09 - 07 - 11

Distributed Systems

Fall '11 , EE 8235

Terminology :

1) Spatially distributed systems : in addition to time , there is
(extended)
a spatial independent variable

2) Infinite dimensional systems
(Distributed parameter systems)
- partial differential equations (PDEs)
- Delay equations

3) Large-scale systems (interconnected systems)
Many degrees of freedom, High dynamical order.

First part of the course : (1) and (2)
Second " " " : (3)

Example

(1) Heat equation in 1D (one spatial direction)
(diffusion)

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

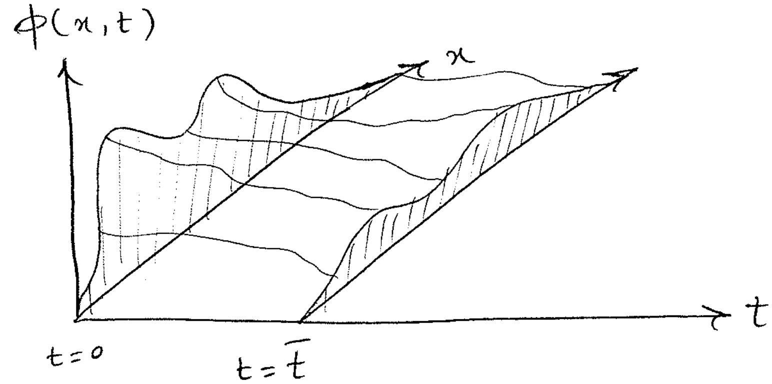
unforced problem (no input)

ϕ ... field of interest
(e.g. temperature distribution)

$\phi(x, t)$

x ... spatial variable
t ... time
} independent variables

A posible objective: find $\phi(x,t)$



x .... a continuous spatial variable

e.g. $x \in [-1,1]$

Note: if x belongs to a finite interval, we will usually normalized it to $[-1,1]$

$$\bar{x} \in [0,L] \xrightarrow{\text{affine}} x \in [-1,1]$$

$$x = a\bar{x} + b$$

$$b = -1 \quad ; \quad a = \frac{2}{L}$$

$$\frac{\partial \phi}{\partial \bar{x}} = \frac{\partial x}{\partial \bar{x}} \frac{\partial \phi}{\partial x} = a \frac{\partial \phi}{\partial x}$$

Derivatives change after the transformation

Question: what do we need to uniquely determine $\phi(x,t)$ from $\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$?

Answer: Need initial and boundary conditions

I.C. : $\phi(x, t=0) = \phi_0(x)$

(an initial temperature distribution)

B.C. : Many possibilities, but need two B.C.

$\downarrow$

degree of $\frac{\partial}{\partial x}$

(a) $\phi(x = \pm 1, t) = 0$ (Dirichlet)

(b) $\frac{\partial \phi}{\partial x}(x = \pm 1, t) = 0$ (Neumann)

(c) Combination of (a) and (b)
(Linear)

<u>For the case of forced Heat equation</u>

$$\frac{\partial \phi}{\partial t}(x,t) = \frac{\partial^2}{\partial x^2}\phi(x,t) + u(x,t)$$

$\downarrow$

spatially and temporally distributed <u>input</u>

$\downarrow$

\+ I.C. + B.C.

Can be disturbance
or Control

<u>Boundary input :</u>

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

I.C. $\phi(x,0) = \phi_0(x)$

B.C. $\begin{cases} \phi(x=-1, t) = u(t) \\ \phi(x=1, t) = 0 \end{cases}$

Examples of distributed Systems

$$\dot{x} = Ax + B_1 d + B_2 u$$

x ... state

d ... disturbance

u ... input control

$$\phi_t = \phi_{xx} + d \qquad (*)$$

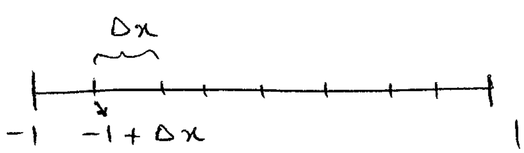
$$\phi(x=-1,t) = u(t)$$

$$\phi(x=1,t) = 0$$

Approximate second derivative with central difference

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{x=\bar{x}} \simeq \frac{\phi(\bar{x}+\Delta x) - 2\phi(\bar{x}) + \phi(\bar{x}-\Delta x)}{2\Delta x}$$

This yields a finite dimensional approximation of $(*)$ with the state given by



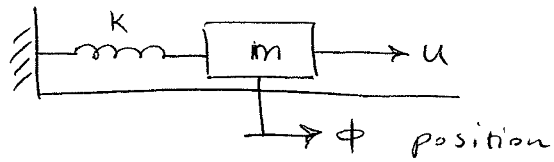
$$\phi_n = \phi(x = -1 + n\Delta x)$$

$$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix}$$

$$\dot{\Phi} = A\Phi + B_2 u + B_1 d$$

$$A = \frac{1}{2\Delta x}\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ & & \ddots & \ddots \\ & & & 1 \quad -2 \end{bmatrix} \; ; \; B_2 = \frac{1}{2\Delta x}\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Mass - Spring System



$$m\ddot{\phi}(t) + K\phi(t) = u(t)$$

2-norm

if $f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \in \mathbb{C}^n$

$$\|f\|_2^2 = f^* f = \bar{f}_1 \cdot f_1 + \bar{f}_2 \cdot f_2 + \cdots \bar{f}_n \cdot f_n$$

if $f = f(x) \in L_2[-1,1]$

$$\|f\|_2^2 = \int_{-1}^{1} f^*(x) f(x)\, dx$$

Notation

$$\phi_t(x,t) = \phi_{xx}(x,t) + u(x,t)$$

$\phi(x,t)$ ..... field (function) ϕ evaluated at time t and position x

$\phi(x,t) \in \mathbb{R}$ .... scalar

$\psi(t) = \phi(\cdot, t)$ .... at any fixed t $\psi(t)$ is a function in a certain Hilbert space.

Abstract notation

$$\frac{d\psi(t)}{dt} = \mathcal{A}\psi(t) + \mathcal{B}u(t)$$

$$\psi(t) \in \mathcal{H} \stackrel{e.g.}{=} \left\{ f, \int_{-\infty}^{\infty} f^*(x) f(x)\, dx < +\infty \right\}$$

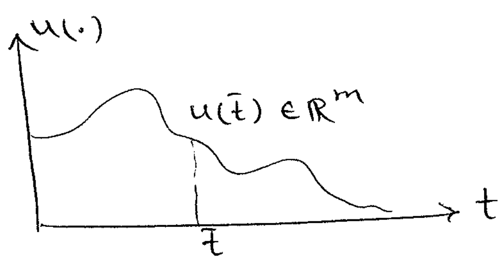
$$\psi(t) = \phi(\cdot, t) \iff [\psi(t)](x) = \phi(x,t)$$

Side note:

e.g. finite-dimensional system

$\dot{\psi} = A\psi + Bu$

input $u \in L_2(0,\infty)$



Wave equation

$$\phi_{tt} = \phi_{xx} + u$$

Define states of the system such that we are left with a first order system of equations in time.

$$\psi_1 = \phi$$
$$\psi_2 = \phi_t$$

Then
- $\psi_{1t} = \phi_t$
- $\psi_{2t} = \phi_{tt}$

So, $\psi_{1t} = \psi_2$

$$\psi_{2t} = \phi_{tt} = \phi_{xx} + u = \psi_{1xx} + u$$

$$\Rightarrow \quad \psi_{1t} = 0 \cdot \psi_1 + I \cdot \psi_2 + 0 \cdot u \quad (1)$$

$$\psi_{2t} = \frac{d^2}{dx^2}\psi_1 + 0 \cdot \psi_2 + I \cdot u \quad (2)$$

$$\phi = I \cdot \psi_1 + 0 \cdot \psi_2 + 0 \cdot u \quad (3)$$

(1) & (2) ... state equations (1st order in time diff. equations)

(3) eu. ------ output equation (static in time equation that tells you how to obtain output of interest from the states and inputs)

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u(t)$$

$$\phi(t) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}$$

Note :

Energy of wave : $E(t) = \frac{1}{2}\int_{-1}^{1} \left(\phi_x^2(x,t) + \phi_t^2(x,t)\right) dx$

$= \frac{1}{2}\int_{-1}^{1} \left(\psi_{1x}^2(x,t) + \psi_2^2(x,t)\right) dx$

We will show (in HW) that

$\begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} \in \begin{bmatrix} L_2[-1,1] \\ L_2[-1,1] \end{bmatrix}$ is not a good candidate for state space.

So, selection of state space for wave equation is more subtle than for diffusion equation.

L_2 norm of $\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$

$$\left\| \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \right\|_{L^2}^2 = \left\langle \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \right\rangle_{L^2}$$

$$= \int_{-1}^{1} \begin{bmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{bmatrix}^* \begin{bmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{bmatrix} dx$$

$$= \int_{-1}^{1} \begin{bmatrix} \bar{\psi}_1(x,t) & \bar{\psi}_2(x,t) \end{bmatrix} \begin{bmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{bmatrix} dx$$

$$= \int_{-1}^{1} \left(\psi_1^2(x,t) + \psi_2^2(x,t) \right) dx$$

Even though the above norm is a valid L_2 norm of the state, this norm is not the norm $\underbrace{(\text{wave energy})}_{E(t) \text{ above}}$ that we are interested in.

Notation

$$(\Delta + I)^2 \phi = (\Delta + I)(\Delta + I)\phi$$

in 1D :

$$\Delta = \frac{\partial^2}{\partial x^2}$$

Then: $(\Delta + 1)^2 = \left(\frac{\partial^2}{\partial x^2} + 1\right)\left(\frac{\partial^2}{\partial x^2} + 1\right) =$

$$\frac{\partial^4}{\partial x^4} + 2\frac{\partial^2}{\partial x^2} + 1$$

$$(\Delta + I)^2 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^2 \phi}{\partial x^2} + \phi$$

$x \in (-\infty, +\infty)$

Fourier Transform :

$$\hat{\phi} - 2K^2\hat{\phi} + K^4\hat{\phi} = (1 - K^2)^2\hat{\phi}$$

09-15-11

Cauchy Sequence

$$\lim_{m,n \to \infty} \| v_m - v_n \| \longrightarrow 0$$

$\| \cdot \|$ ... notion of distance between elements of the space.

Finite dimensional space :

$\mathbb{C}^n$ or $\mathbb{R}^n$

$$v \in \mathbb{C}^n \iff v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \; ; \; v_i \in \mathbb{C}$$

Inner product on $\mathbb{C}^n$:

$$\langle u, v \rangle_{\mathbb{C}^n} = \underbrace{u^*}_{\downarrow} v = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}^* \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Complex-Conjugate transpose of u

$$= [\bar{u}_1 \; \ldots \; \bar{u}_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} \bar{u}_i \, v_i$$

Inner product is Linear in second argument:

$$\langle u, v + w \rangle_{\mathbb{C}^n} = \langle u, v \rangle_{\mathbb{C}^n} + \langle u, w \rangle_{\mathbb{C}^n}$$

it is Conjugate Linear in its first argument:

$$\langle \alpha u_1, v \rangle_{\mathbb{C}^n} = \alpha^* \langle u_1, v \rangle_{\mathbb{C}^n}$$

$$= \bar{\alpha} \langle u, u \rangle_{\mathbb{C}^n}$$

$$\langle u, v \rangle_{\mathbb{C}^n} = \overline{\langle v, u \rangle_{\mathbb{C}^n}}$$

Banach Space : Complete normed space

(But, there is no notion of inner product that induces the norm).

example of Banach space : ℓ_p.

$$\ell_p = \left\{ \{f_n\}_{n \in \mathbb{Z}} , \sum_{n=-\infty}^{\infty} |f_n|^p < +\infty \right\}$$

$p = \infty$: $\ell_\infty \rightarrow \sup |f_n|$

$$\dot{\psi}_n(t) = a_n \psi_n(t)$$

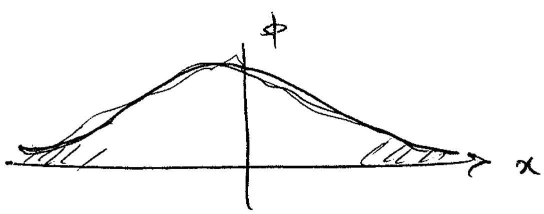
$$\Rightarrow \psi_n(t) = e^{a_n t} \psi_n(0) \quad \text{(solution)}$$

$$\begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} = \begin{bmatrix} e^{a_1 t} & & \\ & \ddots & \\ & & e^{a_n t} \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix}$$

$$\phi_t(x,t) = \phi_{xx}(x,t) + u(x,t)$$

$$\phi \in L_2(-\infty, \infty)$$

Need two boundary conditions, but the fact that the operators act on elements of $L_2(-\infty,\infty)$ means that ϕ should vanish at $\pm\infty$. Otherwise it can't be square-integrable.



$$\langle v_m , \sum_{n=1}^{\infty} \dot{\alpha}_n(t) v_n \rangle$$

$$= \sum_{n=1}^{\infty} \dot{\alpha}_n(t) \underbrace{\langle v_m , v_n \rangle}_{\delta_{m,n} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}}$$

$$= \dot{\alpha}_m(t)$$

Fourier Transform : $\mathcal{F}$

$$
\begin{tikzcd}
L_2(-\infty,\infty) \supset \mathcal{D}(\mathcal{A}) \arrow[r, "\mathcal{A} = \frac{\partial^2}{\partial x^2}"] \arrow[d, shift right, "\mathcal{F}"'] & L_2(-\infty,\infty) \arrow[d, shift right] \\
L_2(-\infty,\infty) \supset \mathcal{D}(\hat{\mathcal{A}}) \arrow[r, "\hat{\mathcal{A}}(k) = -k^2", "\text{multiplication operator}"'] \arrow[u, shift right] & L_2(-\infty,\infty) \arrow[u, shift right, "\mathcal{F}^{-1}"']
\end{tikzcd}
$$

F.T. brings $\phi_t = \phi_{xx} + u$ to

$$\dot{\hat{\phi}}(k,t) = -k^2 \hat{\phi}(k,t) + \hat{u}(k,t)$$

Continuum of decoupled scalar states (parameterized by $k \in \mathbb{R}$)

wave-number

F.T. "diagonalizes" generator of our dynamics

(i.e. operator $\mathcal{A} = \left.\frac{d^2}{dx^2}\right|_{L_2(-\infty,\infty)}$)

