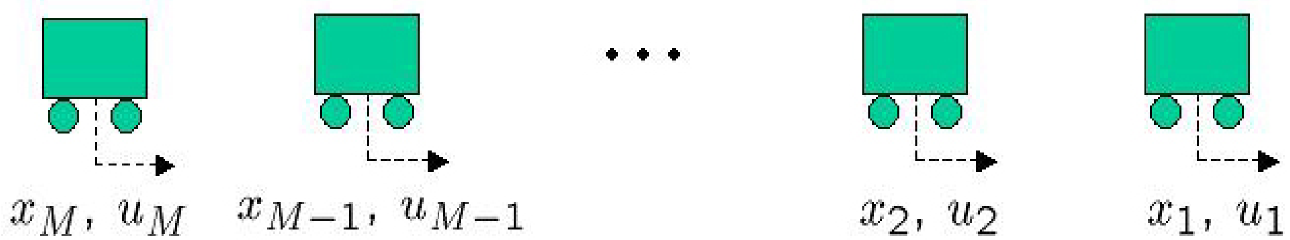
LOCALIZED:



is it enough to look only at nearest neighbors?

Optimal control of vehicular platoons

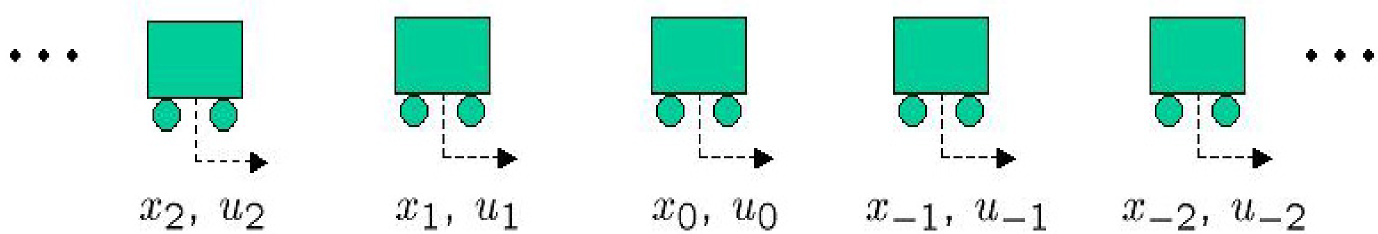
- FINITE PLATOONS



Levine & Athans, IEEE TAC '66

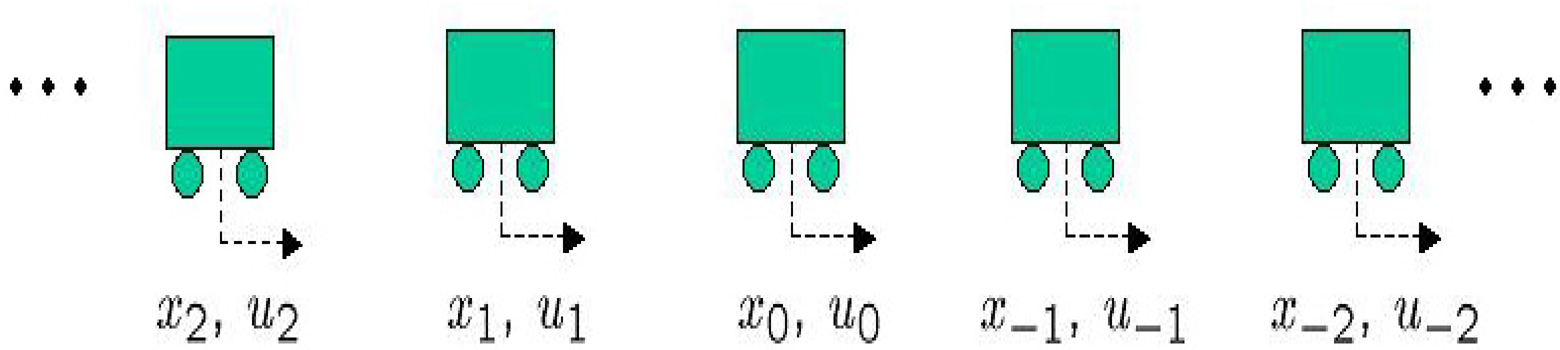
Melzer & Kuo, IEEE TAC '71

- INFINITE PLATOONS



Melzer & Kuo, Automatica '71

Control objective



DYNAMICS OF n-TH VEHICLE: $\ddot{x}_n \; = \; u_n$

CONTROL OBJECTIVE:

desired cruising velocity	v_d	$:=$	const.
inter-vehicular distance	L	$:=$	const.

☞ COUPLING ONLY THROUGH FEEDBACK CONTROLS

ABSOLUTE DESIRED TRAJECTORY

$$x_{nd}(t) \; := \; v_d t \; - \; nL$$

Optimal control of finite platoons

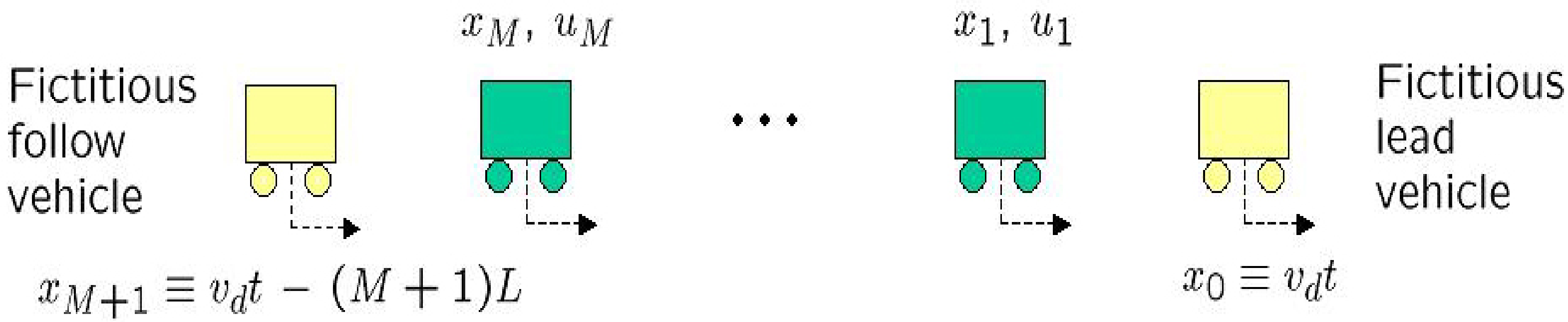
absolute position error: $p_n(t) \;:=\; x_n(t) \;-\; v_d t \;+\; nL$

absolute velocity error: $v_n(t) \;:=\; \dot{x}_n(t) \;-\; v_d$

$$\Downarrow$$

$$\begin{bmatrix} \dot{p}_n \\ \dot{v}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n, \qquad n \in \{1, \ldots, M\}$$

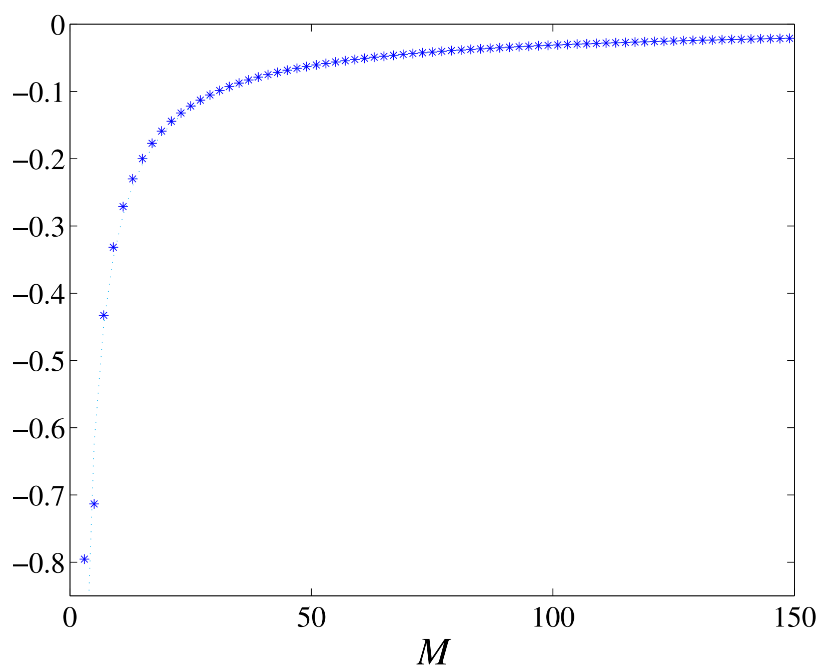
$$J \;:=\; \int_0^\infty \left(\sum_{n=1}^{M+1} (p_n(t) \;-\; p_{n-1}(t))^2 \;+\; \sum_{n=1}^{M} (v_n^2(t) \;+\; u_n^2(t)) \right) \mathrm{d}t$$



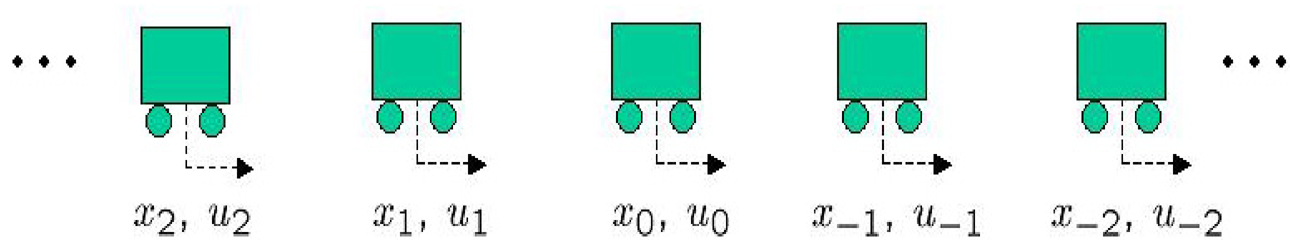
$$\begin{bmatrix} \dot{p}_n \\ \dot{v}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n, \quad n \in \{1, \ldots, M\}$$

$$J := \int_0^\infty \left(\sum_{n=1}^{M+1} (p_n(t) - p_{n-1}(t))^2 + \sum_{n=1}^{M} (v_n^2(t) + u_n^2(t)) \right) \mathrm{d}t$$

- $\max Re\,(\lambda\{A_{cl}\})$:



Optimal control of infinite platoons



MAIN IDEA: EXPLOIT SPATIAL INVARIANCE

$$\begin{bmatrix} \dot{p}_n \\ \dot{v}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n, \quad n \in \mathbb{Z}$$

$$J := \int_0^\infty \sum_{n \in \mathbb{Z}} \left((p_n(t) - p_{n-1}(t))^2 + v_n^2(t) + u_n^2(t) \right) \mathrm{d}t$$

$\downarrow$ SPATIAL $\mathcal{Z}_\theta$-TRANSFORM

$$A_\theta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q_\theta = \begin{bmatrix} 2(1 - \cos\theta) & 0 \\ 0 & 1 \end{bmatrix}, \quad 0 \le \theta < 2\pi$$

⋆ **pair (Q_θ, A_θ) not detectable at $\theta = 0$**

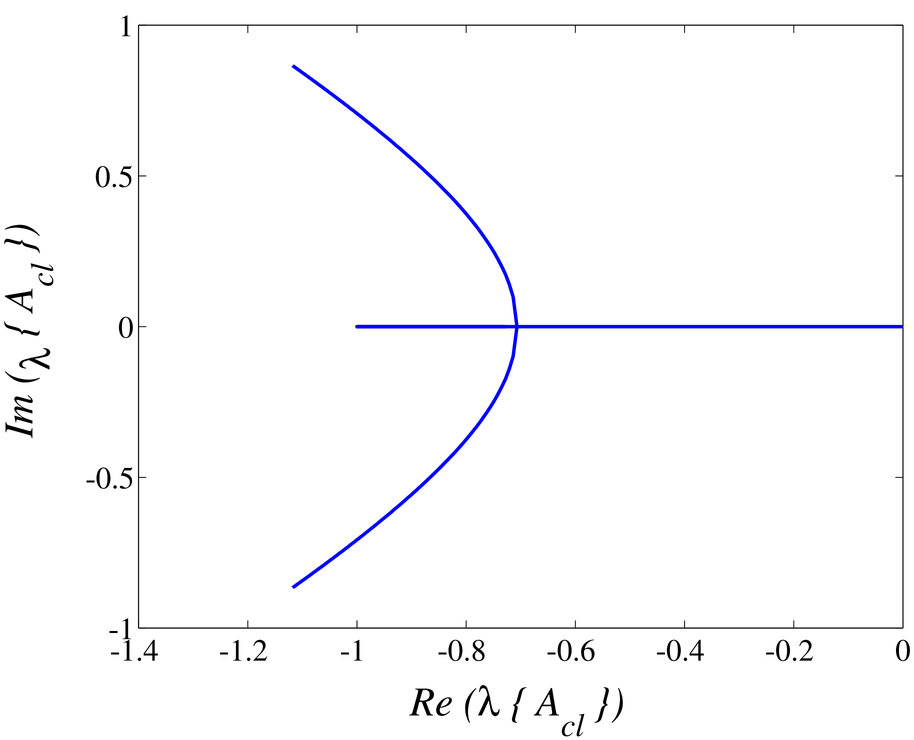
POSSIBLE FIX: PENALIZE ABSOLUTE POSITION ERRORS IN J

$$\begin{bmatrix} \dot{p}_n \\ \dot{v}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n, \quad n \in \mathbb{Z}$$

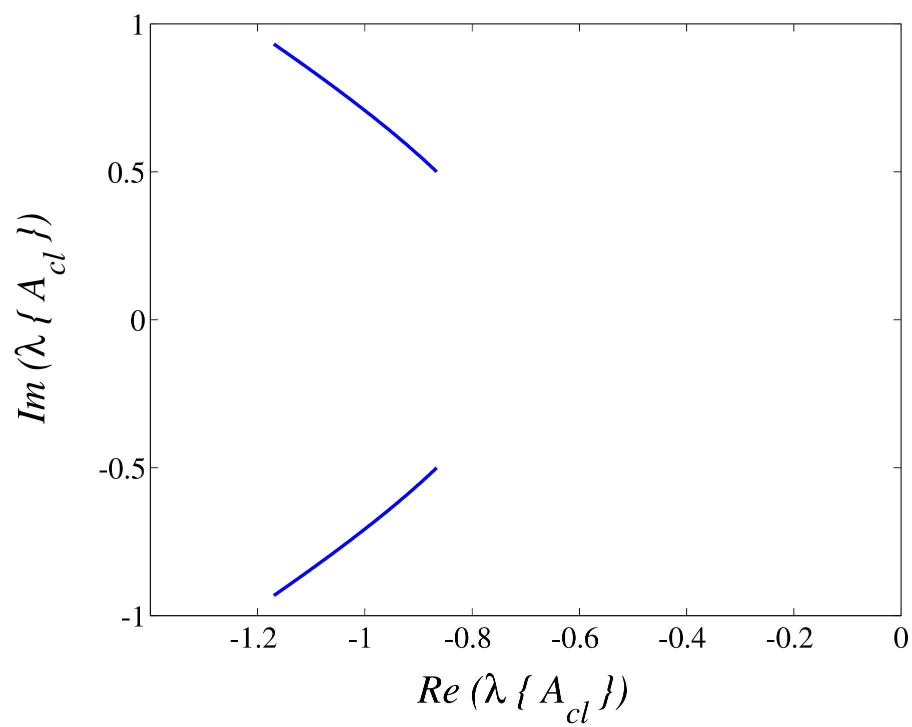
$$J := \int_0^\infty \sum_{n \in \mathbb{Z}} \left(q\, p_n^2(t) + (p_n(t) - p_{n-1}(t))^2 + v_n^2(t) + u_n^2(t) \right) \mathrm{d}t$$

CLOSED-LOOP SPECTRUM:

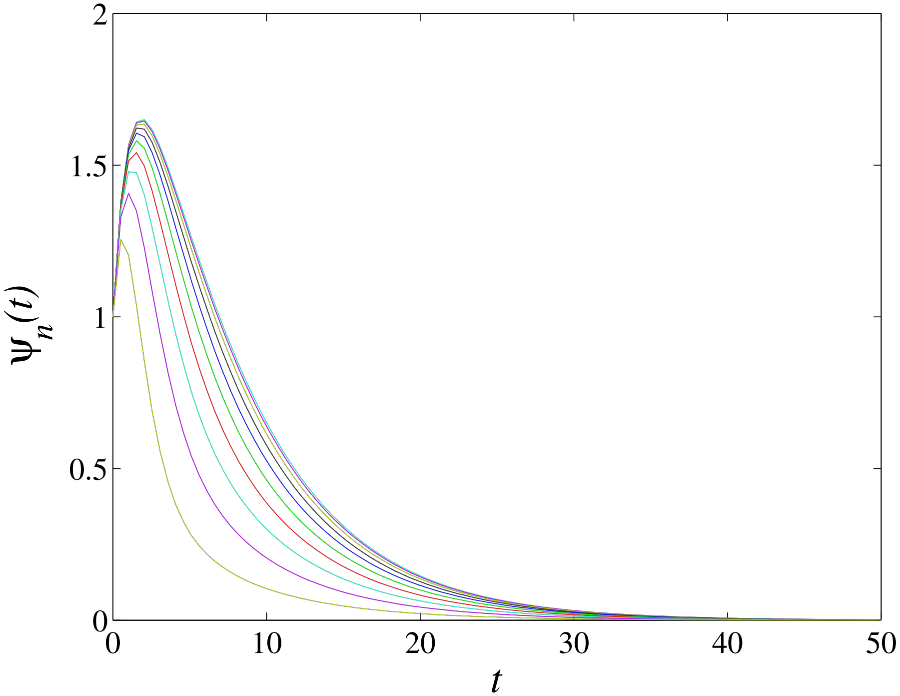
$q = 0$:



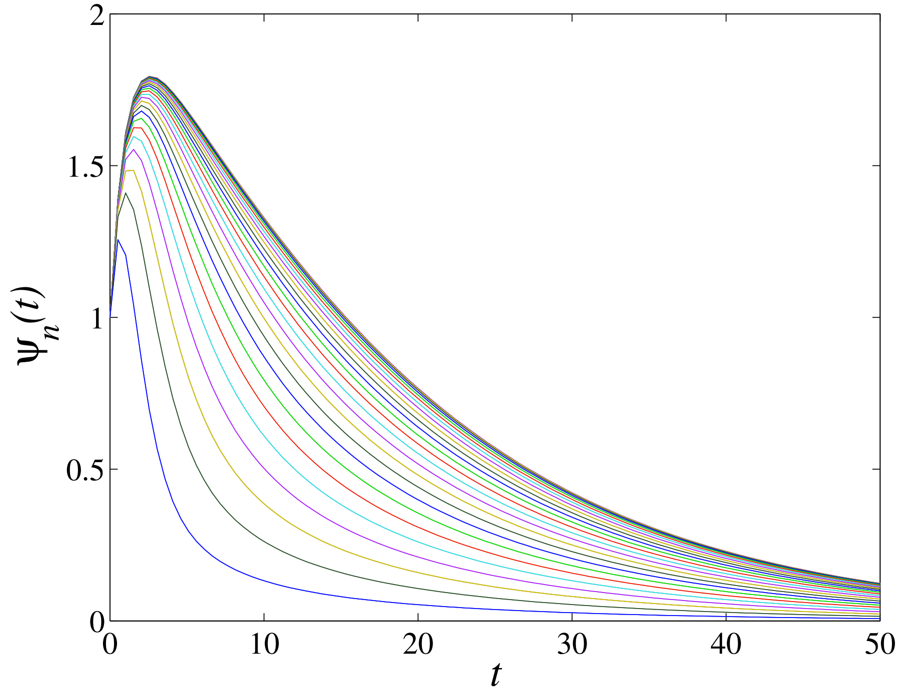
$q = 1$:



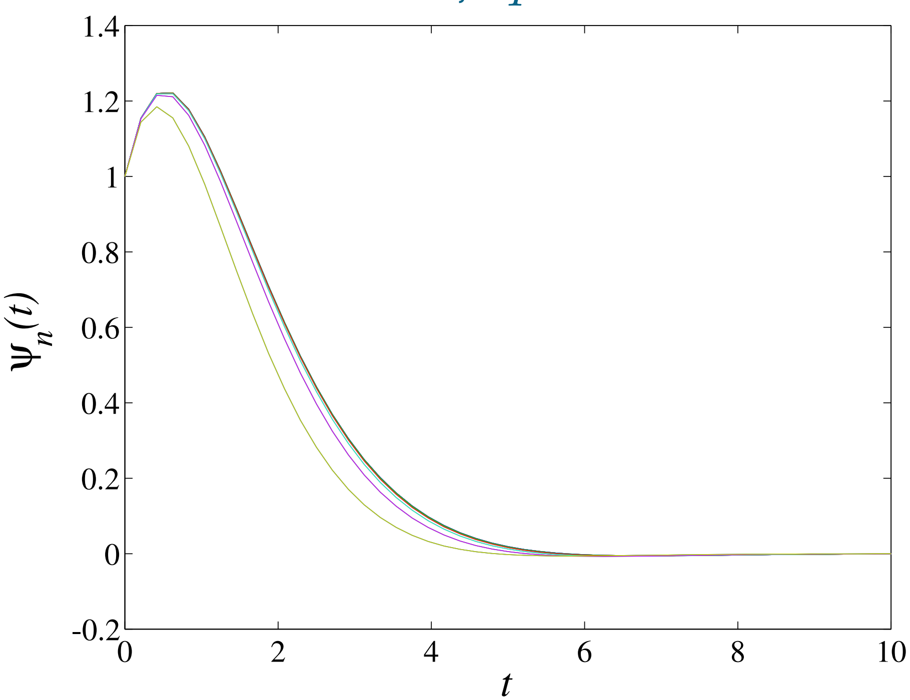
$M = 20, \; q = 0$:



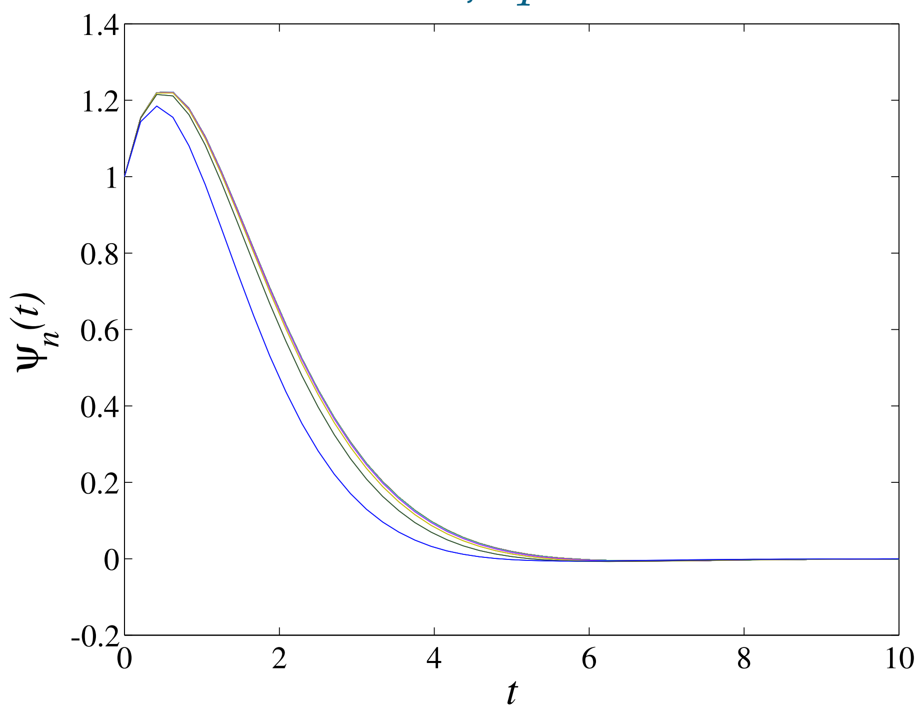
$M = 50, \; q = 0$:



$M = 20, \; q = 1$:



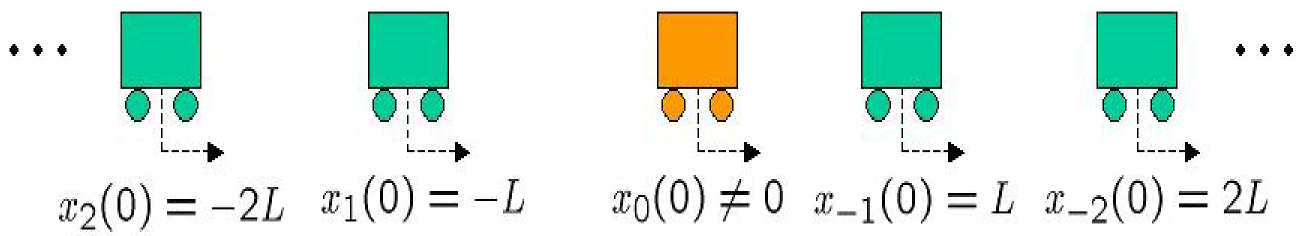
$M = 50, \; q = 1$:



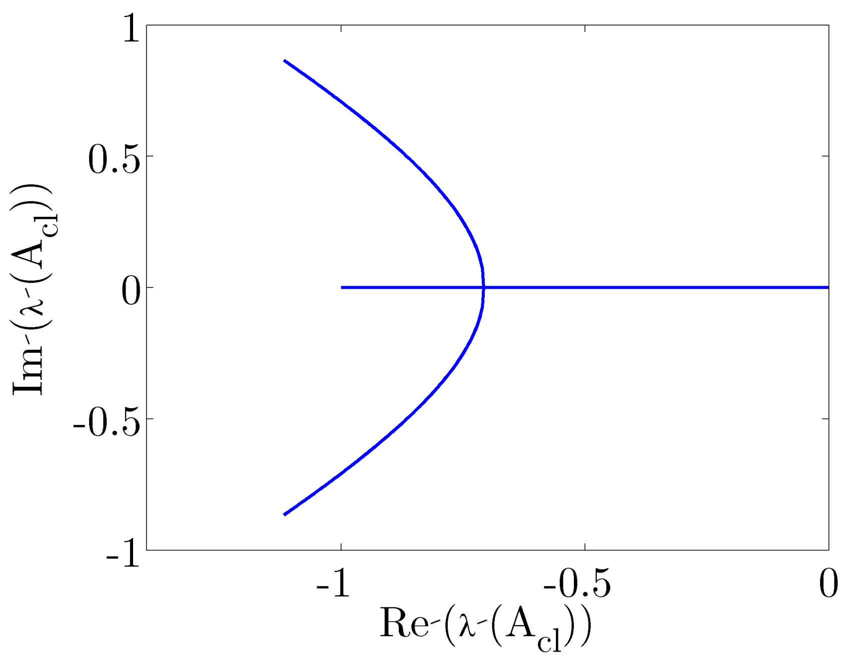
‘Problematic’ initial conditions

- INFINITE PLATOONS:
  non-zero mean initial conditions cannot be driven to zero

$$\sum_{n \in \mathbb{Z}} p_n(0) \;\neq\; 0 \quad \Rightarrow \quad \lim_{t \to \infty} \sum_{n \in \mathbb{Z}} p_n(t) \neq 0$$



☞ **many modes have very slow rates of convergence**

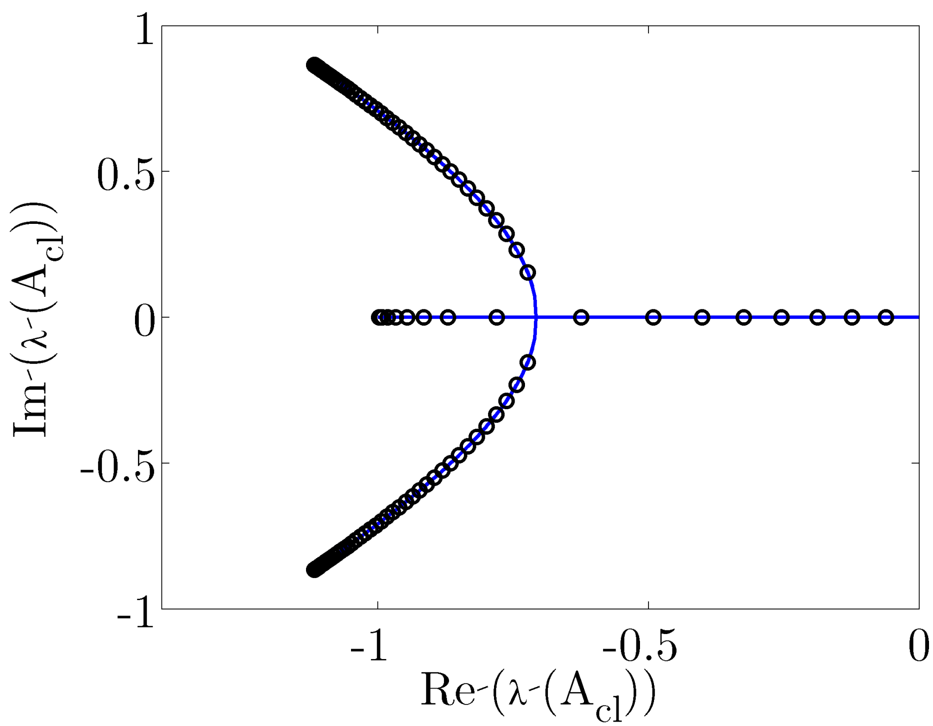


$$\begin{bmatrix} \dot{p} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

$$J = \int_0^\infty \left(p^T(t)\, Q_p\, p(t) + q_v\, v^T(t)\, v(t) + r\, u^T(t)\, u(t) \right) \mathrm{d}t$$

$$Q_p = Q_p^T = V \Lambda V^* > 0, \;\; q_v \geq 0, \;\; r > 0$$

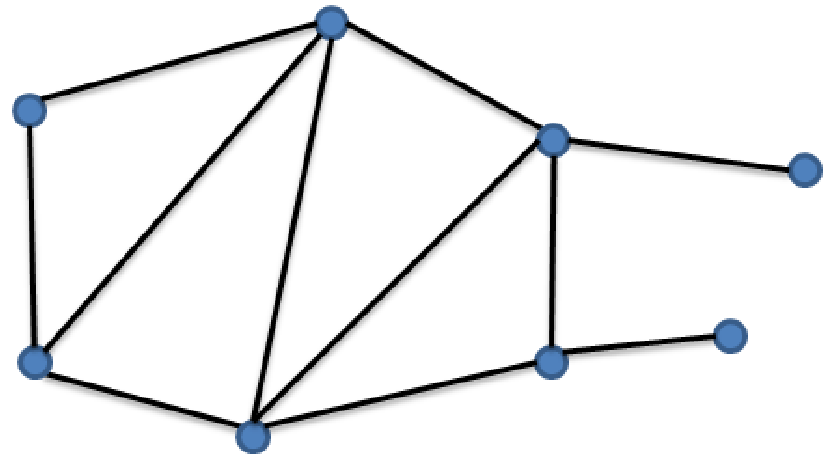
- spectrum of large-but-finite platoon dense in the spectrum of infinite platoon