Heat equation

$\phi_t = \phi_{xx}$ with B.C. $\phi(x=\pm 1, t) = 0$

I.C. $\phi(x, t=0) = f(x)$

$$\phi(x,t) = \sum_{n=1}^{\infty} \alpha_n(t) v_n(x)$$

$$\langle v_n, v_m \rangle = \int_{-1}^{1} v_n^*(x)\, v_m(x)\, dx = \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Cont'd $\rightarrow$

$$\sum_{n=1}^{\infty} \dot{\alpha}_n(t) v_n(x) = \sum_{n=1}^{\infty} \alpha_n(t) \underbrace{v_n''(x)}$$

use the fact that

$$v_n''(x) = \lambda_n v_n(x)$$

$$\sum_{n=1}^{\infty} \dot{\alpha}_n(t) v_n(x) = \sum_{n=1}^{\infty} \lambda_n \alpha_n(t) v_n(x)$$

$$\left\langle v_m, \sum_{n=1}^{\infty} \dot{\alpha}_n(t) v_n(x) \right\rangle = \left\langle v_m, \sum_{n=1}^{\infty} \lambda_n \alpha_n(t) v_n(x) \right\rangle$$

$$\sum_{n=1}^{\infty} \dot{\alpha}_n \underbrace{\langle v_n, v_m \rangle}_{\delta_{n,m}} = \sum_{n=1}^{\infty} \lambda_n \alpha_n(t) \underbrace{\langle v_n, v_m \rangle}_{\delta_{n,m}}$$

$$\boxed{\dot{\alpha}_m(t) = \lambda_m \alpha_m(t)}$$

Eigen-function expansion brings

$$\begin{cases} \phi_t = \phi_{xx} \\ \phi(\pm 1, t) = 0 \end{cases}$$

into

$$\frac{d}{dt} \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \end{bmatrix}$$

$$\begin{tikzcd}
L_2[-1,1] \supset \mathcal{D}(\mathcal{A}) \arrow[r, "\mathcal{A} = \frac{d^2}{dx^2}"] \arrow[d, "\text{Basis expansion}"'] & L_2[-1,1] \\
\ell_2(\mathbb{N}) \supset \mathcal{D}(\hat{\mathcal{A}}) \arrow[r, "\hat{\mathcal{A}} = \operatorname{diag}\{\lambda_n\}_{n \in \mathbb{N}}", "\text{multiplication operator}"'] & \ell_2(\mathbb{N}) \arrow[u]
\end{tikzcd}$$

Remaining Task

Find dependence of

$\alpha_n(0)$ on $\phi(x,0) = f(x)$ initial condition

So far: $\phi(x,t) = \sum_{n=1}^{\infty} e^{\lambda_n t} \alpha_n(0) v_n(x)$

$$f(x) = \phi(x,0) = \sum_{n=1}^{\infty} \alpha_n(0) v_n(x)$$

$$\langle v_m, f \rangle = \sum_{n=1}^{\infty} \alpha_n(0) \underbrace{\langle v_m, v_n \rangle}_{\delta_{m,n}}$$

$$\boxed{\alpha_m(0) = \langle v_m, f \rangle}$$

Then,

$$\phi(x,t) = \sum_{n=1}^{\infty} e^{\lambda_n t} v_n(x) \langle v_n, f \rangle$$

$$= \sum_{n=1}^{\infty} e^{\lambda_n t} v_n(x) \int_{-1}^{1} v_n^*(\xi) f(\xi)\, d\xi$$

$$= \int_{-1}^{1} \underbrace{\sum_{n=1}^{\infty} e^{\lambda_n t} v_n(x) v_n^*(\xi)}_{T(x,\xi,t)} f(\xi)\, d\xi$$

$$\boxed{\phi(x,t) = \int_{-1}^{1} T(x,\xi,t) f(\xi)\, d\xi}$$

Abstractly

$$\phi(x,t) = [\,T(t)\cdot f\,](x)$$

propagator of the dynamics ↲ ↓ initial Condition

$T(t)$ is an operator with a Kernel representation $T(x,\xi,t)$

Finite dimensional

$$\boxed{\frac{d\psi}{dt} = A\psi} \qquad (1)$$

$$A = v\Lambda v^* \iff v^* A v = \Lambda$$

introduce a coordinate transformation:

$$\phi = v^*\psi \longrightarrow \begin{bmatrix}\phi_1\\ \vdots \\ \phi_n\end{bmatrix} = \begin{bmatrix} v_1^* \\ \vdots \\ v_n^*\end{bmatrix}\begin{bmatrix}\psi_1\\ \vdots \\ \psi_n\end{bmatrix}$$

$$\psi = v^{-*}\phi = v\phi$$

$$v^*\left(v\frac{d\phi}{dt} = A v\phi\right) \Rightarrow$$

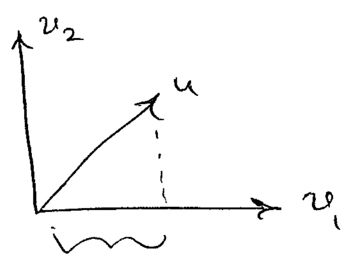
$$v^* v\frac{d\phi}{dt} = v^* A v\phi \Rightarrow \boxed{\frac{d\phi}{dt} = \Lambda\phi}$$

$$\begin{tikzcd}
\psi \in \mathbb{R}^n \arrow[r, "A"] \arrow[d, "v^*"'] & \mathbb{R}^n \\
\phi \in \mathbb{R}^n \arrow[r, "\Lambda = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}"] & \mathbb{R}^n \arrow[u]
\end{tikzcd}
\quad \Rightarrow \quad \psi(t) = e^{At}\psi(0)$$

Summary

$$A u = \sum_{i=1}^{n} \lambda_i v_i \underbrace{\langle v_i , u \rangle}_{\text{Projection of } u \text{ on } v_i}$$

for $u \in \mathbb{R}^2$



projection of u on v_1

Then, $$e^{At} u = \sum_{i=1}^{n} e^{\lambda_i t} v_i \langle v_i , u \rangle \qquad (1)$$

Note : if $u = v_m$

$$e^{At} u = e^{At} v_m = \sum_{i=1}^{n} e^{\lambda_i t} v_i \underbrace{\langle v_i , v_m \rangle}_{\delta_{i,m}}$$

$$= e^{\lambda_m t} \cdot v_m$$

In the case of Heat equation

$$\phi(x,t) = \sum_{n=1}^{\infty} e^{\lambda_n t} \cdot v_n(x) \langle v_n , f \rangle \qquad (2)$$

(1) & (2) are Conceptually similar.

The main difference is the inner - product that is used in (1) or (2).

Finite dimensional

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

given $\left.\begin{array}{l} f \in \mathbb{R}^n \\ g \in \mathbb{R}^m \end{array}\right\}$ $g = Af$

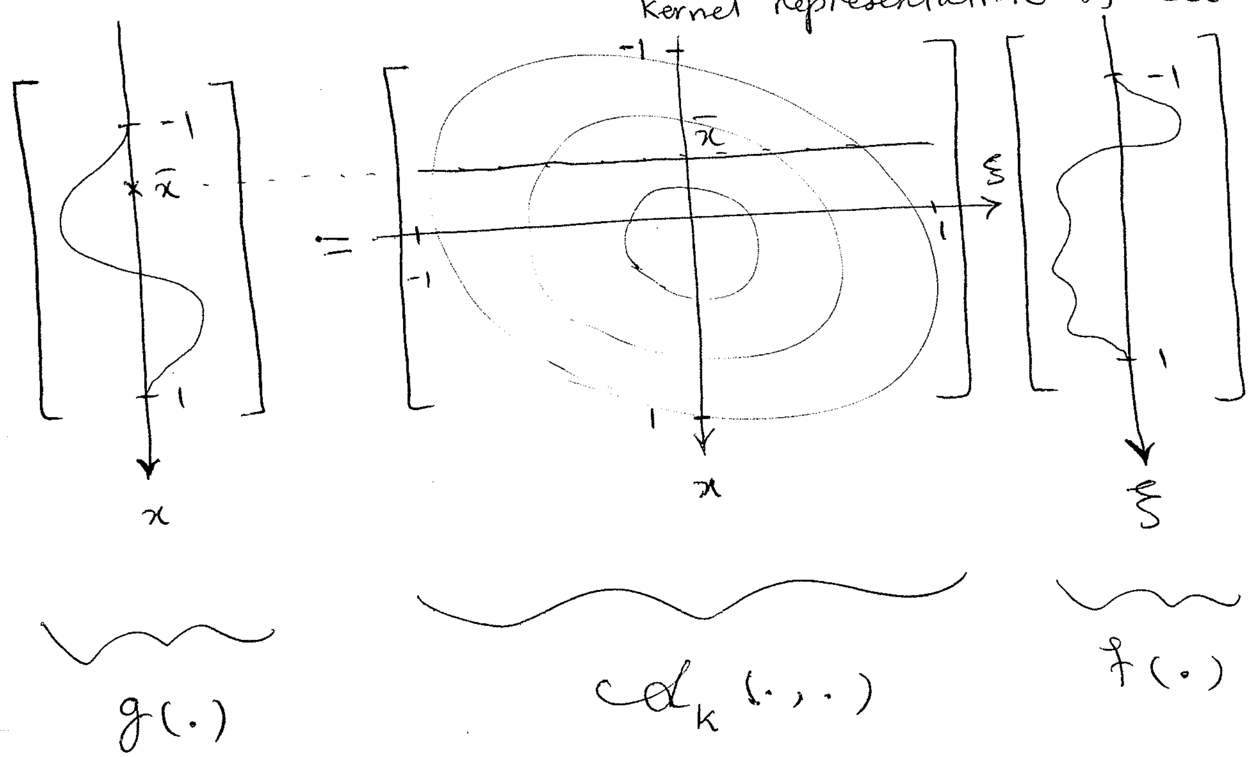
$g \in \mathbb{R}^2 \; ; \; f \in \mathbb{R}^3$

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Infinite dimensional

$\mathcal{A} : L_2[-1,1] \longrightarrow L_2[-1,1]$

$$g(x) = [\mathcal{A} f](x) = \int_{-1}^{1} \underbrace{\mathcal{A}_k(x, \xi)}_{\text{Kernel representation of } \mathcal{A}} f(\xi)\, d\xi$$



$$I : L_2[-1,1] \longrightarrow L_2[-1,1]$$

$$g(x) = [If](x) = f(x) =$$

$$g(x) = \int_{-1}^{1} \delta(x-\xi) f(\xi)\, d\xi = f(x)$$

Multiplication operator : $M_a : L_2[-1,1] \longrightarrow L_2[-1,1]$

$$g(x) = [M_a f](x) = a(x) \cdot f(x)$$

$$g(x) = \int_{-1}^{1} a(x)\delta(x-\xi) f(\xi)\, d\xi = a(x) \cdot f(x)$$

<u>Note</u> kernel representation:

Kernel is a distribution, it doesn't have to be a function.

<u>Side note</u> <u>Bounded operator</u>

$$\mathcal{A} : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$$

$\mathcal{A}$ bounded means

$\exists$ a constant $C < +\infty$ s.t. $\|\mathcal{A} f\|_2 \leq C \|f\|_1$ for all $f \in \mathcal{H}_1$

$$\|f\|_1^2 = \langle f, f \rangle_{\mathcal{H}_1}$$

$$\|\mathcal{A} f\|_2^2 = \langle \mathcal{A} f, \mathcal{A} f \rangle_{\mathcal{H}_2}$$

Example of unbounded operator

$$\mathcal{A} = \frac{d}{dx} \quad ; \quad x \in [-1, 1]$$

$$f_n(x) = \sin(nx) \quad ; \quad n = 1, 2, \ldots$$

$$f_n'(x) = n \cos(nx)$$

$$\| \mathcal{A} f_n \|^2 = \| f_n' \|^2 = \underbrace{n^2 \| f \|^2}$$

Cannot be written as or bounded by $c \| f \|^2$

$$\| \mathcal{A} f_n \|^2 = n^2 \| f \|^2 \not\leq c \| f \|^2$$

An operator norm (an induced norm)

$$\| \mathcal{A} \| = \sup_{f \neq 0} \frac{\| \mathcal{A} f \|_2}{\| f \|_2}$$

<u>Recall</u> $A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$

$$\| A \|_F^2 = \operatorname{trace}(A^* A) = \operatorname{trace}(A A^*)$$

where $\operatorname{trace}(M) = \sum_{i=1}^{n} m_{ii}$

$\| \cdot \|_F$ .... not an induced norm !

$$\mathcal{A} : L_2[-1,1] \longrightarrow L_2[-1,1]$$

$$\|\mathcal{A}\|_{HS}^2 = \int_{-1}^{1}\int_{-1}^{1} \underbrace{\text{trace}\,(\mathcal{A}_K^*(x,\xi)\,\mathcal{A}_K(x,\xi)}_{\|\mathcal{A}_K(x,\xi)\|_F^2}\, dx\, d\xi$$

$$= \text{trace}\,(\mathcal{A}^*\mathcal{A})$$

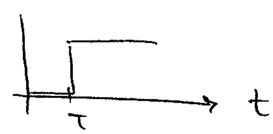
infinite dimensions

Linear System

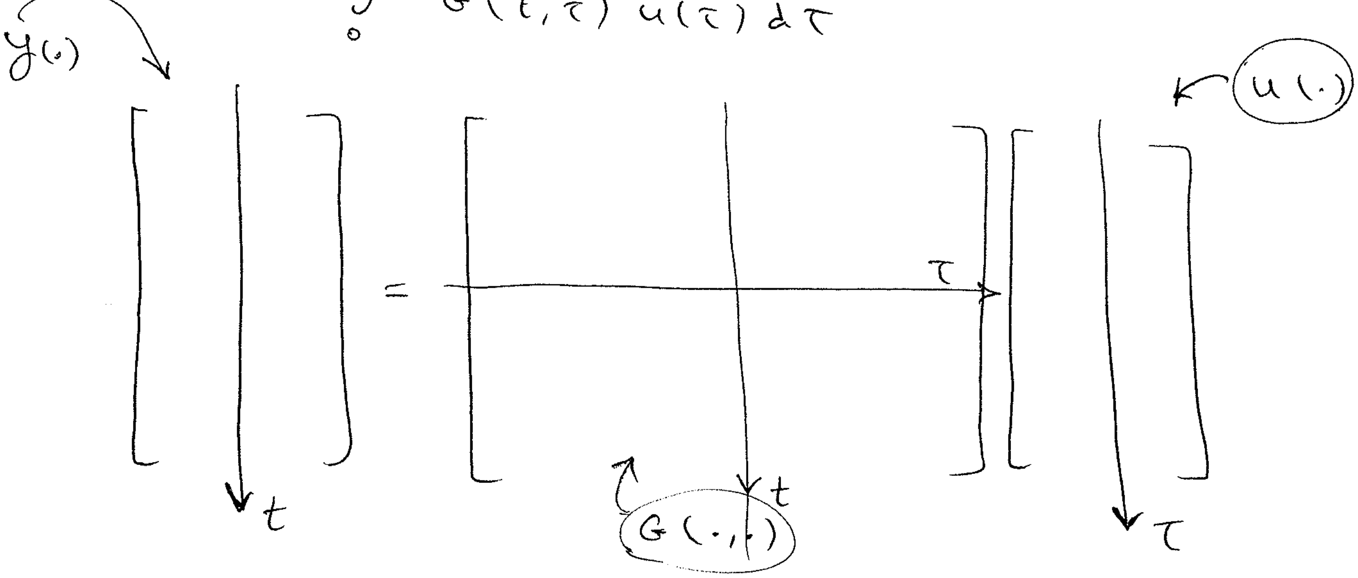
$$\left.\begin{aligned} \dot{x} &= A(t)\,x + B(t)\,u \\ y &= C(t)\,x \end{aligned}\right\} \quad \begin{aligned} & t \in [0,T] \\ & x(0) = 0 \end{aligned}$$