- Key: entries into ARE jointly unitarily diagonalizable by V

Jovanović & Bamieh, IEEE TAC '05

Consensus by distributed computation



- Relative information exchange with neighbors
  - ⋆ Simple distributed averaging algorithm

$$\dot{x}_i(t) \;=\; -\sum_{j\,\in\,\mathcal{N}_i}\Big(x_i(t) \;-\; x_j(t)\Big)$$

- Questions
  - ⋆ Will the network asymptotically equilibrate?

$$\lim_{t\,\to\,\infty} x_n(t) \;\stackrel{?}{=}\; \bar{x}(t) \;:=\; \frac{1}{N}\sum_{n\,=\,1}^{N} x_n(t)$$

  - ⋆ Quantify performance (e.g., rate of convergence, response to disturbances)

Convergence to deviation from average

- Write dynamics as

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_N(t) \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix} + \begin{bmatrix} d_1(t) \\ \vdots \\ d_N(t) \end{bmatrix}$$

$$\dot{x}(t) = A\,x(t) + d(t)$$

- Let A be such that
  - ⋆ All rows and columns sum to zero

$$\begin{aligned} A\,\mathbb{1} &= 0 \cdot \mathbb{1} \\ \mathbb{1}^T A &= 0 \cdot \mathbb{1}^T \end{aligned}$$

  - ⋆ $\mathbb{1} := \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$ is an equilibrium point, $A\,\mathbb{1} = 0$
  - ⋆ All other eigenvalues of A have negative real parts

$$\bar{x}(t) \; := \; \frac{1}{N}\left(x_1(t) \, + \, \cdots \, + \, x_N(t)\right) \; = \; \frac{1}{N} \mathbb{1}^T x(t)$$

- Deviation from average

scalar form: $$\tilde{x}_n(t) \; = \; x_n(t) \, - \, \bar{x}(t)$$

vector form: $$\begin{bmatrix} \tilde{x}_1(t) \\ \vdots \\ \tilde{x}_N(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix} - \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \underbrace{\frac{1}{N} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix}}_{\bar{x}(t)}$$

$$\tilde{x}(t) \; = \; \left(I \, - \, \frac{1}{N} \mathbb{1}\mathbb{1}^T \right) x(t)$$

$$\downarrow$$

$$x(t) \; = \; \underbrace{\tilde{x}(t)}_{\in \, \mathbb{1}^{\perp}} \, + \, \mathbb{1} \, \bar{x}(t)$$

$$\{u_1, \ldots, u_{N-1}\} \; - \; \text{orthonormal basis of } \mathbb{1}^{\perp}$$

- Write $\tilde{x}(t)$ as

$$\tilde{x}(t) \; = \; \psi_1(t)\, u_1 \; + \; \cdots \; + \; \psi_{N-1}(t)\, u_{N-1} \; = \; \underbrace{\left[\begin{array}{ccc} u_1 & \cdots & u_{N-1} \end{array}\right]}_{U} \underbrace{\left[\begin{array}{c} \psi_1(t) \\ \vdots \\ \psi_{N-1}(t) \end{array}\right]}_{\psi(t)}$$

- Coordinate transformation

$$x(t) \; = \; \tilde{x}(t) \; + \; \mathbb{1}\, \bar{x}(t) \; = \; \left[\begin{array}{cc} U & \mathbb{1} \end{array}\right] \left[\begin{array}{c} \psi(t) \\ \bar{x}(t) \end{array}\right]$$

$$\Updownarrow$$

$$\left[\begin{array}{c} \psi(t) \\ \bar{x}(t) \end{array}\right] \; = \; \left[\begin{array}{c} U^* \\ \frac{1}{N}\, \mathbb{1}^T \end{array}\right] x(t)$$

$$\dot{x}(t) \; = \; A\, x(t) \; + \; d(t)$$

- In new coordinates

$$\begin{bmatrix} U & \mathbb{1} \end{bmatrix} \begin{bmatrix} \dot{\psi}(t) \\ \dot{\bar{x}}(t) \end{bmatrix} \; = \; A \begin{bmatrix} U & \mathbb{1} \end{bmatrix} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix} \; + \; d(t)$$

$$\begin{aligned} \begin{bmatrix} \dot{\psi}(t) \\ \dot{\bar{x}}(t) \end{bmatrix} \; &= \; \begin{bmatrix} U^* \\ \frac{1}{N}\mathbb{1}^T \end{bmatrix} A \begin{bmatrix} U & \mathbb{1} \end{bmatrix} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix} \; + \; \begin{bmatrix} U^* \\ \frac{1}{N}\mathbb{1}^T \end{bmatrix} d(t) \\ &= \; \begin{bmatrix} U^* A\, U & U^* A\, \mathbb{1} \\ \frac{1}{N}\mathbb{1}^T A\, U & \frac{1}{N}\mathbb{1}^T A\, \mathbb{1} \end{bmatrix} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix} \; + \; \begin{bmatrix} U^* \\ \frac{1}{N}\mathbb{1}^T \end{bmatrix} d(t) \end{aligned}$$

- Use structure of A to obtain

$$\begin{aligned} \dot{\psi}(t) \; &= \; U^* A\, U\, \psi(t) \; + \; U^* d(t) \\ \dot{\bar{x}}(t) \; &= \; 0 \cdot \bar{x}(t) \; + \; \frac{1}{N}\mathbb{1}^T d(t) \end{aligned}$$

Spatially invariant systems over circle

- Circulant A-matrix

$$\begin{aligned} \dot{x}(t) &= A\,x(t) \;+\; d(t) \\ z(t) &= \left(I \,-\, \frac{1}{N}\,\mathbb{1}\mathbb{1}^T\right) x(t) \end{aligned}$$

- Use DFT to obtain

$$\begin{aligned} \dot{\hat{x}}_k(t) &= \hat{a}_k\,\hat{x}_k(t) \;+\; \hat{d}_k(t) \\ \hat{z}_k(t) &= (1 \,-\, \delta_k)\,\hat{x}_k(t) \end{aligned}$$

- Variance of the network (i.e., the H_2 norm from d to z)
  - ⋆ solve Lyapunov equation and sum over spatial frequencies

$$\|H\|_2^2 \;=\; -\sum_{k\,=\,1}^{N-1} \frac{1}{(\hat{a}_k \,+\, \hat{a}_k^*)}$$

An example

- Nearest neighbor information exchange

$$\dot{x}_n(t) \;=\; -\,(x_n(t) \,-\, x_{n-1}(t)) \;-\; (x_n(t) \,-\, x_{n+1}(t)) \,+\, d_n(t), \quad n \in \mathbb{Z}_N$$

- Use DFT to obtain

$$\begin{aligned} \dot{\hat{x}}_k(t) &= -2\left(1 - \cos\left(\tfrac{2\pi k}{N}\right)\right)\hat{x}_k(t) \,+\, \hat{d}_k(t) \\ \hat{z}_k(t) &= (1 - \delta_k)\,\hat{x}_k(t) \end{aligned}$$

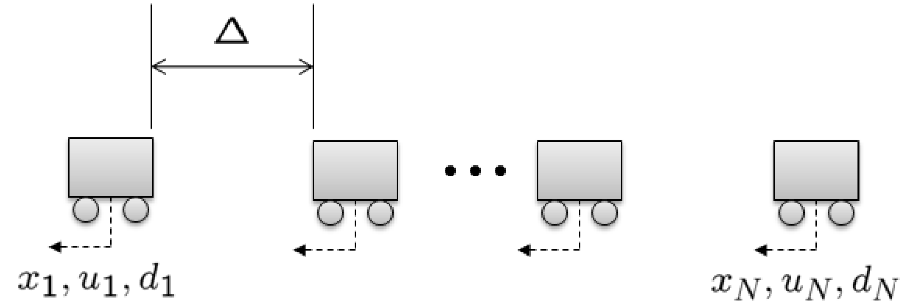
Variance per node

$$\frac{1}{N}\,\|H\|_2^2 \;=\; \frac{1}{N}\sum_{k\,=\,1}^{N-1} \frac{1}{4\left(1 \,-\, \cos\left(\frac{2\pi k}{N}\right)\right)} \;=\; \frac{1}{N}\sum_{k\,=\,1}^{N-1} \frac{1}{8\,\sin^2\left(\frac{\pi k}{N}\right)} \;=\; \frac{N^2 - 1}{24\,N}$$

- Will the scaling trends change if we $\left\{ \begin{array}{l} \text{use information from more neighbors?} \\ \text{work in 2D or 3D?} \end{array} \right.$

Problem setup: double-integrator vehicles

$$\ddot{x}_n = \underset{\text{control}}{\underset{\uparrow}{u_k}} + \underset{\text{disturbance}}{\underset{\uparrow}{d_n}}$$



- Desired trajectory: $\begin{cases} \bar{x}_n := v_d\, t + n\,\Delta \\ \text{constant velocity} \end{cases}$

- Deviations:

$$p_n := x_n - \bar{x}_n, \qquad v_n := \dot{x}_n - v_d$$

- Controls:

$$u = -K_p\, p - K_v\, v$$

- Closed loop:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K_p & -K_v \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} d(t)$$