$$y(t) = C(t)\,x(t) = C(t)\int_0^t \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau$$

$$= \int_0^T C(t)\,\Phi(t,\tau)\,B(\tau)\,\mathbb{1}(t-\tau)\,u(\tau)\,d\tau$$



$$= \int_0^T G(t,\tau)\,u(\tau)\,d\tau$$

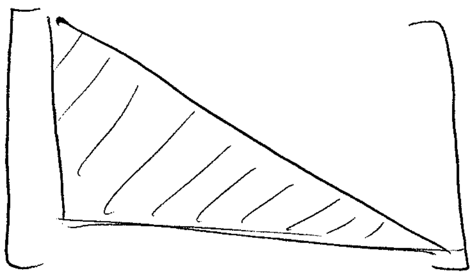


① Causality : G is lower triangular

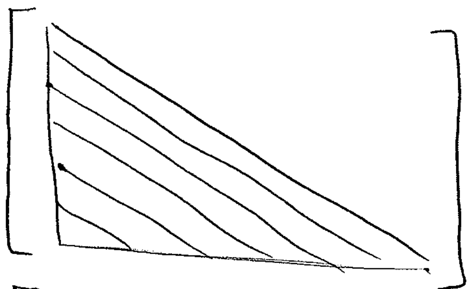
② Time-invariance : G is Toeplitz

③ Time-periodic : $G(t+T, \tau+T) = G(t,\tau)$

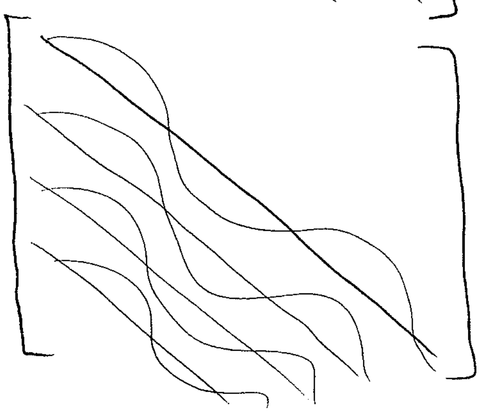
①



②

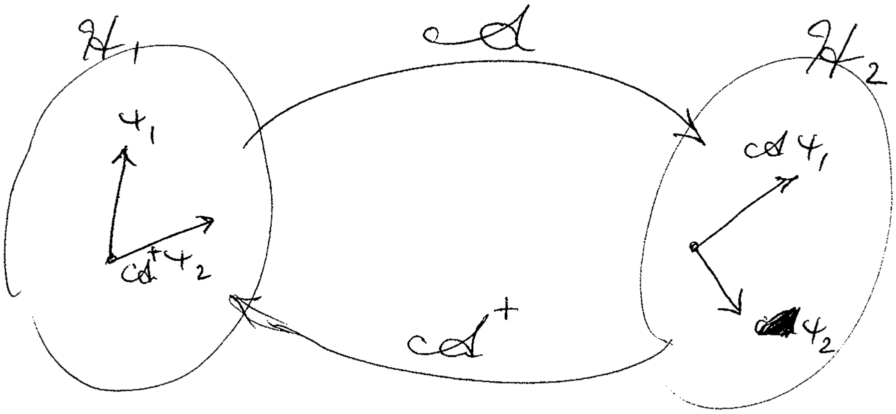


③



The kernel $G(\cdot,\cdot)$ for the linear system can have a distribution part if there is a D term in the system dynamics ; $y = Cx + Du$

$$G(t,\tau) = C(t)\,\Phi(t,\tau)\,B(\tau) + D(\tau)\,\delta(t-\tau)$$



Adjoints

$$\langle \psi_2 , \mathcal{A}\psi_1 \rangle_{\mathcal{H}_2} = \langle \mathcal{A}^+\psi_2 , \psi_1 \rangle_{\mathcal{H}_1}$$

for all $\psi_1 \in \mathcal{H}_1$, $\psi_2 \in \mathcal{H}_2$

$\exists\ \phi_1 \in \mathcal{H}_1$

$$\langle \psi_2 , \mathcal{A}\psi_1 \rangle_{\mathcal{H}_2} = \langle \phi_1 , \psi_1 \rangle_{\mathcal{H}_1}$$ for all $\psi_1 \in \mathcal{H}_1$, $\psi_2 \in \mathcal{H}_2$

$\mathcal{A}^+\psi_2 = \phi_1$

$\boxed{\text{Adjoint is unique if inner product is fixed.}}$

<u>Example</u> $\quad A : \mathbb{C}^n \longrightarrow \mathbb{C}^m \qquad \langle f, g \rangle = f^* \cdot g$

$$\langle \psi_2 , A\psi_1 \rangle_{\mathbb{C}^m} = \langle A^+\psi_2 , \psi_1 \rangle_{\mathbb{C}^n}$$

$$\psi_2^* A \psi_1 = (A^+\psi_2)^* \psi_1 = \langle A^+\psi_2 , \psi_1 \rangle_{\mathbb{C}^n}$$

$$\Rightarrow \quad \boxed{A^+ = A^*}$$

in finite dimensions,

adjoint is equal to complex conjugate transpose.

$$\boxed{\mathcal{L}(Q) = \int_0^\infty e^{At} B Q B^* e^{A^* t}\, dt}$$

$$AP + PA^* = -BQB^*$$

$$\langle R, Q \rangle = \text{trace}\,(R^* Q)$$

$$\langle R, \mathcal{L}(Q) \rangle = \langle \mathcal{L}^+(R), Q \rangle$$

Let $B = I$,

$$\langle R, \int_0^\infty e^{At} Q e^{A^* t}\, dt \rangle = \text{trace}\left(R^* \int_0^\infty e^{At} Q e^{A^* t}\, dt\right)$$

$$= \text{trace}\left(\int_0^\infty e^{A^* t} R^* e^{At}\, dt . Q \right)$$

$$= \langle \int_0^\infty e^{A^* t} R e^{At}\, dt, Q$$

$\downarrow$

R is ~~H~~ Hermitian

$$\boxed{\mathcal{L}^+(R) = \int_0^\infty e^{A^* t} R e^{At}\, dt}$$

09-27-11

$$\mathcal{L} : \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}$$

$$\boxed{\begin{aligned} \mathcal{L}(Q) &= \int_0^\infty e^{At} B Q B^* e^{A^*t}\, dt = P \\ AP + PA^* &= -BQB^* \end{aligned}}$$

$$\begin{tikzcd} \mathbb{C}^{n\times n} \arrow[r, bend left, "\mathcal{L}"] & \mathbb{C}^{n\times n} \arrow[l, bend left, "\mathcal{L}^\dagger"] \end{tikzcd}$$

Appropriate inner product on the space of symetric matrices:

$$\langle R, Q \rangle = \text{trace}(R^* Q)$$

This inner product induces Ferobinuis norm.

$$\langle R, \mathcal{L} Q \rangle = \langle \mathcal{L}^\dagger R, Q \rangle$$

$$\langle R, \mathcal{L} Q \rangle = \text{trace}\left(R^* \int_0^\infty e^{At} B Q B^* e^{A^*t}\, dt\right) =$$

[use Linearity of integral and trace operators]

$$= \int_0^\infty \text{trace}(R^* e^{At} B Q B^* e^{A^*t})\, dt =$$

[use trace(MN) = trace(NM)]

$$= \int_0^\infty \text{trace}(B^* e^{A^*t} R^* e^{At} B Q)\, dt =$$

$$= \text{trace}\left(B^* \int_0^\infty e^{A^*t} R^* e^{At}\, dt\, B\, Q\right)$$

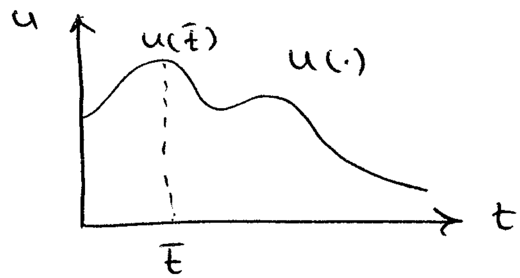
$$= \left\langle B^* \int_0^\infty e^{A^*t} R e^{At}\, dt\, B, Q \right\rangle$$

Thus, $\boxed{\begin{aligned} \mathcal{L}^\dagger R &= B^* V(R) B \\ A^* V + VA &= -R \end{aligned}}$

Ex. $\dot{x} = Ax + Bu \quad , \quad x(0) = 0 \quad , \quad x(t) \in \mathbb{R}^n$

$$x(T) = \int_0^T e^{A(T-\tau)} B \cdot u(\tau)\, d\tau$$

$$= [\mathcal{L} u](T)$$



$\mathcal{L} : L_2[0,T] \longrightarrow \mathbb{R}^n$

Controllability given a state, can we find input to bring the state to given state at a given time.

$x(T) \in \mathbb{R}^n \quad ; \quad u \in L_2[0,T]$

$$\langle x, \mathcal{L} u \rangle_{\mathbb{R}^n} = \langle \mathcal{L}^T x, u \rangle_{L^2[0,T]}$$

$$\langle x, \mathcal{L} u \rangle_{\mathbb{R}^n} = x^* \int_0^T e^{A(T-\tau)} B u(\tau)\, d\tau =$$

$$= \int_0^T x^* e^{A(T-\tau)} B\, u(\tau)\, d\tau$$

$$= \int_0^T \left(B^* e^{A^*(T-\tau)} x \right)^* u(\tau)\, d\tau$$

$$= \langle \underbrace{B^* e^{A^*(T-\tau)}}_{\mathcal{L}^\dagger} x , u \rangle_{L^2[0,T]}$$

$$\boxed{\mathcal{L}^\dagger = B^* e^{A^*(T-t)}}$$

$$\begin{tikzcd} L_2[0,T] \arrow[r, bend left, "\mathcal{L}"] & \mathbb{R}^n \arrow[l, bend left, "\mathcal{L}^\dagger"] \end{tikzcd}$$

$$\boxed{\mathcal{A}\mathcal{A}^{\dagger}} = \int_0^T e^{A(T-\tau)} BB^* e^{A^*(T-\tau)} \, d\tau$$

$$= \boxed{\int_0^T e^{At} BB^* e^{A^* t} \, dt}$$

Constant Matrix

Controllability Gramian

$$\begin{tikzcd} \mathbb{R}^n & L_2[0,T] \arrow[l, "\mathcal{A}"'] & \mathbb{R}^n \arrow[l, "\mathcal{A}^{\dagger}"'] \arrow[ll, bend left, "\mathcal{A}\mathcal{A}^{\dagger}"] \end{tikzcd}$$

<u>Ex.</u> $\mathcal{A} : L_2[a,b] \longrightarrow L_2[a,b]$

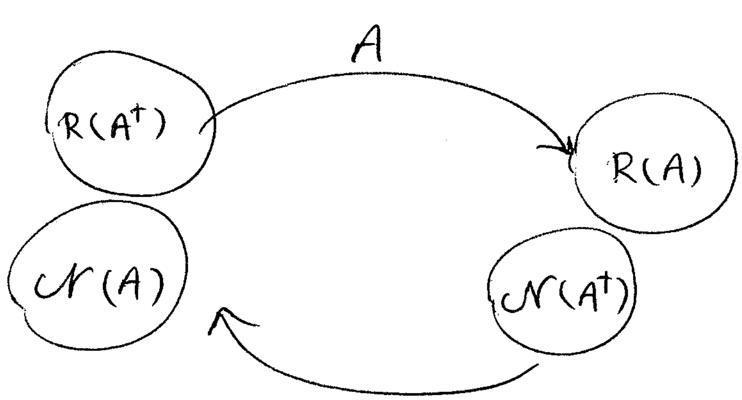
↳ bounded operator with kernel representation

$$[\mathcal{A}f](x) = \int_a^b A_k(x,\xi) f(\xi) \, d\xi$$

$$[\mathcal{A}^{\dagger}g](x) = \int_a^b A_k^*(\xi,x) f(\xi) \, d\xi$$

<u>Finite Dimensions</u>

$A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$



$\mathcal{N}(A^{\dagger}) = [R(A)]^{\perp}$

$$x(T) \leftarrow \boxed{\mathcal{A}} \leftarrow u$$

$$x(T) = [\mathcal{A}u](T) \iff [\ ]_{n\times 1} = \int_0^T \left[\xrightarrow{\quad A_k \quad} \right] \left[\ \Big\downarrow_{0}^{T}\ \right]$$

System Controllable

$$\Updownarrow$$

$$R(\mathcal{A}) = \mathbb{R}^n$$

$$R(\mathcal{A}) = R(\mathcal{A}\mathcal{A}^+) \longleftarrow \text{use}$$

if Controllable, then interested in finding u with Smallest energy.

$$\min \|u\|^2_{L_2[0,T]}$$

u is given by psuedo-inverse of $\mathcal{A}$

$$\boxed{u = \mathcal{A}^+(\mathcal{A}\mathcal{A}^+)^{-1} x(T)}$$

Adjoint of unbounded operators:

Ex $\quad \mathcal{A} : L_2[-1,1] \supset \mathcal{D}(\mathcal{A}) \longrightarrow L_2[-1,1]$

$$\mathcal{A} = \frac{d}{dx} \ ; \quad \mathcal{D}(\mathcal{A}) = \{ f \in L_2 \ ; \ f' \in L_2 ,\ f(-1) = 0 \}$$

$$\langle g, \mathcal{A} f \rangle_{L_2} = \langle \mathcal{A}^+ g, f \rangle_{L^2}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$f \in \mathcal{D}(\mathcal{A}) \qquad g \in \mathcal{D}(\mathcal{A}^+)$$

$$\langle g, cd\,f\rangle_{L_2} = \langle g, \frac{df}{dx}\rangle_{L_2} = \int_{-1}^{1} g^* \frac{df}{dx}(x)\,dx$$

[use integration by parts]

$$= g(x)f(x)\Big]_{-1}^{1} - \int_{-1}^{1} \frac{dg^*}{dx}(x)\, f(x)\,dx$$

$$= g(1)f(1) - g(-1)\underbrace{\cancel{f(-1)}}_{0} + \int_{-1}^{1} \left(-\frac{dg^*}{dx}(x)\right) f(x)\,dx$$

$f \in \mathcal{D}(cd)$

$$= \underbrace{g(1)f(1) + \langle -\frac{d}{dx} g, f\rangle_{L_2[-1,1]}}_{\langle cd^{\dagger} g, f\rangle_{L_2[-1,1]}}$$

? want

Candidate for adjoint :

$$\boxed{\begin{aligned} &cd^{\dagger} = -\frac{d}{dx} \quad \text{with} \\ &\mathcal{D}(cd^{\dagger}) = \left\{ g \in L_2[-1,1];\ \frac{dg}{dx} \in L_2[-1,1],\ g(1) = 0 \right\} \end{aligned}}$$

Note : $\frac{d}{dx}$ is not invertable unless we specify that

e.g. $\mathcal{D}(\frac{d}{dx})$ is functions with b.c. $f(-1) = 0$.

This is to restrict the Null space of $\frac{d}{dx}$ to $f = 0$, (instead of $f = \text{Const.}$ without b.c.) so that $\frac{d}{dx}$ is invertible.

Eigen-values of self-adjoint operators are real.

$$\lambda \|\psi\|^2 = \langle \psi, \lambda\psi \rangle = \langle \psi, \mathcal{A}\psi \rangle = \langle \mathcal{A}\psi, \psi \rangle$$

$$= \bar{\lambda} \|\psi\|^2$$

$$(\lambda - \bar{\lambda}) \|\psi\|^2 = 0 \implies \lambda = \bar{\lambda}.$$

Self-adjoint operator $\mathcal{L}$

09-29-11

$$\langle f, \mathcal{L} g \rangle = \langle \mathcal{L} f, g \rangle$$

for all $f, g \in \mathcal{D}(\mathcal{L})$; $\mathcal{D}(\mathcal{L}^+) = \mathcal{D}(\mathcal{L})$

(Boundary Conditions matter)

Ex.

$$\begin{cases} \mathcal{L} = \dfrac{d}{dx} & \text{with } g(-1) = 0 \\ \mathcal{L}^+ = -\dfrac{d}{dx} & \text{with } f(1) = 0 \end{cases}$$

Not self-adjoint

Ex.

$$\boxed{\mathcal{L} = j\frac{d}{dx} \quad \text{with } g(-1) = 0}$$

; $j = \sqrt{-1}$

$$\left\langle f, j\frac{d}{dx} g \right\rangle = j f(x) g(x) \Big|_{-1}^{1} + \left\langle j \frac{d}{dx} f, g \right\rangle$$

$$= j f(1) g(1) - j f(-1) \underbrace{\cancel{g(-1)}}_{0} + \left\langle j \frac{d}{dx} f, g \right\rangle$$

$\downarrow$

Let $f(1) = 0$

$$\boxed{\mathcal{L}^+ = j\frac{d}{dx} \quad \text{with } f(1) = 0}$$

! $\mathcal{L}$ and $\mathcal{L}^+$ have the same symbol $j\frac{d}{dx}$.
But their domains are different.
So $j\frac{d}{dx}$ is not self-adjoint with the above domains.

We showed that eigenvalues of a self-adjoint operator are <u>real</u>.

Now, we show that eigen-vectors corresponding to two different eigenvalues are orthogonal.

$\mathcal{L}$ ... self-adjoint

$\mathcal{L} v = \lambda v$

① $\lambda \in \mathbb{R}$

③ $\lambda_n, \lambda_m, \quad \lambda_n \neq \lambda_m \Rightarrow \langle v_n, v_m \rangle = 0$

$$\lambda_m \langle v_n, v_m \rangle = \langle v_n, \lambda_m v_m \rangle = \underline{\langle v_n, \mathcal{L} v_m \rangle = \langle \mathcal{L} v_n, v_m \rangle} = \langle \lambda_n v_n, v_m \rangle$$

$$= \lambda_n \langle v_n, v_m \rangle$$

$$\Rightarrow \left.\begin{array}{l} \langle v_n, v_m \rangle (\lambda_m - \lambda_n) = 0 \\ \lambda_m \neq \lambda_n \end{array}\right\} \Rightarrow \boxed{\langle v_n, v_m \rangle = 0}$$

<u>Ex</u> $\boxed{\mathcal{L} = \frac{d^2}{dx^2} \oplus g(\pm 1) = 0}$

$$\langle f, \mathcal{L} g \rangle = \langle \mathcal{L}^+ f, g \rangle$$

$$\langle f, g'' \rangle = f(x) g'(x) \Big|_{-1}^{1} - \langle f', g' \rangle$$

$$= f(x) g'(x) \Big|_{-1}^{1} - f'(x) g(x) \Big|_{-1}^{1} + \langle f'', g \rangle$$

$$= f(1) g'(1) - f(-1) g'(-1) - f'(1) \underbrace{\cancel{g(1)}}_{0} + f'(-1) \underbrace{\cancel{g(-1)}}_{0} + \langle f'', g \rangle$$

Since $g'(\pm 1)$ is arbitrary, we need $\underline{f(\pm 1) = 0}$.

$\boxed{\mathcal{L}^+ = \frac{d^2}{dx^2} \oplus f(\pm 1) = 0}$ $\quad \mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L}^+)$

So, $\mathcal{L}$ is self-adjoint.

Ex. $\left.\begin{array}{l} f'(x) = g(x) \\ f(-1) = 0 \end{array}\right\} \Rightarrow f(x) - \underbrace{\cancel{f(-1)}}_{0} = \int_{-1}^{x} g(\xi)\, d\xi$

$[\mathcal{d} f](x) = g(x) \Rightarrow f(x) = [\mathcal{d}^{-1} g](x) =$

$\int_{-1}^{x} g(\xi)\, d\xi =$

$\int_{-1}^{1} \mathbb{1}(x-\xi)\, g(\xi)\, d\xi$

$f(x) = \int_{-1}^{x} \underbrace{K(x,\xi)}_{1}\, g(\xi)\, d\xi$

$\mathcal{d}^{-1}$ bounded operator (because its kernel is bounded)

$(\mathcal{d}^{-1})^{+}$ .... $h(x) = \int_{x}^{1} q(\xi)\, d(\xi)$

$\mathcal{d}^{-1} : L_2[-1,1] \longrightarrow L_2[-1,1]$

$$\begin{tikzcd} L_2 \arrow[r, bend left, "\mathcal{d}^{-1}"] & L_2 \arrow[l, bend left, "(\mathcal{d}^{-1})^{+}"] \end{tikzcd}$$

$\mathcal{d}^{-1} : g \longrightarrow f$

$(\mathcal{d}^{-1})^{+} : q \longrightarrow h$

$h(x) = \int_{x}^{1} q(\xi)\, d\xi = [(\mathcal{d}^{-1})^{+} q](x)$

$= [\mathcal{B} q](x)$

$h'(x) = -q(x)$

b.c. $h(1) = 0 = \int_{1}^{1} q(\xi)\, d\xi$

Eigenvalue decomposition of $\left.\frac{d^2}{dx^2}\right|_{v(\pm1)=0}$

$$\underbrace{\frac{d^2 v}{dx^2}}_{\mathcal{L} v} = \lambda v \quad ; \quad (v(\pm 1) = 0)$$

$v'' - \lambda v = 0$

$s^2 - \lambda = 0 \Rightarrow s = \pm\sqrt{\lambda}$

$\lambda \in \mathbb{R}$ because $\mathcal{L}$ self-adjoint

$v(x) = a e^{\sqrt{\lambda} x} + b e^{-\sqrt{\lambda} x}$

Constants to be determined such that $v(\pm 1) = 0$.

λ : real $\begin{cases} \lambda > 0 \\ \lambda < 0 \end{cases}$

$\boxed{\text{if } \lambda > 0}$

B.C. $\begin{cases} a e^{\sqrt{\lambda}} + b e^{-\sqrt{\lambda}} = 0 & (x = 1) \\ a e^{-\sqrt{\lambda}} + b e^{\sqrt{\lambda}} = 0 & (x = -1) \end{cases}$

$$\underbrace{\begin{bmatrix} e^{\sqrt{\lambda}} & e^{-\sqrt{\lambda}} \\ e^{-\sqrt{\lambda}} & e^{\sqrt{\lambda}} \end{bmatrix}}_{M} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

For non-trivial solution, we need

$$\det(M) = 0 = e^{2\sqrt{\lambda}} - e^{-2\sqrt{\lambda}} = 0$$

$\Rightarrow e^{4\sqrt{\lambda}} = 1$; only option $\lambda = 0$

$\Rightarrow v(x) = a + bx$

but, cannot satisfy b.c.

So $\lambda \geq 0$ cannot be an eigenvalue of $\left.\mathcal{L}\right|_{v(\pm1)=0}$.

Therefore, $\lambda < 0$.

$\lambda < 0 \Rightarrow s^2 = \lambda = -|\lambda|$

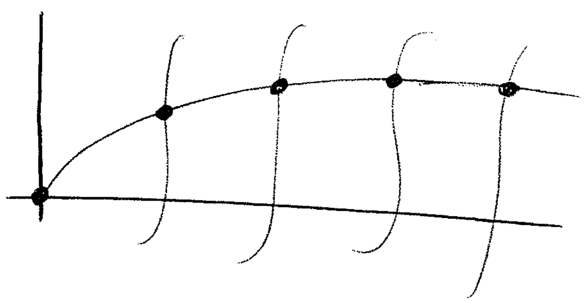
$\Rightarrow s = \pm j\sqrt{|\lambda|}$

$v(x) = a \sin(\sqrt{\lambda}\, x) + b \cos(\sqrt{\lambda}\, x)$

$\lambda = -\left(\frac{+n\pi}{2}\right)^2 \; ; \; n \in \{1, 2, \dots\}$

$v_n(x) = \sin\left(\frac{n\pi}{2}(x+1)\right)$

<u>HW.</u> $\mathcal{A} = \frac{d^2}{dx^2}$ with $\begin{cases} v(-1) = 0 \\ v(1) = v'(1) \end{cases}$



<u>Alternative.</u>

Bring $\frac{d^2 v}{dx^2} - \lambda v = 0$ to state-space form.

$v'' - \lambda v = 0$

$\left.\begin{array}{l} \psi_1 = v \\ \psi_2 = v' \end{array}\right\}$ states $\quad \begin{array}{l} \psi_1' = v' = \psi_2 \\ \psi_2' = v'' = \lambda v = \lambda \psi_1 \end{array}$

$$\begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix}}_{A} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \longrightarrow \begin{array}{l} \psi' = A\psi \\ v = C\psi \; ; \; C = [1 \;\; 0] \end{array}$$

$$\left.\begin{array}{l} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{N_1} \begin{bmatrix} \psi_1(-1) \\ \psi_2(-1) \end{bmatrix} + \\ \qquad \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{N_2} \begin{bmatrix} \psi_1(1) \\ \psi_2(1) \end{bmatrix} \end{array}\right\} \rightarrow 0 = N_1 \psi(-1) + N_2 \psi(1)$$

$$\begin{cases} \psi' = A\psi \\ 0 = N_1 \psi(-1) + N_2 \psi(1) \end{cases} \quad ; \quad v = C\psi$$

$$\psi(x) = e^{A(x-(-1))}\,\psi(-1) = e^{A(x+1)}\,\psi(-1)$$

Problem : donot know $\psi(-1) = \begin{bmatrix} \psi_1(-1) \\ \psi_2(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ ? \end{bmatrix}$

use BCs :

$$N_1 \psi(-1) + N_2 \psi(1) = N_1 \psi(-1) + N_2 e^{2A} \psi(-1)$$

$$= \underbrace{(N_1 + N_2 e^{2A})}\,(\psi(-1)) = 0$$

$$\det(N_1 + N_2 e^{2A}) = 0$$

$$\downarrow$$

gives λ .

Resolvent

10 - 04 - 11

$$\frac{\partial \Psi}{\partial t} = \mathcal{A}\Psi + u$$

Apply Lapplas transform : $s\Psi(s) - \Psi(0) = \mathcal{A}\Psi + u$

$$\Rightarrow \quad \Psi(s) = (sI - \mathcal{A})^{-1}(\Psi(0) + u)$$

In finite dimensions : $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

$$y(s) = \underbrace{\left[C(sI - A)^{-1}B\right]}_{\text{Transfer function}} u(s) + C(sI - A)^{-1}x(0)$$

$(sI - A)^{-1} = R_s(A)$ ---- resolvent of A

Ex $\quad A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

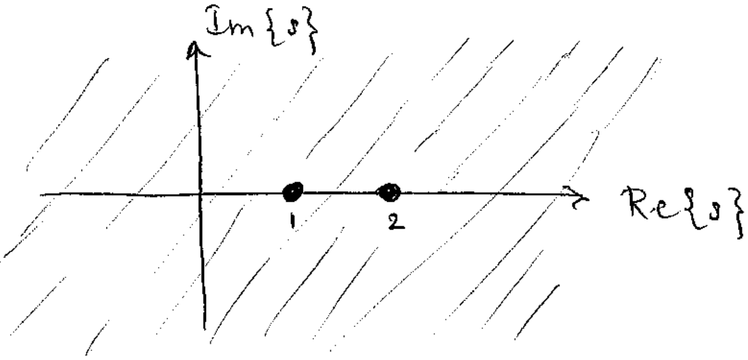
$$sI - A = \begin{bmatrix} s-1 & 0 \\ -3 & s-2 \end{bmatrix}$$

$$\det(sI - A) = (s-1)(s-2)$$

$sI - A$ invertible $\iff$ $s \neq 1$, $s \neq 2$

resolvent set of A : $\rho(A) = \{ s \in \mathbb{C} ;\ s \neq 1,\ s \neq 2 \}$

Spectrum of A : $\sigma(A) = \mathbb{C} \setminus \rho(A) = \{ s = 1, s = 2 \}$



Shaded region denotes $\rho(A)$.

$\sigma(A)$ is only the two points $s = 1, s = 2$.