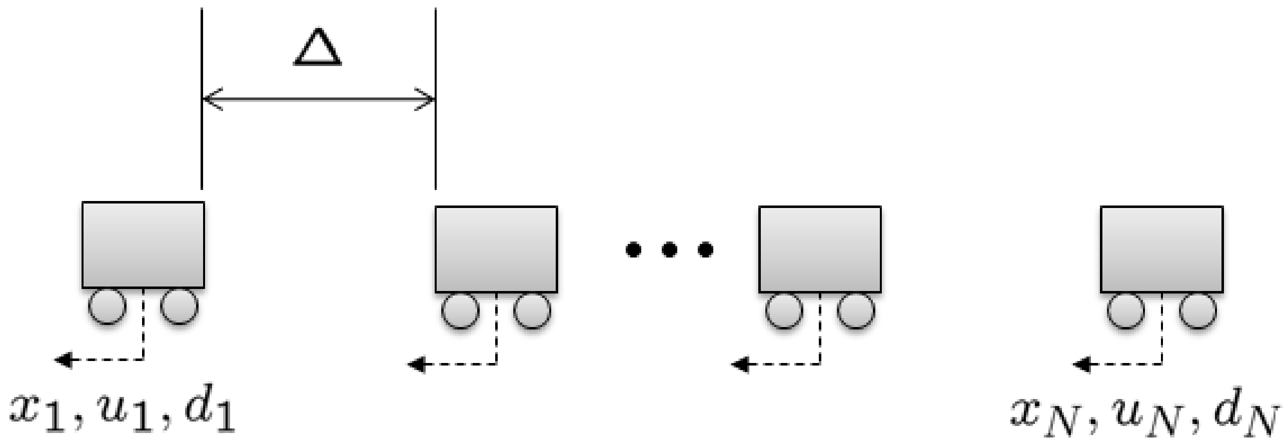
$K_p,\ K_v$: feedback gains

Structured feedback design

Example: design K_p and K_v to use nearest neighbor feedback

e.g. use a simple rule like:

$$\begin{aligned} u_n \;=\; & -K_p^+ \left(x_{n+1} - x_n - \Delta\right) \;-\; K_p^- \left(x_n - x_{n-1} - \Delta\right) \\ & -K_v^+ \left(v_{n+1} - v_n\right) \;-\; K_v^- \left(v_n - v_{n-1}\right) \end{aligned}$$

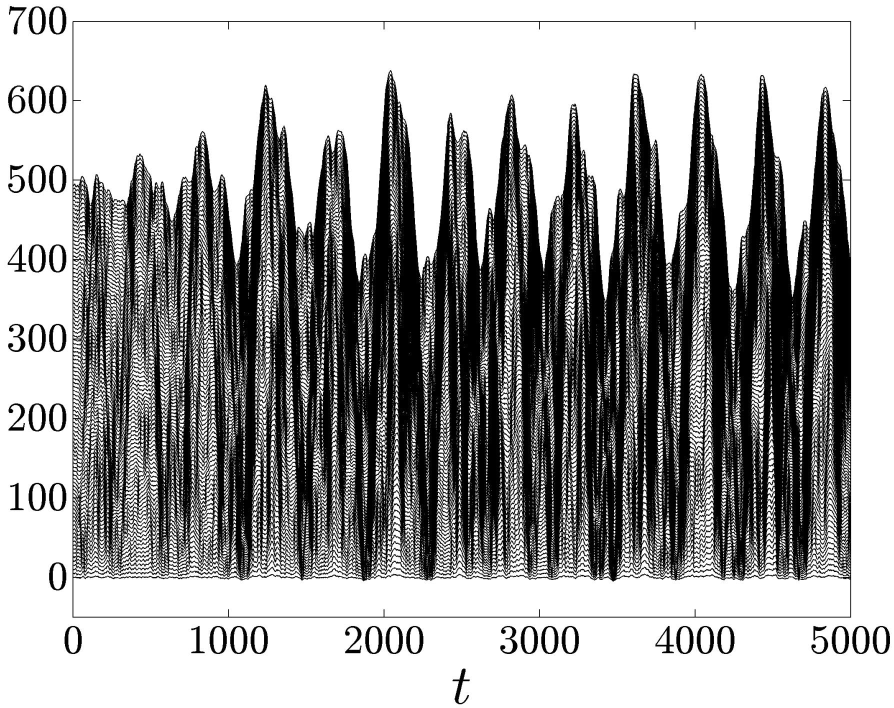


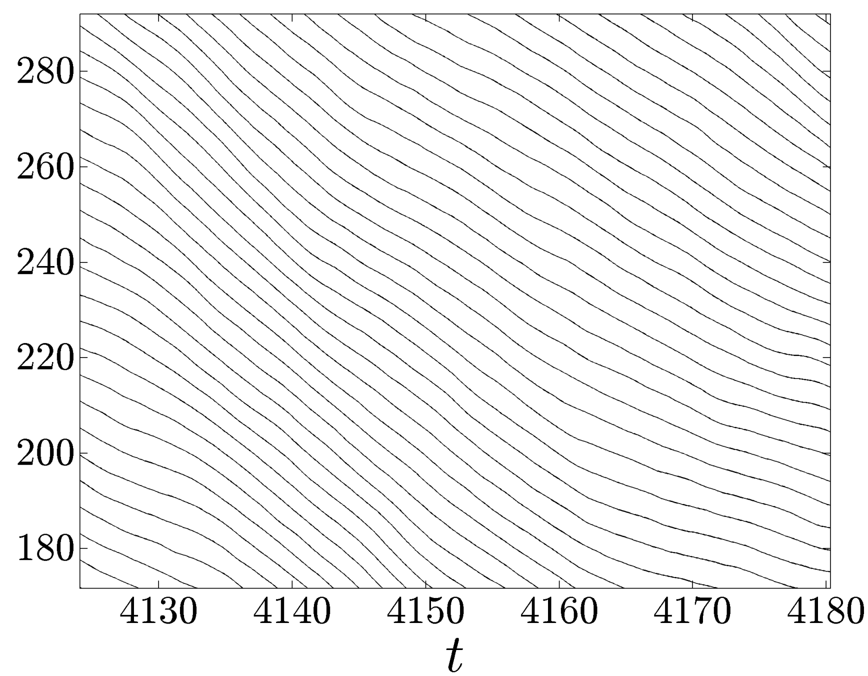
select K_p and K_v to guarantee global stability

Incoherence phenomenon

LOCAL FEEDBACK: GLOBAL STABILITY

$N = 100$ VEHICLES



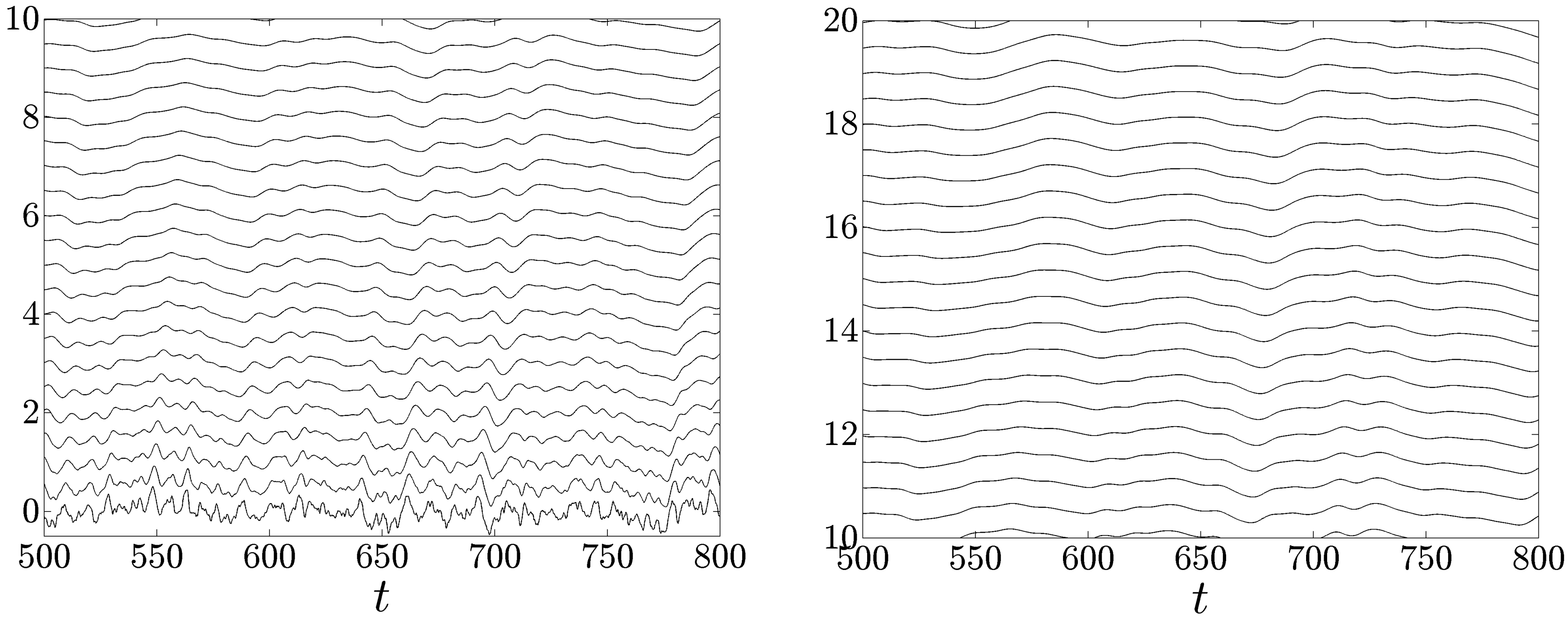


poor macroscopic performance: not string instability!

- ⋆ high frequency disturbance quickly regulated
- ⋆ low frequency disturbance penetrates further into formation

random disturbance acting on lead vehicle

$N = 100$ VEHICLES



Bamieh, Jovanović, Mitra, Patterson, IEEE TAC '11 (to appear)

Role of dimensionality

$M = N^d$ vehicles arranged in d-dimensional torus $\mathbb{Z}_N^d$

$$\ddot{x}_{(n_1,\ldots,n_d)} \;=\; u_{(n_1,\ldots,n_d)} \;+\; w_{(n_1,\ldots,n_d)}, \qquad n_i \in \mathbb{Z}_n$$

desired trajectory: $\bar{x}_k \;:=\; vt \;+\; k\Delta$

- STRUCTURAL FEATURES:
  - ⋆ spatial invariance
  - ⋆ locality
  - ⋆ mirror symmetry

- RELATIVE VS. ABSOLUTE MEASUREMENTS

$$\begin{aligned} u_n \;=\; & -K_p^+\,(x_{n+1} - x_n - \Delta) \;-\; K_p^-\,(x_n - x_{n-1} - \Delta) \;- \\ & K_v^+\,(v_{n+1} - v_n) \;-\; K_v^-\,(v_n - v_{n-1}) \;- \\ & K_p^0\,(x_n \;-\; (v_d t + n\Delta)) \;-\; K_v^0\,(v_n \;-\; v_d) \end{aligned}$$

Performance measures

- Microscopic: local position deviation $(x_{n+1} - x_n - \Delta)$
- Macroscopic: deviation from average or long range deviation

How does variance per vehicle scale with system size?