So, in finite dimensions, spectrum of A is only a point spectrum:

$$\sigma(A) = \sigma_p(A)$$

In infinite dimensions:

$$\sigma(A) = \underbrace{\sigma_p(A)}_{\text{Point spectrum}} \cup \underbrace{\sigma_c(A)}_{\text{Continuous spectrum}} \cup \underbrace{\sigma_r(A)}_{\text{residual spectrum}}$$

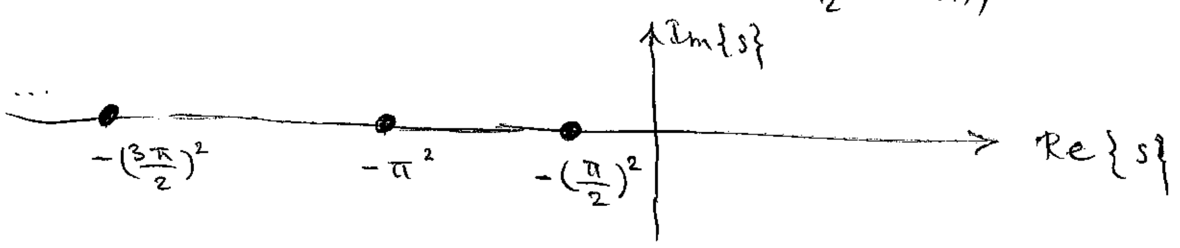
$$\sigma_p(A) = \{ s \in \mathbb{C},\ \text{s.t. } (sI - \mathcal{A}) \text{ is not } \underbrace{1 \text{ to } 1}_{\text{(invertable)}} \}$$

These points are called eigenvalues with eigen-vectors $v \in \mathcal{N}(sI - \mathcal{A})$

Example: $\mathcal{A} = \frac{d^2}{dx^2}$; $v(\pm 1) = 0$

$$\sigma_p(\mathcal{A}) = \{ -(\tfrac{n\pi}{2})^2 ;\ n = 1, 2, \ldots \}$$

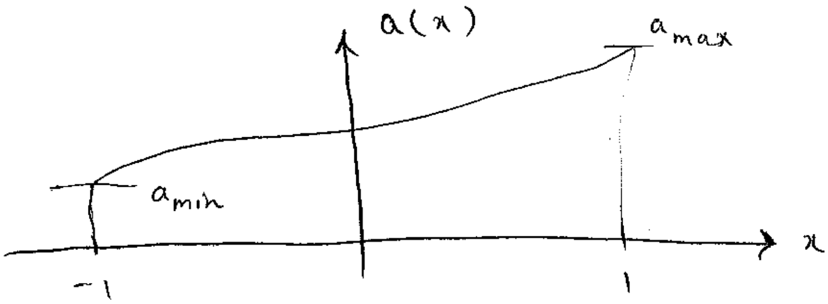
$$v_n(x) = \sin(\tfrac{n\pi}{2}(x+1))$$

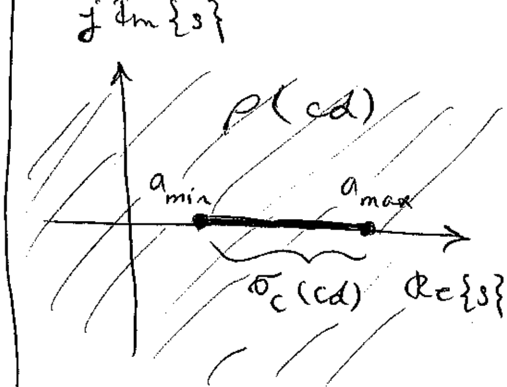


Example $\boxed{\text{Continuous spectrum}}$

$$[\mathcal{A} f](x) = a(x) f(x)$$

Multiplication operator on $L_2[-1,1]$





$$g(x) = a(x)\cdot f(x)$$

Want to study: $(\lambda I - \mathcal{A}) f = g$

$$(\lambda - a(x)) f(x) = g(x) \longrightarrow f(x) = \frac{1}{\lambda - a(x)} g(x)$$

$$= [(\lambda I - \mathcal{A})^{-1} g](x)$$

$$\mathcal{D}(R_\lambda(\mathcal{A})) = \left\{ g,\ \text{s.t. } (\lambda I - \mathcal{A})^{-1} g \in L_2 \right\}$$

Easy to see that $\mathcal{D}(\mathcal{A})$ is dense in L_2

Q: under what conditions $R_\lambda(\mathcal{A})$ is bounded?

$$\|f\|_{L_2}^2 = \int_{-1}^{1} \frac{1}{(\lambda - a(x))^2} g^2(x)\,dx \le \sup_x \frac{1}{(\lambda - a(x))^2} \|g\|_{L_2}^2$$

Note: $\left(\sup_x \frac{1}{(\lambda - a(x))^2} < \infty \right) \Longleftrightarrow \left(\lambda \notin (a_{min}, a_{max}) \right)$

Then
$$\boxed{\begin{aligned} \rho(\mathcal{A}) &= \{ \lambda \in \mathbb{C},\ \text{s.t. } \lambda \notin (a_{min}, a_{max}) \} \\ \sigma(\mathcal{A}) &= \sigma_c(\mathcal{A}) = \{ \lambda \in \mathbb{C},\ \lambda \in (a_{min}, a_{max}) \} \end{aligned}}$$

$\underline{\underline{\text{Ex}}}$ $\boxed{\text{Residual spectrum}}$

$$\mathcal{A} : \ell_2 \longrightarrow \ell_2$$

$$g = [\mathcal{A} f] = [S_r f] = \{ f_{n-1} \}$$

Right shift

$$\begin{bmatrix} g_1 \\ g_2 \\ \vdots \end{bmatrix} = S_r \begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}$$

$$(\lambda I - \mathcal{A}) f = (\lambda I - S_r) f = \lambda \begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix} - \begin{bmatrix} 0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \lambda f_1 \\ \lambda f_2 - f_1 \\ \lambda f_3 - f_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

if $\lambda \neq 0 \Rightarrow f = 0$ $(\lambda I - \mathcal{A})$ invertable.

if $\lambda = 0 \Rightarrow -S_r f = g = \begin{bmatrix} 0 \\ -f_1 \\ -f_2 \\ \vdots \end{bmatrix}$

$$\Rightarrow f = (\lambda I - \mathcal{A})^{-1} g \Big|_{\lambda = 0}$$

$$= -\lambda_\ell \, g$$

So inverse of $(\lambda I - S_r)\Big|_{\lambda = 0}$ exists, it is $(-S_\ell)$,

But its domain in not dense in ℓ_2.

$$R_0(\mathcal{A}) = (\lambda I - \mathcal{A})^{-1}\Big|_{\lambda = 0} = -S_\ell$$

$$\mathcal{D}(R_0(\mathcal{A})) = \left\{ g \in \ell_2 ;\ g = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}^{\perp} \right\} \text{ or } g = \begin{bmatrix} 0 \\ g_1 \\ g_2 \\ \vdots \end{bmatrix}$$

$$g = \begin{bmatrix} 0 \\ g_1 \\ g_2 \\ \vdots \end{bmatrix}$$

Consider sequences of the form $\begin{bmatrix} 0 \\ g_{1n} \\ g_{2n} \\ \vdots \end{bmatrix}$

So, this can never recover an element of ℓ_2 in the direction of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$.

__Thus__ $\mathscr{D}(R_0(\mathcal{L}))$ is not dense in ℓ_2.

$$\boxed{\sigma_r(\mathcal{L}) = \{\lambda = 0\}}$$

$\mathcal{L} : H \supset \mathscr{D}(\mathcal{L}) \longrightarrow H$

$\mathcal{L}^{-1}$: Compact, normal

$\downarrow$

finite HS norm :

$$\int_a^b \int_a^b \operatorname{trace}\left(A_K(x,\xi) A_K^*(x,\xi) \right) dx\, d\xi < +\infty$$

$$\begin{cases} \mathcal{L}^{-1} v_n = \dfrac{1}{\lambda_n} v_n \\ \mathcal{L} v_n = \lambda_n v_n \end{cases}$$

$\boxed{\text{Matrix } M:}$

$$M = U \Sigma V^* = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

$$U U^* = U^* U = I$$

$$V V^* = V^* V = I$$

$$M M^* = U \Sigma \underbrace{V^* V}_{I} \Sigma^* U^* = U \Sigma \Sigma^* U^*$$

$$M M^* u_i = \sigma_i^2 u_i \quad ; \quad U = [u_1, \ldots, u_m]$$

$$M^* M v_i = \sigma_i^2 v_i$$

$$M f = \sum_{i=1}^{r} \sigma_i u_i \langle v_i, f \rangle$$

$$\|M\|_{2i} = \underbrace{\sigma_1}_{\text{maximum singular value of } M} \quad \ldots \quad \text{induced 2-norm}$$

$$\|M\|_F^2 = \sum_{i=1}^{r} \sigma_i^2$$

↑ overview of SVD in finite dimensions.

Eigenvalue - Decomposition

Ex $v'' = av, \quad v(\pm 1) = 0$

$$\begin{cases} \psi_1 = v \\ \psi_2 = v' \end{cases} \qquad \begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}}^{A} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{N_1} \begin{bmatrix} \psi_1(-1) \\ \psi_2(-1) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{N_2} \begin{bmatrix} \psi_1(1) \\ \psi_2(1) \end{bmatrix}$$

$$\boxed{\begin{aligned} \psi(x) &= e^{A(x+1)} \psi(-1) \\ 0 &= N_1 \psi(-1) + N_2 \psi(1) \\ &= (N_1 + N_2 e^{2A}) \psi(-1) \end{aligned}}$$

$$\boxed{\text{Non-trivial solutions: } \det(N_1 + N_2 e^{2A}) = 0}$$

---

① Continuity with respect to initial condition:

i.c. $f, g \in \mathbb{H}$

$\| f - g \|$ small $\Longrightarrow \| T(t)(f-g) \|$ small

(small changes in initial condition, don't change response by much.)

---

③ Strong Continuity:

For fixed initial condition, small change in time shouldn't change the responce by much.

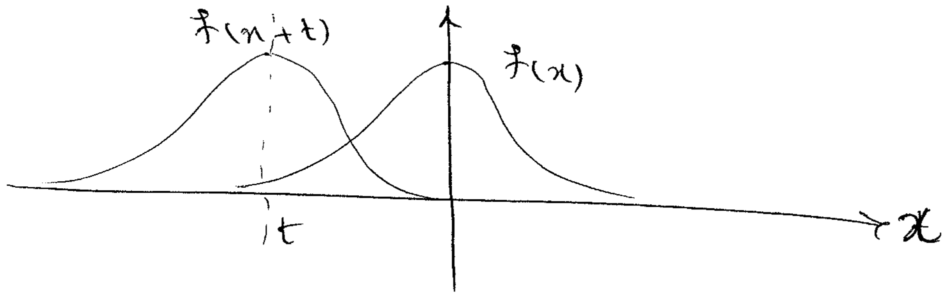
$$\lim_{\Delta t \to 0} \| T(t+\Delta t)\psi(0) - \psi(t) \| = \quad \text{(by semi-group property)}$$

$$= \lim_{\Delta t \to 0} \| T(t) (T(\Delta t) - I) \psi(0) \| \leq \quad \text{(by boundedness of } T(t) \text{ that follows from ①)}$$

$$\leq M \underbrace{\lim_{\Delta t \to 0} \| T(\Delta t)\psi(0) - \psi(0) \|}_{=0 \quad \text{strong Continuity}} = 0$$

Ex Transport operator

$$\phi(x,t) = [T(t)f](x) = f(x+t)$$



Properties of $T(t)$:

① $T(0) \Rightarrow \phi(x,0) = f(x) = [T(0)f](x)$

$\Rightarrow T(0) = I$ ✓

② $T(t_1+t_2) = T(t_2)T(t_1)$

③ Is $T(t)$ bounded?

if $f \in L_2 \Rightarrow \phi(\cdot,t) \in L_2$

Yes! why? $\int_{-\infty}^{\infty} (f(x+t))^2 dx = \int_{-\infty}^{\infty} (f(x))^2 dx$

$T(t)$ bounded with norm:

$$\|T(t)\| = \sup_{\substack{f \in L_2 \\ f \neq 0}} \frac{\|T(t)f\|}{\|f\|} = 1 < \infty$$

④ Strong Continuity:

$$\lim_{t \to 0^+} \|T(t)f - f\|^2 = 0 \qquad \forall f \in L_2(-\infty,\infty)$$

$$\int_{-\infty}^{\infty} (f(x+t) - f(x))^2 dx$$

Fact any function $f \in L_2$ can be approximated (in L_2 sense) by a continuous function h with compact support.

$$\int_{-\infty}^{\infty} (f(x+t) - f(x))^2 \, dx$$

$$= \int_{-\infty}^{\infty} (T(t) f(x) - f(x))^2 \, dx$$

$$= \| T(t) f - f \|^2 = \| T(t)(f - h + h) - (f - h + h) \|^2$$

$$= \| T(t)(f-h) + (T(t)h - h) - (f-h) \|^2 \leq$$

$$\underbrace{\| T(t)(f-h) \|^2}_{< \varepsilon/3} + \underbrace{\| T(t)h - h \|^2}_{< \varepsilon/3} + \underbrace{\| f - h \|^2}_{< \varepsilon/3}$$

As $t \to 0$ Can be made smaller than ε.

boundedness of $T(t)$

the fact above

$$\lim_{t \to 0} \int_{-\infty}^{\infty} (h(x+t) - h(x))^2 \, dx = 0$$

Ex Infinite number of decoupled scalar states

$\mathcal{A} : \ell_2 \longrightarrow \ell_2$

$$\frac{d}{dt} \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \end{bmatrix}$$

① if $\mathcal{A}$ is bounded ?

$g = \mathcal{A} f \Rightarrow g_n = a_n f_n$

$\{f_n\} \in \ell_2 \overset{?}{\Longrightarrow} \{g_n\} \in \ell_2$

$$\sum_{n=1}^{\infty} |g_n|^2 = \sum_{n=1}^{\infty} |a_n f_n|^2 = \sum_{n=1}^{\infty} |a_n|^2 |f_n|^2 \leq \sup_n |a_n|^2 \sum_{n=1}^{\infty} |f_n|^2$$

$$= \sup_n |a_n|^2 \, \|f\|^2_{\ell_2}$$

<u>Last time</u>

$$\dot{\Psi}(t) = \mathcal{A}\Psi(t) \; ; \quad \mathbb{H} = \ell_2(\mathbb{N})$$

$$\mathcal{A} = \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{bmatrix} = \text{diag}\{a_n\}_{n\in\mathbb{N}}$$

We've shown that

if $\sup_n |a_n|^2 < +\infty \Rightarrow \mathcal{A}$: bounded from ℓ_2 to ℓ_2

(i.e. $g = \mathcal{A}f$, $f \in \ell_2 \Rightarrow g \in \ell_2$)

<u>Ex</u> Second derivative $\quad a_n = -\left(\frac{n\pi}{2}\right)^2$

$$\sup_n |a_n| = \sup_n \left|\left(\frac{n\pi}{2}\right)^2\right| \not< +\infty$$

unbounded operator.

Let $\mathcal{D}(\mathcal{A}) = \left\{ f \in \ell_2 \text{ s.t. } \sum_{n=1}^{\infty} |a_n f_n|^2 < +\infty \right\}$

<u>Important property to check</u>:

Boundedness of $T(t)$ ... State-transition operator (C_0_semigroup)

$$T(t) = \begin{bmatrix} e^{a_1 t} & & \\ & e^{a_2 t} & \\ & & \ddots \end{bmatrix} ; \quad g = T(t) f = \left\{ e^{a_n t} f_n \right\}_{n\in\mathbb{N}}$$

<u>Q:</u> determine conditions on $\{a_n\}_{n\in\mathbb{N}}$ s.t.

$$f \in \ell_2 \Rightarrow g \in \ell_2$$

Answer: $$\|g\|_{\ell_2}^2 = \sum_{n=1}^{\infty} |e^{a_n t} \cdot f_n|^2 = \sum_{n=1}^{\infty} |e^{a_n t}|^2 \cdot |f_n|^2$$

$$\leq \underbrace{\sup_n |e^{a_n t}|^2}_{\downarrow} \cdot \underbrace{\sum_{n=1}^{\infty} |f_n|^2}_{\|f\|_{\ell_2}^2}$$

What is important for bounding this term is for fixed 't'. What happens as $t \to \infty$ is a question of stability.

Note: **Aside**

$$a_n = \mathrm{Re}(a_n) + j\,\mathrm{Im}(a_n)$$

$$|e^{a_n t}|^2 = |e^{\mathrm{Re}(a_n)t} e^{j\,\mathrm{Im}(a_n)t}|^2$$

$$= e^{2\mathrm{Re}(a_n)t} \underbrace{|e^{j\,\mathrm{Im}(a_n)t}|^2}_{1}$$

$$\|g\|_{\ell_2}^2 \leq e^{2\mathrm{Re}(a_n)t} \|f\|_{\ell_2}^2$$

if $\sup_n \mathrm{Re}(a_n) < M < +\infty$, then

So-called Half-plane Condition

$$\|g\|_{\ell_2}^2 \leq M \cdot \|f\|_{\ell_2}^2 \Rightarrow g \in \ell_2$$

So for fixed t,

$$T(t) : \ell_2 \longrightarrow \ell_2 \quad \text{is bounded.}$$

Example $\quad a_n = -\left(\frac{n\pi}{2}\right)^2$

$\mathcal{A}$ is unbounded. but,

$$\sup_n \operatorname{Re}(a_n) = \sup_n -\left(\frac{n\pi}{2}\right)^2 < \underbrace{M}_{\text{Some positive number}} < +\infty$$

So, $T(t)$ is bounded.

Example $\quad$ backward-in-time heat equation

$$\psi_t = -\psi_{xx} \implies a_n = \left(\frac{n\pi}{2}\right)^2$$

$$\sup_n \operatorname{Re}(a_n) = +\infty \qquad \text{(unbounded)}$$

Finally Determine conditions on $\{a_n\}_{n\in\mathbb{N}}$ s.t.

$$\lim_{t\to 0^+} \|T(t)\underbrace{\psi(\cdot)}_{f} - \underbrace{\psi(\cdot)}_{f}\| = 0 \qquad \forall\, \psi(\cdot) \in \ell_2$$

$$\|(T(t)-I)f\|^2_{\ell_2} = \sum_{n=1}^{\infty} |(e^{a_n t}-1)f_n|^2$$

$$= \sum_{n=1}^{N} |(e^{a_n t}-1)f_n|^2 + \sum_{n=N+1}^{\infty} |(e^{a_n t}-1)f_n|^2$$

$$\leq \sup_{1\leq n\leq N} |e^{a_n t}-1|^2 \underbrace{\sum_{n=1}^{N} |f_n|^2}_{\leq \|f\|^2_{\ell_2}} + \sup_{n>N} |e^{a_n t}-1|^2 \sum_{n=N+1}^{\infty} |f_n|^2$$

<u>Summary</u> if half-plane condition is satisfied:

$$\sup_n \operatorname{Re}(a_n) < M < +\infty$$

then

$$\lim_{t\to 0^+} \| T(t)\psi(0) - \psi(0)\| = 0$$

Thus, $T(t)$ generates a C_0 - semigroup (read: equivalent of Matrix exponential on ℓ_2)

In the proof of Hille-Yosida Theorem:

$$\left[\begin{array}{l} \text{Implicit Euler: } \dfrac{d\psi}{dt} = \mathcal{A}\psi \\ \Rightarrow \dfrac{\psi(t+\Delta t) - \psi(t)}{\Delta t} = \mathcal{A}\psi(t+\Delta t) \\ \psi(t+\Delta t) = (I - \underbrace{\Delta t}_{\frac{t}{N}}\mathcal{A})^{-1}\psi(t) \end{array}\right]$$

A method for computing $T(t) = \lim_{N\to\infty} (I - \frac{t}{N}\mathcal{A})^{-N}$.

Hille-Yosida ... Necessary & Sufficient Conditions (difficult to use)

Lumer-Phillips ... Sufficient Conditions (easy to use)

Example   Lumer-Phillips

$$\mathcal{A} = \frac{d}{dx} \quad ; \quad L_2[-1,1] \quad ; \quad f(1) = 0$$

$$\langle \psi, \mathcal{A}\psi \rangle = \langle \psi, \psi' \rangle = \psi(x)\psi(x)\Big|_{-1}^{1} - \langle \psi', \psi \rangle$$

$$= \psi(1)\cancelto{0}{\psi(1)} - \psi(-1)\psi(-1) - \langle \psi', \psi \rangle$$

$$2\,\mathrm{Re}\{\langle \psi, \mathcal{A}\psi \rangle\} = -\psi^2(-1) \leqslant 0$$

$$\left.\begin{array}{l} \mathrm{Re}\{\langle \psi, \mathcal{A}\psi \rangle\} \leqslant e^{0t} \\ \text{similarly:} \\ \mathrm{Re}\{\langle \psi, \mathcal{A}^{+}\psi \rangle\} \leqslant e^{0t} \end{array}\right\} \Rightarrow T(t) \text{ generates } C_0\text{-semigroup.}$$

Last time

Examples of C_0-semigroups

Hille-Yosida and Lumer-Phillips Theorems

Compare implicit Euler with explicit Euler

$$\frac{\partial \psi}{\partial t} = c\mathcal{A}\psi \xrightarrow[\text{Euler}]{\text{Implicit}} \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t} = c\mathcal{A}\psi(t+\Delta t)$$

Evaluate right-hand-side one step ahead

$$\Downarrow$$

$$\boxed{\psi(t+\Delta t) = (I - \Delta t\, c\mathcal{A})^{-1}\psi(t)}$$

$$\frac{\partial \psi}{\partial t} = c\mathcal{A}\psi \xrightarrow[\text{Euler}]{\text{Explicit}} \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t} = c\mathcal{A}\psi(t)$$

Evaluate right-hand-side at current time

$$\Downarrow$$

$$\boxed{\psi(t+\Delta t) = (I + \Delta t\, c\mathcal{A})\psi(t)}$$

Note

$(I - \Delta t\, c\mathcal{A})$ unbounded .... differential operators

$(I - \Delta t\, c\mathcal{A})^{-1}$ bounded .... inverse of differential operators

Implicit Euler involves composition with bounded operators for propagating the state ψ forward in time.

Euler - Bernoulli beam

$$\phi_{tt}(x,t) = -\phi_{xxxx}(x,t)$$

$$\phi(x,0) = f(x) \; ; \quad \phi_t(x,0) = g(x)$$

$$\phi(\pm 1, t) = 0$$

$$\phi_{xx}(\pm 1, t) = 0$$

Abstract evolution model :

$$\begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix} = \begin{bmatrix} \phi(\cdot, t) \\ \phi_t(\cdot, t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \dot{\psi}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\frac{d^4}{dx^4} & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}$$

Dynamical generator :

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & 0 \end{bmatrix} \; ; \quad \mathcal{A}_0 = \frac{d^4}{dx^4}$$

$$\mathcal{D}(\mathcal{A}_0) = \left\{ f \in L_2[-1,1], \; \frac{d^4 f}{dx^4} \in L_2[-1,1], \; f(\pm 1) = f''(\pm 1) = 0 \right\}$$

Positive operator:

self-adjoint operator $\mathcal{A} : \mathcal{H} \supset \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}$ is **Positive**

$$\langle \psi, \mathcal{A}\psi \rangle > 0 \quad \text{for all non-zero } \psi \in \mathcal{D}(\mathcal{A})$$

matrices: $P = P^*$ is positive if

$x^* P x > 0, \quad \forall x \neq 0$ --- positive definite

$x^* P x \geq 0, \quad \forall x$ --- positive semi-definite

$P = P^{1/2} P^{1/2}$

$P^{1/2} = (P^{1/2})^* > 0$

operator $\mathcal{A}$ is **coercive** if

$\exists\, \varepsilon > 0$

$$\langle \psi, \mathcal{A}\psi \rangle \geq \varepsilon \|\psi\|^2 \quad \forall\, \psi \in \mathcal{D}(\mathcal{A})$$

In matrices, coercivity is always satisfied

$P = P^*$

$x^* P x \geq \underbrace{\lambda_{min}}_{\text{minimum eigenvalue of } P} \|x\|^2$

Square-root $\mathcal{A}^{1/2}$ of self-adjoint $\mathcal{A}$

$$\begin{cases} \mathcal{D}(\mathcal{A}^{1/2}) \supset \mathcal{D}(\mathcal{A}) \\ \mathcal{A}^{1/2}\psi \in \mathcal{D}(\mathcal{A}^{1/2}) \\ \mathcal{A}^{1/2}\mathcal{A}^{1/2}\psi = \mathcal{A}\psi \end{cases}$$

reference [Kato]

Examples of positive, self-adjoint operators

$$\mathcal{A}_0 = \frac{-d^2}{dx^2}\ ; \quad \mathcal{D}(\mathcal{A}_0) = \left\{ f \in L_2[-1,1];\ \frac{d^2 f}{dx^2} \in L_2[-1,1],\ f(\pm 1) = 0 \right\}$$

$$\mathcal{A}_0 = -\frac{d^4}{dx^4}\ ; \quad \mathcal{D}(\mathcal{A}_0) = \left\{ f \in L_2[-1,1];\ \frac{d^4 f}{dx^4} \in L_2[-1,1],\ f(\pm 1) = f''(\pm 1) = 0 \right\}$$

$\mathcal{D}(\mathcal{A}_0^{1/2})$: determined from the following requirement

$$\langle \mathcal{A}_0^{1/2} f, \mathcal{A}_0^{1/2} g \rangle = \langle f, \mathcal{A}_0 g \rangle, \quad \forall g \in \mathcal{D}(\mathcal{A}_0)$$

<u>Example</u> $\mathcal{A}_0 = \frac{d^4}{dx^4}\ ; \quad g(\pm 1) = g''(\pm 1) = 0$

$$\langle f, \mathcal{A}_0 g \rangle = \langle \mathcal{A}_0^{1/2} f, \mathcal{A}_0^{1/2} g \rangle \quad \text{for all } g \in \mathcal{D}(\mathcal{A}_0)$$

$$\langle f, \frac{d^4}{dx^4} g \rangle = \langle f, g^{(4)} \rangle = f(x) g^{(3)}(x)\Big|_{-1}^{1} - \langle f', g^{(3)} \rangle =$$

$$= \underbrace{\underbrace{f(x) g^{(3)}(x)}_{\text{arbitrary}}\Big|_{-1}^{1}}_{\text{Need } f(\pm 1) = 0} - \underbrace{f'(x) \cancel{g''(x)}^{\,0}\Big|_{-1}^{1}}_{g''(\pm 1) = 0} + \langle f'', g'' \rangle$$

$$= \langle f'', g'' \rangle \quad \text{if} \quad f(\pm 1) = 0$$

Thus,

$$\mathcal{A}_0^{1/2} = -\frac{d^2}{dx^2}\ ; \quad \mathcal{D}(\mathcal{A}_0^{1/2}) = \left\{ f \in L_2[-1,1],\ f'' \in L_2[-1,1],\ f(\pm 1) = 0 \right\}$$

Want $\mathcal{A}_0^{1/2}$ to be positive operator.

$\boxed{\text{E-values of } \frac{d^2}{dx^2}\Big|_{f(\pm 1)=0} \text{ are } -\left(\frac{n\pi}{2}\right)^2}$

Adjoint of $\mathcal{A}$ with respect to the energy inner product

$\langle \cdot , \cdot \rangle_e$

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & -a_1 I \end{bmatrix}$$

$$\langle \phi_1 , \phi_2 \rangle_e = \left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} , \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_e =$$

$$= \langle \mathcal{A}_0^{1/2} f_1 , \mathcal{A}_0^{1/2} f_2 \rangle_2 + \langle g_1 , g_2 \rangle_2$$

Definition: $\langle \phi_1 , \mathcal{A} \phi_2 \rangle_e = \langle \mathcal{A}^\dagger \phi_1 , \phi_2 \rangle_e$

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} , \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & -a_1 I \end{bmatrix} \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_e = \left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} , \begin{bmatrix} g_2 \\ -\mathcal{A}_0 f_2 - a_1 g_2 \end{bmatrix} \right\rangle_e$$

$\mathcal{A}_0^{1/2}$

$$= \langle \mathcal{A}_0^{1/2} f_1 , \mathcal{A}_0^{1/2} g_2 \rangle_2 + \langle g_1 , -\mathcal{A}_0 f_2 - a_1 g_2 \rangle_2$$

[using the slides: guess for $\mathcal{A}^\dagger$]

$$= \langle \mathcal{A}_0 f_1 , g_2 \rangle_2 + \langle -\mathcal{A}_0^{1/2} g_1 , \mathcal{A}_0^{1/2} f_2 \rangle_2$$

$$- \langle a_1 g_1 , g_2 \rangle_2$$

$$\Rightarrow \mathcal{A}^\dagger = \begin{bmatrix} 0 & -I \\ \mathcal{A}_0 & -a_1 I \end{bmatrix}$$

Spectral decomposition for wave equation

$$\phi_{tt} = \phi_{xx} \quad w/ \quad \phi(\pm 1) = 0$$

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{bmatrix}$$

$$\mathcal{A} v = \lambda v$$

$$\left\{\begin{array}{l} v_2 = \lambda v_1 \\ v_1'' = \lambda v_2 \\ v_1(\pm 1) = 0 \end{array}\right. \Rightarrow \left.\begin{array}{l} v_1'' = \lambda^2 v_1 \\ v_1(\pm 1) = 0 \end{array}\right\}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Compare with

$$\left\{\begin{array}{l} v'' = s v \\ v(\pm 1) = 0 \end{array}\right.$$

Heat equation

Know

$$s_n = -\left(\frac{n\pi}{2}\right)^2 \; ; \quad n = 1, 2, \ldots$$

$$v_n = \sin\left(\frac{n\pi}{2}(x+1)\right)$$

So, $\lambda_n^2 = -\left(\frac{n\pi}{2}\right)^2$

$$\Rightarrow \boxed{\lambda_n = \pm j \left(\frac{n\pi}{2}\right) \; ; \quad n = 1, 2, \ldots}$$

- There are two sets of eigen-vectors.