- relative position & absolute velocity feedback:

MICROSCOPIC ERROR:

bounded for any dimension d

ASYMPTOTIC SCALING OF MACROSCOPIC ERROR:

$d = 1$	M
$d = 2$	$\log M$
$d \geq 3$	bounded!

$\star$ Same scaling obtained in standard consensus problem

- relative position & relative velocity feedback:

ASYMPTOTIC SCALING OF MICROSCOPIC ERROR:

$d = 1$	M
$d = 2$	$\log M$
$d = 3$	bounded

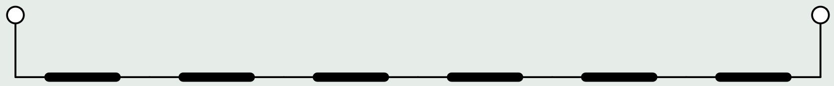
ASYMPTOTIC SCALING OF MACROSCOPIC ERROR:

$d = 1$	M^3
$d = 2$	M
$d = 3$	$M^{1/3}$

Only local feedback: **large 'tight formations' in 1D not possible!**

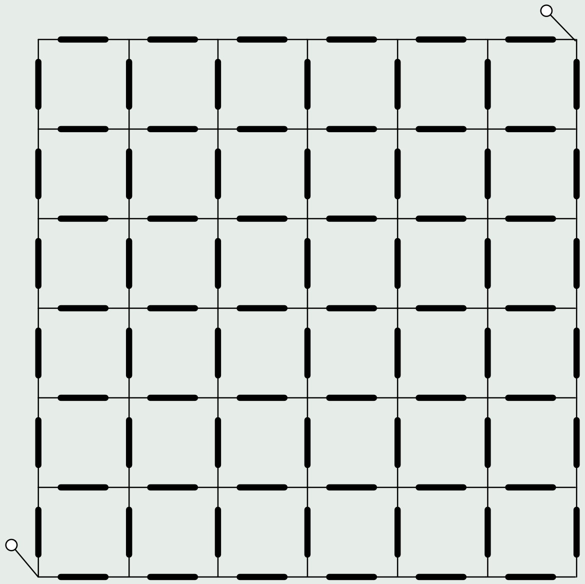
Resistive network analogy

1D :



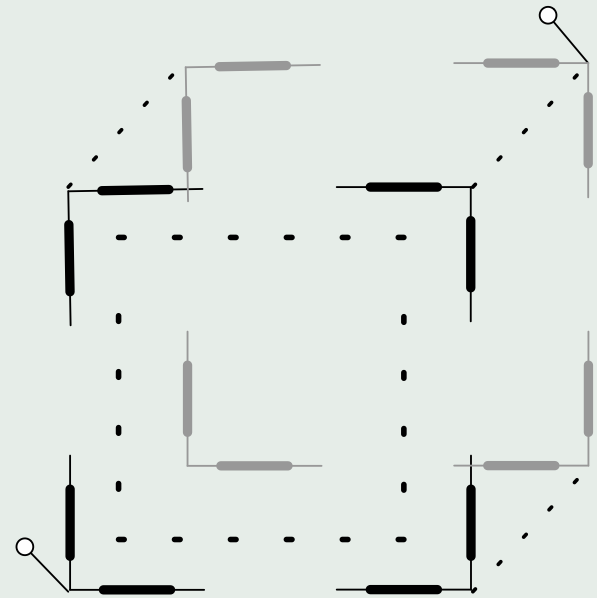
Net resistance $= R\,M$

2D :



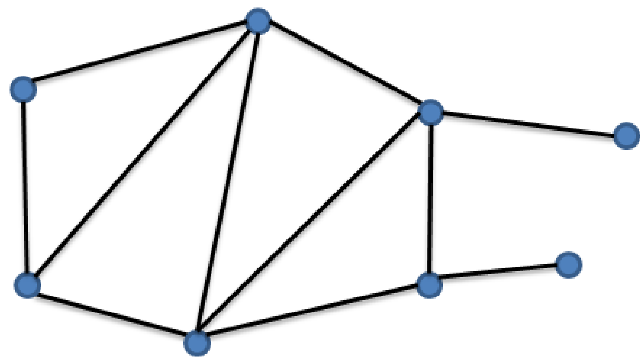
Net resistance $= O\left(\log(M)\right)$

3D :



Net resistance is *bounded*!

Lecture 27: Optimal control of undirected graphs



- Single-integrator dynamics

$$\dot{x}_i = u_i + d_i$$

- Relative information exchange with neighbors

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} k_{ij} \Big(x_i(t) - x_j(t) \Big)$$

- Closed-loop dynamics

$$\dot{x}(t) = -L(k)\, x(t) + d(t)$$

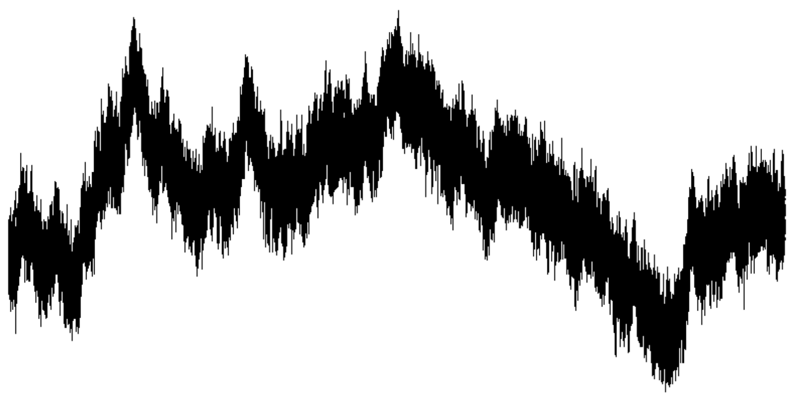
- Structured matrix L depends on $\left\{ \begin{array}{l} \text{graph topology} \\ \text{vector of feedback gains } k \end{array} \right.$

- Independent of graph topology and feedback gains

$$L(k)\,\mathbb{1} \;=\; 0 \cdot \mathbb{1}$$

Average mode

$$\bar{x}(t) \;=\; \frac{1}{N}\sum_{i\,=\,1}^{N} x_i(t): \quad \text{undergoes random walk}$$



If other modes are stable, $x_i(t)$ fluctuates around $\bar{x}(t)$

deviation from average: $\tilde{x}_i(t) \;=\; x_i(t) \;-\; \bar{x}(t)$

steady-state variance: $\lim_{t\,\to\,\infty} \mathcal{E}\left(\tilde{x}^T(t)\,\tilde{x}(t)\right)$

Optimal control problem

What graph topologies lead to small variance?

How to design feedback gains to minimize variance?

$$
\begin{aligned}
\dot{x}(t) &= -L(k)\,x(t) \;+\; d(t) \\
z(t) &= \begin{bmatrix} \tilde{x}(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} I - \frac{1}{N}\mathbb{1}\mathbb{1}^T \\ -\,L(k) \end{bmatrix} x(t)
\end{aligned}
$$

- Setup:
  - ⋆ Undirected graphs: bi-directional interaction between nodes
  - ⋆ Symmetric feedback gains

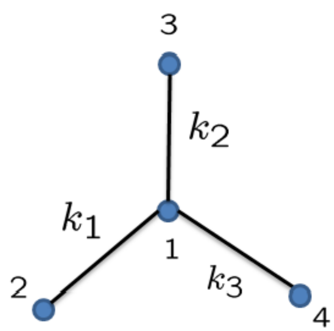
$$
k_{ij} \;=\; k_{ji} \;\;\Rightarrow\;\; L(k) \;=\; L^T(k)
$$

Incidence matrix

- Edge $l \sim (i, j)$: connects nodes i and j
  - ⋆ Define $e_l \in \mathbb{R}^N$ with only two nonzero entries

$$(e_l)_i \; = \; 1 \qquad\qquad (e_l)_j \; = \; -1$$

Incidence matrix: $E \; = \; \left[\, e_1 \; \cdots \; e_m \, \right]$



$$E \; = \; \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad E^T x \; = \; \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \\ x_1 - x_4 \end{bmatrix}, \quad E^T \mathbb{1} \; = \; 0$$

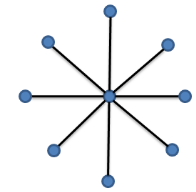
Edge $l \sim (i, j)$: $k_l \; := \; k_{ij} \; = \; k_{ji}$

Laplacian: $L(K) \; = \; E \, K \, E^T \; = \; \sum_{l\,=\,1}^{m} k_l \, e_l \, e_l^T$

Structured feedback gain: $K \; = \; \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_m \end{bmatrix}$

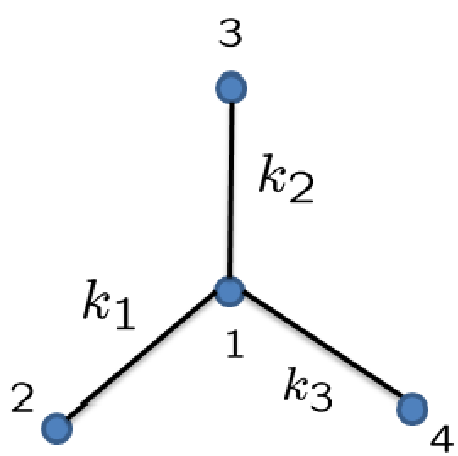
Tree graphs

- Trees: connected graphs with no cycles

path  star 

Incidence matrix of a tree $E_t \in \mathbb{R}^{N \times (N-1)}$

$$E_t = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$



$$E_t = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- Coordinate transformation

$$\begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} E_t^T \\ \frac{1}{N} \mathbb{1}^T \end{bmatrix}}_{T} x(t) \;\Leftrightarrow\; x(t) = \underbrace{\begin{bmatrix} E_t \,(E_t^T E_t)^{-1} & \mathbb{1} \end{bmatrix}}_{T^{-1}} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix}$$

In new coordinates

$$\begin{aligned} \begin{bmatrix} \dot{\psi}(t) \\ \dot{\bar{x}}(t) \end{bmatrix} &= -\begin{bmatrix} E_t^T \\ \frac{1}{N} \mathbb{1}^T \end{bmatrix} E_t \, K \, E_t^T \begin{bmatrix} E_t \,(E_t^T E_t)^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix} + \begin{bmatrix} E_t^T \\ \frac{1}{N} \mathbb{1}^T \end{bmatrix} d(t) \\ &= \begin{bmatrix} -E_t^T E_t \, K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix} + \begin{bmatrix} E_t^T \\ \frac{1}{N} \mathbb{1}^T \end{bmatrix} d(t) \end{aligned}$$

$$z(t) = \begin{bmatrix} I - \frac{1}{N} \mathbb{1}\mathbb{1}^T \\ -\, E_t \, K \, E_t^T \end{bmatrix} \begin{bmatrix} E_t \,(E_t^T E_t)^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix}$$