<u>Summary</u>

$$\lambda_n = j\left(\frac{n\pi}{2}\right) \; ; \quad n = \pm 1, \pm 2, \ldots$$

$$\lambda_{n} = -\lambda_n \quad , \quad \text{use} \quad \sin(-x) = -\sin(x)$$

$$v_n(x) = \begin{bmatrix} \frac{1}{\lambda_n} \sin\left(\frac{n\pi}{2}(x+1)\right) \\ \sin\left(\frac{n\pi}{2}(x+1)\right) \end{bmatrix} \begin{array}{l} \longrightarrow \text{Same for } \pm n \\ \longrightarrow \text{Changes sign when } n \to -n \end{array}$$

Normalization is done such that $\langle v_n, v_m \rangle_e = \delta_{n,m}$

Undamped Wave Equation $(a_1 = 0)$ $\boxed{10-18-11}$

$$\phi_{tt} = \phi_{xx} \quad , \quad \phi(\pm 1) = 0$$

$$\begin{bmatrix} \psi_{1t} \\ \psi_{2t} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & -a_1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

Abstractly:

$$\begin{bmatrix} \psi_{1t} \\ \psi_{2t} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_0 & -a_1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

Lumer–Phillips: sufficient conditions for well-posedness

$$\left\{ \begin{array}{ll} \mathrm{Re}\{\langle \psi, \mathcal{A}\psi \rangle_e\} \leq w \|\psi\|_e^2 & , \forall \psi \in \mathcal{D}(\mathcal{A}) \\ \mathrm{Re}\{\langle \psi, \mathcal{A}^+\psi \rangle_e\} \leq w \|\psi\|_e^2 & , \forall \psi \in \mathcal{D}(\mathcal{A}^+) \end{array} \right.$$

$$\langle \psi, \mathcal{A}\psi \rangle_e = \left\langle \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} \psi_2 \\ -\mathcal{A}_0\psi_1 - a_1\psi_2 \end{bmatrix} \right\rangle_e$$

$$= \langle \mathcal{A}_0^{1/2}\psi_1, \mathcal{A}_0^{1/2}\psi_2 \rangle_2 + \langle \psi_2, -\mathcal{A}_0\psi_1 \rangle_2 + \langle \psi_2, -a_1\psi_2 \rangle_2$$

$$= \langle \psi_1, \mathcal{A}_0\psi_2 \rangle_2 - \langle \psi_2, \mathcal{A}_0\psi_1 \rangle_2 - a_1 \langle \psi_2, \psi_2 \rangle_2$$

$$\mathrm{Re}\{\langle \psi, \mathcal{A}\psi \rangle_e\} = -a_1 \langle \psi_2, \psi_2 \rangle_2 \leq -a_1 \|\psi\|_e^2$$

$$\leq 0 \|\psi\|_e^2$$

same holds for $\mathrm{Re}\{\langle \psi, \mathcal{A}^+\psi \rangle_e\}$

Note: Worst case happens for undamped equation $(a_1 = 0)$

Thus:

well-posed for $\psi \in \mathcal{H} = \begin{bmatrix} \mathcal{D}(\mathcal{A}_0^{1/2}) \\ L_2[-1,1] \end{bmatrix}$

Check orthonormality of the eigen-functions of the undamped wave equation:

$$v_n = \begin{bmatrix} v_{1n} \\ v_{2n} \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_n}\phi_n(x) \\ \phi_n(x) \end{bmatrix}$$

$$\phi_n(x) = \sin\left(\frac{n\pi}{2}(x+1)\right), \quad \lambda_n = j\frac{n\pi}{2}, \quad n \in \mathbb{N}$$

$$\langle v_m, v_n \rangle_e = \langle v_{1m}, \underbrace{cd_0}_{-\frac{d^2}{dx^2}} v_{1n} \rangle_2 + \langle v_{2m}, v_{2n} \rangle_2$$

$$v_{1n}'' = \frac{1}{\lambda_n}\phi_n'' = \frac{1}{\lambda_n}\underbrace{\lambda_n^2}_{\lambda_n\bar{\lambda}_n}\phi_n = \bar{\lambda}_n \phi_n(x)$$

So,

$$\langle v_m, v_n \rangle_e = \frac{\bar{\lambda}_n}{\bar{\lambda}_m}\langle \phi_m, \phi_n \rangle_2 + \underbrace{\langle \phi_m, \phi_n \rangle_2}_{\delta_{m,n}}$$

$$= \left(\frac{\bar{\lambda}_n}{\bar{\lambda}_m} + 1\right)\delta_{m,n}$$

$$= \begin{cases} 2 & ; \quad m = n \\ 0 & ; \quad m \neq n \end{cases}$$

$\frac{1}{2}\langle v_m, v_n \rangle_e$ gives you energy.

Check Completeness Need to show that the orthogonal complement of span $\{v_n\}_{n \in \mathbb{Z}\setminus\{0\}}$ is zero.

i.e. $$\boxed{\langle \theta, v_n \rangle_e = 0 \Rightarrow \theta = 0}$$

Note Since $\{v_n\}$ is orthonormal, it generates a Reisz basis. (on $\begin{bmatrix} L_2[-1,1] \\ L_2[-1,1] \end{bmatrix}$, it does not !)

Navier - Stokes Equations (NSE)

(1) $$\underline{v}_t + (\underline{v}\cdot\nabla)\underline{v} = -\nabla p + \frac{1}{Re}\Delta\underline{v}$$

(2) $$\underline{v} = \underline{V} + \underline{\tilde{v}}$$

$\underline{\tilde{v}}$ → fluctuations

$\underline{V}$ → mean velocity (equilibrium point)

$\underline{v}$ → total velocity

Linearized NSE $\left\{ \begin{array}{l} \text{Put (2) into (1)} \\ \text{neglect } (\underline{\tilde{v}}\cdot\nabla)\underline{\tilde{v}} \end{array} \right.$

(3) $$\underline{\tilde{v}}_t + (\underline{\tilde{v}}\cdot\nabla)\underline{V} + (\underline{V}\cdot\nabla)\underline{\tilde{v}} = -\nabla p + \frac{1}{Re}\Delta\underline{\tilde{v}}$$

Continuity : $\nabla\cdot\underline{\tilde{v}} = 0$

$$\left\{ \begin{array}{lll} \text{Pressure driven flow:} & \underline{V} = \begin{bmatrix} U(y) = 1-y^2 & 0 & 0 \end{bmatrix}^T \\ \text{Shear driven flow:} & \underline{V} = \begin{bmatrix} U(y) = y & 0 & 0 \end{bmatrix}^T \end{array} \right.$$

Linearized NSE $\quad \underline{\tilde{v}} = [u \;\; v \;\; w]^T$

$$\left\{ \begin{array}{lll} u_t + U(y)\,u_x + U'(y)\,v & = & -p_x + \frac{1}{Re}\Delta u \\ v_t + U(y)\,v_x & = & -p_y + \frac{1}{Re}\Delta v \\ w_t + U(y)\,w_x & = & -p_z + \frac{1}{Re}\Delta w \\ u_x + v_y + w_z = 0 & & \text{(Constraint)} \end{array} \right.$$

Note :

$$\begin{bmatrix} u_t \\ v_t \\ w_t \\ 0\,p_t \end{bmatrix} = \bar{\mathcal{A}} \begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix}$$

! Not in standard evolution form.

set $\partial_x = 0$.

$$\begin{cases} u_t + U'(y) v = \frac{1}{Re} \Delta u \\ v_t = -p_y + \frac{1}{Re} \Delta v \\ w_t = -p_z + \frac{1}{Re} \Delta w \\ v_y + w_z = 0 \end{cases}$$

<u>Let</u> $v = -\psi_z$, $w = \psi_y$;

then $v_y + w_z = 0$ Automatically !

<u>obtain</u>

$$\begin{cases} \Delta \psi_t = \frac{1}{Re} \Delta^2 \psi \\ u_t = \frac{1}{Re} \Delta u - U' \psi_z \\ \Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{cases}$$

$$\begin{bmatrix} \psi_t \\ u_t \end{bmatrix} = \begin{bmatrix} \frac{1}{Re} \mathcal{L} & 0 \\ C_p & \frac{1}{Re} S \end{bmatrix} \begin{bmatrix} \psi \\ u \end{bmatrix}$$

$$\left. \begin{array}{lll} \text{Orr–Sommerfeld:} & \mathcal{L} = & \Delta^{-1} \Delta^2 \\ \text{Squire :} & S = & \Delta \\ \text{Coupling :} & C_p = & -U'(y) \partial_z \end{array} \right\}$$

- Can we cancel $\Delta^{-1} \Delta^2$ to obtain Δ ?!

Q : what is $\Delta^{-1} \Delta^2$?

A : an operator that maps $\Delta^{-1} \Delta^2 : f \longrightarrow g$

$$\Delta^{-1} \Delta^2 f = g \Rightarrow \Delta^2 f = \Delta g$$

F.T in z $\Rightarrow$ $\begin{cases} \Delta = \frac{\partial^2}{\partial y^2} - k_z^2 I \\ \Delta^2 = \frac{\partial^4}{\partial y^4} - 2 k_z^2 \frac{\partial^2}{\partial y^2} + k_z^4 I \end{cases}$

So $\Delta^{-1}\Delta^2$ is an integro-differential operator.

F.T in z:

$$\begin{cases} \Delta = \partial_{yy} - k_z^2 I \\ \Delta^2 = \partial_{yyyy} - 2k_z^2 \partial_{yy} + k_z^4 I \\ C_p = -jk_z U' \end{cases}$$

B.C.s Δ : Dirichlet

Δ^2 : Dirichlet + Neumann

$$\text{Energy} = \frac{1}{2}\left(\langle u,u\rangle + \langle v,v\rangle + \langle w,w\rangle \right)$$

where: $\langle u,u\rangle = \int_{-1}^{1} u^{*}(y,k_z,t)\, u(y,k_z,t)\, dy$

$\downarrow$

$E_u(k_z,t)$ ... energy density at k_z

$$\langle v,v\rangle + \langle w,w\rangle = \langle \psi_z, \psi_z\rangle + \langle -\psi_y, -\psi_y\rangle$$

$$= \langle jk_z\psi, jk_z\psi\rangle + \langle -\psi_y, -\psi_y\rangle$$

$$= \langle \psi, k_z^2\psi\rangle + \langle \psi, -\psi_{yy}\rangle$$

$$= \langle \psi, -\Delta\psi\rangle$$

$$\phi = \begin{bmatrix} \psi \\ u \end{bmatrix}$$

$$\langle u,u\rangle + \langle v,v\rangle + \langle w,w\rangle = \langle u,u\rangle + \langle \psi, -\Delta\psi\rangle$$

$$= \langle \phi, \phi\rangle_e$$

$$= \left\langle \begin{bmatrix} \psi \\ u \end{bmatrix}, \begin{bmatrix} -\Delta & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \psi \\ u \end{bmatrix} \right\rangle$$

Simple finite-dimensional example

$$\begin{bmatrix} \dot\phi_1 \\ \dot\phi_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}}_{A} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}$$

$k \neq 0 \Rightarrow$ A is not normal .... $AA^T \neq A^TA$

If A has a full set of linearly independent eigenvectors, we can still bring A to diagonal form, But the eigenvectors are not going to be orthonormal.

$$\boxed{A v_i = \lambda_i v_i} \longrightarrow A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$AV = V\Lambda \Rightarrow A = V \Lambda V^{-1}$$

where $V^{-1} \neq V^*$

(not unitary)

Let $V^{-1} = W^*$

$$\boxed{A = V \Lambda W^*}$$

Can show $\underbrace{V^{-1}}_{W^*} A = \Lambda W^*$

$$\begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix} \Rightarrow A^* \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \Lambda^*$$

$$\Rightarrow \boxed{A^* w_i = \bar\lambda_i w_i}$$

So, $$Af = \sum_{i=1}^{n} \lambda_i v_i \langle w_i, f \rangle$$

Action of A on f is determined by a linear combination of the right eigenvectors of A (v_i) with coefficients $\underbrace{\lambda_i \langle w_i, f \rangle}_{}$

modal contribution of f ~~in the direction of the i-th left eigenvector of A~~

Important: Eigenvectors are not orthonormal, so even though eigenvalues of A are negative and ~~both~~ all modes decay to zero at large times, there could be large transients.

Systems with inputs

Two input types:

* Additive inputs
* Boundary inputs

Heat equation example for additive inputs

$$\phi_t(x,t) = \phi_{xx}(x,t) + u(x,t)$$
$$\phi(x,0) = \phi_0(x)$$
$$\phi(\pm 1, t) = 0$$

Abstract evolution equation

$$\psi_t(t) = A\psi(t) + u(t)$$

$$A = \frac{d^2}{dx^2}, \quad D(A) = \{ f \in L_2[-1,1],\ f'' \in L_2[-1,1],\ f(\pm 1) = 0 \}$$

Solution $\quad \psi(t) = T(t)\psi(0) + \int_0^t T(t-\tau)u(\tau)\,d\tau$

$$\phi(x,t) = \sum_{n=1}^{+\infty} a_n(t) v_n(x)$$

$$[T(t)\phi(0)](x) = \sum_{n=1}^{+\infty} e^{\lambda_n t} v_n(x) \langle v_n, \phi(0) \rangle$$

$$\dot{a}_n(t) = \lambda_n a_n(t)$$

$$\lambda_n \to -\left(\frac{n\pi}{2}\right)^2$$

$$\int_0^t T(t-\tau)u(\tau)\,d\tau = \sum_{n=1}^{+\infty} \int_0^t e^{\lambda_n(t-\tau)} v_n(x) \langle v_n, u(\tau) \rangle\, d\tau$$

Input-Output maps

$$\psi_t(t) = A\psi(t) + Bu(t)$$
$$\phi(t) = C\psi(t)$$

Input-output mapping

$$\phi(t) = [Hu](t) = \int_0^t C\,T(t-\tau)B\,d\tau$$

* Impulse response

$$\mathcal{H}(t) = (\mathcal{C}\mathcal{T}(t)\mathcal{B})\,\mathbb{1}(t)$$

* Transfer function

$$\mathcal{H}(s) = C(sI - A)^{-1}B$$

* Frequency response

$$\mathcal{H}(j\omega) = C(j\omega I - A)^{-1}B$$

An example

$$\left.\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) + d(t) \\ \phi(\pm 1, t) &= 0 \end{aligned}\right\} \xrightarrow{\text{Laplace transform}} \left\{\begin{aligned} \phi''(x,s) &= s\phi(x,s) - d(x,s) \\ \phi(\pm 1, s) &= 0 \end{aligned}\right.$$

$$\begin{bmatrix} \psi_1'(x,s) \\ \psi_2'(x,s) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} d(x,s)$$

$$\phi(x,s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(x,s) \\ \psi_2(x,s) \end{bmatrix}$$

$$0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(-1,s) \\ \psi_2(-1,s) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(1,s) \\ \psi_2(1,s) \end{bmatrix}$$