Tree graphs: structured optimal H_2 design

$$\begin{aligned}\dot{\psi}(t) &= -\,E_t^T E_t\, K\, \psi(t) \;+\; E_t^T\, d(t)\\ z(t) &= \begin{bmatrix} E_t\,(E_t^T E_t)^{-1} \\ -\,E_t\, K \end{bmatrix} \psi(t)\end{aligned}$$

H_2 norm (from d to z)

$$J(K) \;=\; \frac{1}{2}\,\mathrm{trace}\Big((E_t^T E_t)^{-1} K^{-1} \;+\; K\, E_t^T E_t\Big)$$

Diagonal matrix: $K \;=\; \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_{N-1} \end{bmatrix}$

- Structured optimal feedback gains

$$k_i \;=\; \sqrt{\frac{\big(E_t^T E_t\big)_{ii}^{-1}}{2}}, \quad i \;=\; 1, \ldots, N-1$$

- In Lecture 28, I made a blunder on board while deriving the optimal values of k_i

Here is correct derivation:

⋆ $G := \left(E_t^T E_t\right)^{-1} \;\Rightarrow\;$ diagonal elements of G determined by $G_{ii} = \left(E_t^T E_t\right)_{ii}^{-1}$

⋆ All diagonal elements of $E_t^T E_t$ are equal to 2

$$
\begin{aligned}
E_t^T E_t \;&=\; \begin{bmatrix} e_1 & \cdots & e_{N-1} \end{bmatrix}^T \begin{bmatrix} e_1 & \cdots & e_{N-1} \end{bmatrix} \;=\; \begin{bmatrix} e_1^T \\ \vdots \\ e_{N-1}^T \end{bmatrix} \begin{bmatrix} e_1 & \cdots & e_{N-1} \end{bmatrix} \\
&=\; \begin{bmatrix} e_1^T e_1 & \cdots & e_1^T e_{N-1} \\ \vdots & \ddots & \vdots \\ e_{N-1}^T e_1 & \cdots & e_{N-1}^T e_{N-1} \end{bmatrix} \;=\; \begin{bmatrix} 2 & \cdots & e_1^T e_{N-1} \\ \vdots & \ddots & \vdots \\ e_{N-1}^T e_1 & \cdots & 2 \end{bmatrix}
\end{aligned}
$$

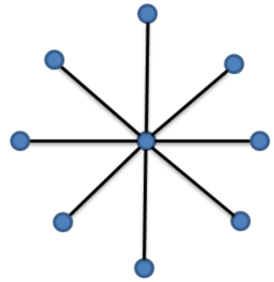
⋆ K – diagonal matrix $\;\Rightarrow\;$ $J(K)$ can be written as

$$
J(K) \;=\; \sum_{i\,=\,1}^{N-1} \left(\frac{G_{ii}}{2k_i} \;+\; k_i \right)
$$

⋆ $J(K)$ in a separable form $\;\Rightarrow\;$ element-wise minimization will do

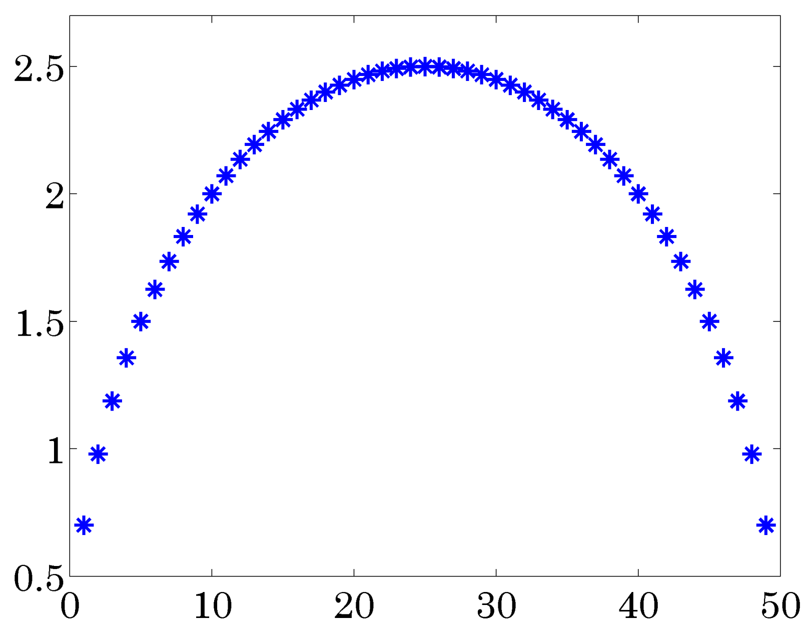
$$
\frac{\mathrm{d}}{\mathrm{d}k_i} \left(\frac{G_{ii}}{2k_i} \;+\; k_i \right) \;=\; -\frac{G_{ii}}{2k_i^2} \;+\; 1 \;=\; 0 \;\Rightarrow\; k_i \;=\; \sqrt{\frac{G_{ii}}{2}}, \;\; i \;=\; 1, \ldots, N-1
$$

Optimal gains for star and path

- Star:  uniform gain $k = \sqrt{\dfrac{N-1}{2N}} \approx \dfrac{1}{\sqrt{2}}$ for large N

- Path: 

$$k_i = \sqrt{\frac{i\,(N-i)}{2N}}$$



largest gains in the center

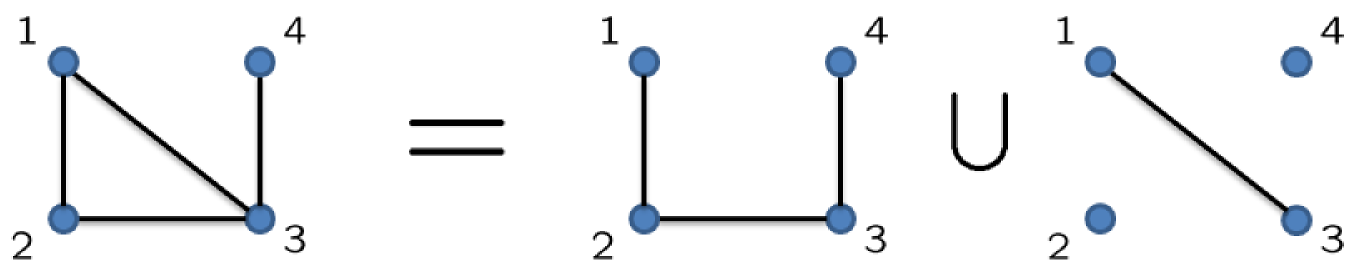
General undirected graphs

- Decompose graph into a tree subgraph and remaining edges

$$\text{Incidence matrix:} \quad E = \begin{bmatrix} E_t & E_c \end{bmatrix}$$

$$\text{Projection matrix:} \quad \Pi = E_t E_t^{+} = E_t \left(E_t^T E_t\right)^{-1} E_t^T$$

$$E_c \in \text{range}(\Pi): \quad E_c = \Pi E_c$$



$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \cup \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} E &= \begin{bmatrix} E_t & E_c \end{bmatrix} = \begin{bmatrix} E_t & \Pi E_c \end{bmatrix} \\ &= E_t \begin{bmatrix} I & (E_t^T E_t)^{-1} E_t^T E_c \end{bmatrix} = E_t M \end{aligned}$$

General graphs: structured optimal H_2 design

$$
\begin{aligned}
\dot{\psi}(t) &= -E_t^T E_t \, M \, K \, M^T \psi(t) + E_t^T d(t) \\
z(t) &= \begin{bmatrix} E_t \left(E_t^T E_t\right)^{-1} \\ -E_t \, M \, K \, M^T \end{bmatrix} \psi(t)
\end{aligned}
$$

tree graphs: $M = I$

H_2 norm (from d to z)

$$
J(K) = \frac{1}{2} \operatorname{trace} \left(\left(E_t^T E_t\right)^{-1} \left(M \, K \, M^T\right)^{-1} + M \, K \, M^T E_t^T E_t \right)
$$

- Main result:
  - ⋆ Closed-loop stability $\Leftrightarrow$ $M \, K \, M^T > 0$

    $\{W_1 > 0, W_2 = W_2^T\}$ then $-W_1 W_2$ Hurwitz $\Leftrightarrow$ $W_2 > 0$

  - ⋆ $M \, K \, M^T > 0 \Rightarrow$ convexity of $J(K)$

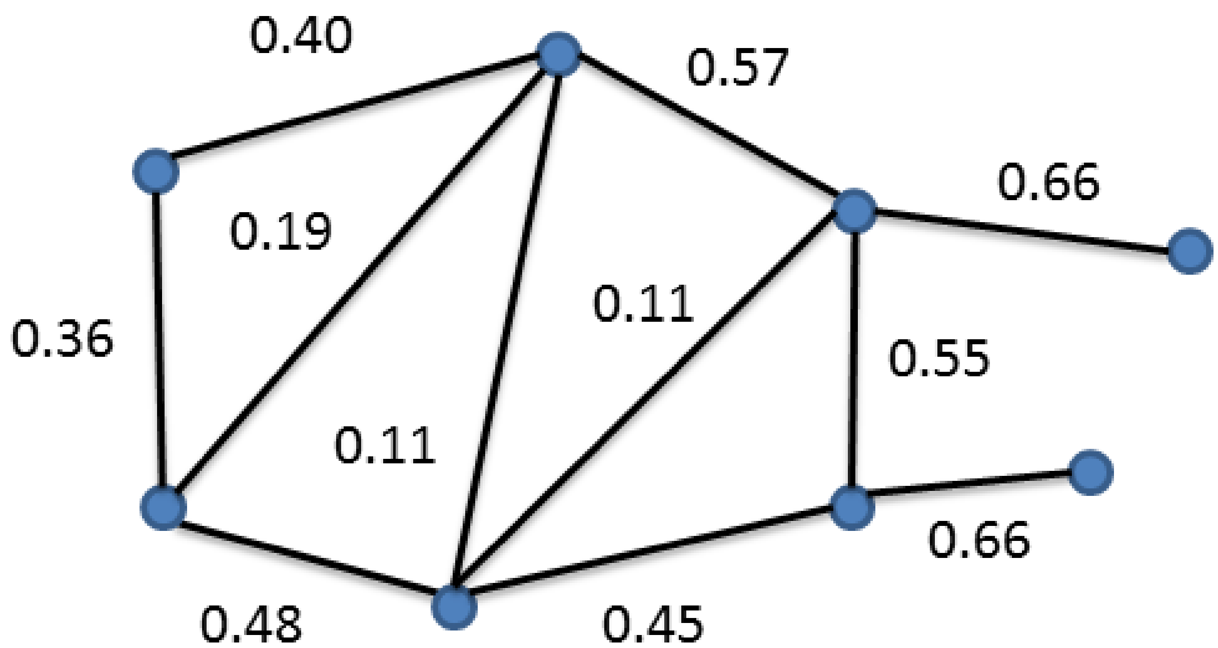
- Semi-definite program

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\,\mathrm{trace}\left(X + M\,K\,M^T E_t^T\,E_t\right) \\ \text{subject to} & \left[\begin{array}{cc} X & (E_t^T E_t)^{-1/2} \\ (E_t^T E_t)^{-1/2} & M\,K\,M^T \end{array}\right] > 0 \\ & K \text{ diagonal} \end{array}$$

- Use CVX to solve it

```
cvx_begin sdp

    variable k(Ne) % vector of unknown feedback gains

    variable X(Nv-1,Nv-1) symmetric;
    X == semidefinite(Nv-1); % Schur complement variable

    Mk = M*diag(k)*M'; % Matrix Mk

    minimize(0.5*trace( q*X + r*Mk*W ))
    subject to [X, invWh; invWh, Mk] > 0;

cvx_end
```

Examples

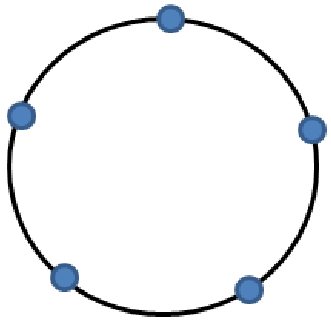


- Compare with performance of uniform gain design

J^*	$J(k = 1)$	$(J - J^*)/J^*$
9.1050	13.1929	45%

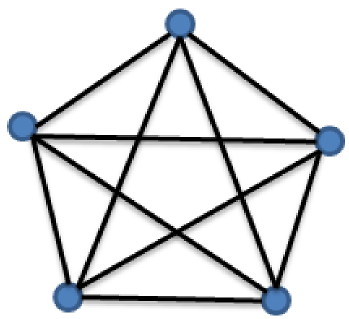
- Analytical results for circle and complete graphs

⋆ Circle



uniform gain $k = \sqrt{\dfrac{N^2 - 1}{24N}}$

⋆ Complete graph



uniform gain $k = \dfrac{2}{N}$

Additional material

- Papers to read
  - ⋆ *Xiao, Boyd, Kim, J. Parallel Distrib. Comput. ’07*
  - ⋆ *Zelazo & Mesbahi, IEEE TAC ’11*
  - ⋆ *Lin, Fardad, Jovanovic, CDC ’10*

Lecture 28: Alternating Direction Method of Multipliers (ADMM)

- Well-suited to $\left\{ \begin{array}{l} \text{distributed optimization} \\ \text{large-scale problems} \end{array} \right.$

- Precursors
  - ⋆ Dual ascent
  - ⋆ Dual decomposition
  - ⋆ Method of multipliers

- Design of optimal sparse feedback gains via ADMM

  Lin, Fardad, Jovanović, IEEE TAC '11 (submitted; also: arXiv:1111.6188v1)

- Online resources
  - ⋆ Stephen Boyd's webpage

    ADMM material (paper, talks, Matlab files)

    ℓ_1 methods for convex-cardinality problems (lectures and videos)

Equality-constrained convex optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & A\,x \;=\; b \end{array}$$

f: $\mathbb{R}^n \;\to\; \mathbb{R}$ – convex function

- Lagrangian

$$\mathcal{L}(x, y) \;=\; f(x) \;+\; y^T\,(A\,x \;-\; b)$$

- dual function

$$g(y) \;=\; \inf_{x}\; \mathcal{L}(x, y)$$

- dual problem

$$\text{maximize} \;\; g(y)$$

Dual ascent

$$\begin{aligned} x\text{-minimization:} \quad x^{k+1} &:= \arg\min_x \mathcal{L}(x, y^k) \\ \text{dual variable update:} \quad y^{k+1} &:= y^k + s^k \left(A\,x^{k+1} - b\right) \end{aligned}$$

- Features
  - ⋆ For properly selected $s^k \Rightarrow g(y^{k+1}) > g(y^k)$
  - ⋆ Requires strong assumptions
  - ⋆ May converge slowly
  - ⋆ Can lead to distributed implementation

Dual decomposition

separable form:

$$f(x) = \sum_{n=1}^{N} f_n(x_n)$$

Lagrangian:

$$\mathcal{L}(x, y) = \sum_{n=1}^{N} f_n(x_n) + y^T \left(\sum_{n=1}^{N} A_n x_n - b \right) = \sum_{n=1}^{N} \mathcal{L}_n(x_n, y) - y^T b$$

decomposition:

$$\mathcal{L}_n(x_n, y) := f_n(x_n) + y^T A_n x_n$$

- **Can be solved in parallel**

DUAL DECOMPOSITION:

$$
\begin{aligned}
x_n^{k+1} &:= \underset{x_n}{\arg\min}\; \mathcal{L}_n(x_n, y^k) \\
y^{k+1} &:= y^k + s^k \left(\sum_{n\,=\,1}^{N} A_n x_n^{k+1} - b \right)
\end{aligned}
$$

- distributed optimization
  - ⋆ broadcast y^k
  - ⋆ update x_n^{k+1} in parallel
  - ⋆ gather $A_n x_n^{k+1}$
- well-suited to large-scale problems
  - ⋆ sub-problems solved iteratively in parallel
  - ⋆ dual variable update provides coordination

Method of multipliers

augmented Lagrangian: $\mathcal{L}_\rho(x, y) \;=\; \mathcal{L}(x, y) \;+\; \dfrac{\rho}{2}\,\|Ax \;-\; b\|_2^2$

METHOD OF MULTIPLIERS:

$$
\begin{aligned}
x^{k+1} &:= \operatorname*{arg\,min}_x \mathcal{L}_\rho(x, y^k) \\
y^{k+1} &:= y^k \;+\; \rho\left(Ax^{k+1} \;-\; b\right)
\end{aligned}
$$

compared to dual ascent:

- advantages:
  - ⋆ convergence under milder assumptions
  - ⋆ brings robustness
- disadvantage
  - ⋆ quadratic term: in general not separable $\Rightarrow$ may not be solved in parallel

OPTIMALITY CONDITIONS:

$$\begin{aligned} \nabla_x \mathcal{L}_\rho(x^\star, y^\star) &= \nabla_x f(x^\star) + A^T y^\star = 0 \\ \nabla_y \mathcal{L}_\rho(x^\star, y^\star) &= A x^\star - b = 0 \end{aligned}$$

- x^{k+1} minimizer of $\mathcal{L}(x, y^k)$

$$\begin{aligned} 0 &= \nabla_x \mathcal{L}(x^{k+1}, y^k) \\ &= \nabla_x f(x^{k+1}) + A^T y^k + \rho A^T (A x^{k+1} - b) \\ &= \nabla_x f(x^{k+1}) + A^T (y^k + \rho(A x^{k+1} - b)) \\ &= \nabla_x f(x^{k+1}) + A^T y^{k+1} \end{aligned}$$

- dual feasibility satisfied at every iteration

- primal feasibility satisfied in the limit

$$\lim_{k \to \infty} A x^k = b$$

Alternating direction method of multipliers

- Converges under mild assumptions
  - ⋆ robust dual decomposition
- Facilitates decomposition
  - ⋆ decomposable method of multipliers
- Proposed in ’70s
- Many modern applications
  - ⋆ distributed computing
  - ⋆ distributed signal processing
  - ⋆ image denoising
  - ⋆ machine learning
  - ⋆ statistics

Boyd et al., Foundations and Trends in Machine Learning ’11

- standard ADMM formulation

$$\begin{array}{ll} \text{minimize} & f(x) \; + \; g(z) \\ \text{subject to} & Ax \; + \; Bz \; = \; c \end{array}$$

augmented Lagrangian

$$\mathcal{L}_\rho(x, z, y) \; = \; f(x) \; + \; g(z) \; + \; y^T(Ax \; + \; Bz \; - \; c) \; + \; \frac{\rho}{2} \|Ax \; + \; Bz \; - \; c\|_2^2$$

ADMM:

$$\begin{array}{rcl} x^{k+1} & := & \operatorname*{arg\,min}_x \, \mathcal{L}_\rho(x, z^k, y^k) \\ z^{k+1} & := & \operatorname*{arg\,min}_z \, \mathcal{L}_\rho(x^{k+1}, z, y^k) \\ y^{k+1} & := & y^k \; + \; \rho \left(Ax^{k+1} \; + \; Bz^{k+1} \; - \; c\right) \end{array}$$

Reduces to method of multipliers if minimization done jointly (over x and z)

OPTIMALITY CONDITIONS:

$$
\begin{aligned}
\nabla_x \mathcal{L}_\rho(x^\star, y^\star, z^\star) &= \nabla_x f(x^\star) \,+\, A^T y^\star &= 0 \\
\nabla_z \mathcal{L}_\rho(x^\star, y^\star, z^\star) &= \nabla_z g(z^\star) \,+\, B^T y^\star &= 0 \\
\nabla_y \mathcal{L}_\rho(x^\star, y^\star, z^\star) &= A x^\star \,+\, B z^\star \,-\, c &= 0
\end{aligned}
$$

- z^{k+1} minimizes $\mathcal{L}(x^{k+1}, z, y^k)$

$$
\begin{aligned}
0 &= \nabla_z g(z^{k+1}) \,+\, B^T y^k \,+\, \rho B^T (A x^{k+1} \,+\, B z^{k+1} \,-\, c) \\
&= \nabla_z g(x^{k+1}) \,+\, B^T y^{k+1}
\end{aligned}
$$

- second dual feasibility satisfied at every iteration
- primal and first dual feasibility satisfied asymptotically