Two point boundary value problems

$$\psi'(x) = A(x)\psi(x) + B(x)d(x)$$

$$\varphi(x) = \mathcal{C}(x)\Psi(x)$$

$$0 = N_a\psi(a) + N_b\psi(b)$$

For boundary inputs

$$\phi_t(x,t) = \phi_{xx}(x,t) + d(x,t)$$

$$\phi(-1,t) = u(t)$$

$$\phi(+1,t) = 0$$

Problem : control does not enter additively into the equation

Define :

$$\psi(x,t) = \phi(x,t) - f(x)\,u(t)$$

$$\phi(x=-1,t) = u(t)$$

$$\phi(x=+1,t) = 0$$

determine $f(x)$ s.t.

$$\psi(x=\pm 1,t) = 0$$

$$\psi(-1,t) = \phi(-1,t) - f(-1)\,u(t) = u(t) - f(-1)\,u(t) \overset{\text{want}}{=} 0 \iff f(-1)=1$$

$$\psi(1,t) = \phi(1,t) - f(1)u(t) = 0 - f(1)u(t) \underset{\text{want}}{=} 0 \Leftarrow f(1)=0$$

Many choices for $f(x)$, for example, $\boxed{f(x) = \frac{1-x}{2}}$

In new coordinates:

$$\psi_t(x,t) + f(x)\,\dot{u}(t) = \psi_{xx}(x,t) + f''(x)\,u(t) + d(x,t)$$

$$\psi(\pm 1,t) = 0$$

New input : $v(t) = \dot{u}(t)$

$$\frac{d}{dt}\begin{bmatrix}\psi(t)\\ u(t)\end{bmatrix} = \begin{bmatrix}A_0 & f''\\ 0 & 0\end{bmatrix}\begin{bmatrix}\psi(t)\\ u(t)\end{bmatrix} + \begin{bmatrix}I\\ 0\end{bmatrix}d(t) + \begin{bmatrix}-f\\ I\end{bmatrix}v(t)$$

$$\phi(t) = [\,I \quad f\,]\begin{bmatrix}\psi(t)\\ u(t)\end{bmatrix}$$

Note : Curtain's book (More general)

$$\phi_t = A\phi$$

$\mathcal{E}\phi(t) = u(t)$, point evaluation functional

$$[\quad \mathcal{E} \quad]\begin{bmatrix} \\ \phi(t) \\ \\ \end{bmatrix} = u(t)$$

Right inverse of $\mathcal{E}$

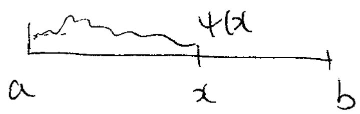
$\mathcal{E}F = I$, in our example, we select $F = f(x)$, $f(-1)=1$, $f(+1)=0$

$$\mathcal{E}F\cdot u(t) = u(t)$$

Two point boundary value problems

$$\psi' = A\psi + Bd \qquad \text{--- (1)}$$

$$r = N_a \psi(a) + N_b \psi(b) \qquad \text{--- (2)}$$



$$\psi(x) = e^{A(x-a)} \underbrace{\psi(a)}_{\uparrow \text{ unknown!}} + \int_a^x e^{A(x-\xi)} Bd(\xi)\, d\xi \qquad \text{--- (3)}$$

evaluation (3) at $x=b$ and plug into (2)

$$\Rightarrow \quad r = \underbrace{\left[N_a + N_b e^{A(b-a)} \right]}_{\text{If invertible}} \psi(a) + N_b \int_a^b e^{A(b-\xi)} Bd(\xi) . \, d\xi$$

If invertible, then $\psi(a) = f(r, d)$

Two-point boundary value problems:

$\boxed{10/27/11}$

$$\Psi'(x) = A(x)\Psi(x) + B(x)u(x),$$
$$\phi(x) = C(x)\Psi(x)$$
$$\nu = N_a \Psi(a) + N_b \Psi(b)$$

Aside:
Curtain and Morris
Automatica 2009
transfer function for infinite dimensional systems.
(spatially distributed)

→ **Solution:**

$$\phi(x) = C(x)\Phi(x,a)\left(N_a + N_b\Phi(b,a)\right)^{-1}\nu +$$
$$+ C(x)\int_a^x \Phi(x,\xi)B(\xi)d(\xi)\,d\xi -$$
$$- C(x)\Phi(x,a)\left(N_a + N_b\Phi(b,a)\right)^{-1} N_b \int_a^b \Phi(b,\xi)B(\xi)\,d(\xi)\,d\xi$$

* formula is useful when we have analytical solutions. That is, it useful for symbolic computation using Mathematica.
* Naive approach: compute $\Phi(x,\xi)$ numerically using marching algorithms.

  This approach may give numerical junk.
* another way: bvp4c and chebfun.

  ↑ bvp4c: two point boundary value problem solver in Matlab

  ↑ chebfun: coming soon: powerful numerical solvers for boundary value problems, and more

Controlability and Observability

ability to steer states

ability to estimate states

important :

- grammians
- operator Lyapunov equations

<u>an example</u> :

$$\begin{aligned} \psi_t(x,t) &= \psi_{xx}(x,t) + b(x)u(t) \\ \phi(t) &= \int_{-1}^{1} c(x)\,\psi(x,t)\,dx \\ \psi(x,0) &= \psi_0(x) \\ \psi(\pm 1, t) &= 0 \end{aligned}$$

diffusion eq on $L_2[-1,1]$ with point actuation and sensing.

$$b(x) = \frac{1}{2\varepsilon}\, \mathbb{1}_{[x_c-\varepsilon,\, x_c+\varepsilon]}(x)$$

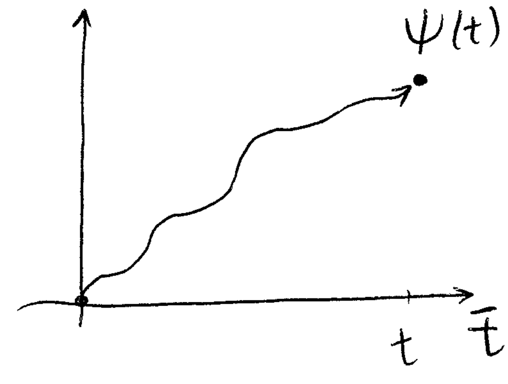
$$c(x) = \frac{1}{2\delta}\, \mathbb{1}_{[x_s-\delta,\, x_s+\delta]}(x)$$

$$\mathbb{1}_{[a,b]}(x) = \begin{cases} 1, & x \in [a,b] \\ 0, & \text{otherwise} \end{cases}$$

Controllability operator and grammian

$$\begin{rcases} \psi_t = A\psi + Bu \\ \psi(0) = 0 \end{rcases}$$ want to study influence of control on $\psi(t)$

$$\psi(t) = \int_0^t T(t-\tau) B u(\tau)\, d\tau$$



Q: Can we choose $u[0,t]$ such that we bring our system from $\psi(0)=0$ to a given $\psi_f = \psi(t)$?

abstractly: $\psi(t) = [R_t u](t) = \int_0^t T(t-\tau) B u(\tau)\, d\tau$

$R_t : L_2([0,t]; \mathcal{U}) \longrightarrow \mathbb{H}$

$\downarrow$

$L_2[0,t] = \{ f ;\ \int_0^t f^*(\tau) f(\tau)\, d\tau < +\infty \}$

$\downarrow$

$L_2([0,t]; \mathbb{C}^n)$

In general:

$$L_2([0,t]; \mathcal{U}) = \{ u ;\ \int_0^t \underbrace{\langle u(\tau), u(\tau) \rangle_{\mathcal{U}}}_{\text{e.g. } \int_{-1}^{1} u^*(x,\tau) u(x,\tau) dx} d\tau < +\infty \}$$

Adjoint: $[R_t^+ \psi](\tau) = B^+ T^+(t-\tau)$, $\tau \in [0,t]$

Controlability grammian:

$$P_t = R_t R_t^+ = \int_0^t T(\tau) B B^+ T^+(\tau)\, d\tau .$$

* exact controlability : $\text{range}(R_t) = \mathbb{H}$
  - rarely-satisfied by infinite dimensional systems
  - never satisfied for systems with finite dimensional $\mathbb{U}$
* approximate controlability : $\overline{\text{range}(R_t)} = \mathbb{H}$

Observability operator and grammian:

$$O_t : \mathbb{H} \longrightarrow L_2([0,t]; \mathbb{Y})$$

$$\phi(t) = [O_t \psi(0)](t) = C T(t) \psi(0)$$

<u>gramian</u>:

$$\mathcal{V}_t = O_t^+ O_t = \int_0^t T^+(\tau) C^+ C T(\tau)\, d\tau .$$ (finite horizon grammian)

* $(A, \cdot, C)$ is approximately obser. on $[0,t]$
$\Longleftrightarrow$ $(A^+, C^+, \cdot)$ is approx. contr. on $[0,t]$.

approx contr on $[0,t]$ $\Longleftrightarrow$

1) $P_t > 0 \Longleftrightarrow \{ \langle \psi, P_t \psi \rangle > 0, \ \forall \psi \neq 0 \in \mathbb{H} \}$

or

2) $\text{null}(R_t^+) = 0 \Longleftrightarrow \{ B^+ T^+(\tau) \psi = 0 \text{ on } [0,t] \Longrightarrow \psi = 0 \}$

Idea: $P_t = R_t R_t^{+}$

$$\langle \psi, P_t \psi \rangle > 0$$

$$\langle \psi, R_t R_t^{+} \psi \rangle = \langle \underbrace{R_t^{+} \psi}_{\phi}, \underbrace{R_t^{+} \psi}_{\phi} \rangle > 0$$

$\rightarrow$ This shows: $\text{null}(R_t^{+}) = \{0\} \iff P_t > 0$

$$\rightarrow (\text{null}(R_t^{+}))^{\perp} = \overline{\text{range}(R_t)} = \mathbb{H}$$

$$R_t : L_2([0,t]; \mathbb{U}) \longrightarrow \mathbb{H}$$
$$R_t^{+} : \mathbb{H} \longrightarrow L_2([0,t]; \mathbb{U})$$

$$\begin{tikzcd} \mathbb{H} & L_2 \arrow[l, "R_t"'] & \mathbb{H} \arrow[l, "R_t^{+}"'] \arrow[ll, bend right, "P_t"'] \end{tikzcd}$$

Infinite horizon grammians:

$$P = R_\infty R_\infty^{+} = \int_0^\infty T(\tau) B B^{+} T^{+}(\tau)\, d\tau$$

$$V = O_\infty^{+} O_\infty = \int_0^\infty T^{+}(\tau) C^{+} C\, T(\tau)\, d\tau.$$

Lyapunov equations:

$$\langle A^{+}\psi_1, P\psi_2 \rangle + \langle P\psi_1, A^{+}\psi_2 \rangle = -\langle B^{+}\psi_1, B^{+}\psi_2 \rangle$$

for all $\psi_1, \psi_2 \in D(A^{+})$

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \qquad \begin{aligned} AP + PA^* &= -BB^* \quad \ldots (1) \\ A^*V + VA &= -C^*C \quad \ldots (2) \end{aligned}$$

Operator versions of (1) and (2) given by

$$AP + PA^\dagger = -BB^\dagger \quad \ldots (3)$$

$$A^\dagger V + VA = -C^\dagger C \quad \ldots (4)$$

where $A^\dagger$, $B^\dagger$, $C^\dagger$ are determined using proper inner products.

Discretization of (3) and (4) does not necessarily give you problem of the form:

$$A_d P_d + P_d A_d^* = -B_d B_d^*.$$

Motivating example : $\mathcal{L} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}$ ⊕ $f(\pm 1) = 0$

Q: Kernel representation of $\mathcal{L}^{-1}$?

$$\begin{cases} Af = g \\ f(\pm 1) = 0 \end{cases} \Rightarrow \begin{cases} f = \mathcal{L}^{-1} g \\ f(\pm 1) = 0 \end{cases}$$

①
$$f_1 = f, \quad f_2 = f' \Rightarrow \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_1(-1) \\ f_2(-1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_1(1) \\ f_2(1) \end{bmatrix}$$

$$f(x) = \int_{-1}^{1} K(x, \xi) g(\xi)\, d\xi = [A^{-1} g](x)$$

② Do eigenvalue decomposition of $\mathcal{L}$ and use the fact that $\mathcal{L}$ is self-adjoint w.r.t.

$$\langle f, g \rangle_w = \int_{-1}^{1} f(x)\, g(x)\, e^{-x}\, dx$$

$$\mathcal{L} v_n = \lambda_n v_n$$

$$f(x) = [\mathcal{L}^{-1} g](x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} v_n(x) \langle v_n, g \rangle_w = \int_{-1}^{1} \underbrace{\sum_{n=1}^{\infty} v_n(x)\, v_n(\xi)\, e^{-\xi}}_{K(x,\xi)} g(\xi)\, d\xi$$

What should we do for problems that are not as simple, meaning that the operator $\mathcal{A}$ is such that eigenvalue decomposition of $\mathcal{A}$ is difficult.

Here, we use tools for numerically solving these problems.

$\boxed{\text{Spectral methods}}$ use global information to approximate derivative operators. We end up having full matrices. Error decays exponentially with the number of discritization points, $\mathcal{O}(c^{-N})$

vs.

$\boxed{\text{Finite-difference methods}}$ use local information. The underlying matrices are sparse. Error decays as $\mathcal{O}(N^{-p})$ where N is the number of discritization points, and p is an integer $p > 0$.

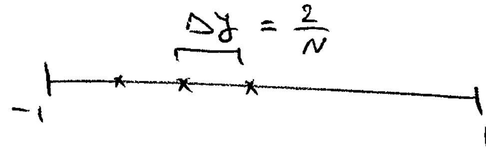
vs.

$\boxed{\text{Psuedo-spectral methods}}$ Accuracy similar to spectral methods. Computationally easier than spectral methods (close to finite-differences).

Example

Operator $\quad U(y)\,\dfrac{d^2}{dy^2}\,\psi$

- Finite-differences @ $\bar{y}$



$$U(\bar{y})\,\frac{\psi(\bar{y}+\Delta y) - 2\,\psi(\bar{y}) + \psi(\bar{y}-\Delta y)}{(\Delta y)^2}$$

$\mathcal{O}((\Delta y)^2) = \mathcal{O}\left(\frac{1}{N^2}\right)$

In matrix form:

$$\mathrm{diag}(U) \cdot T \cdot \underline{\psi}$$

$\downarrow$

Toeplitz $\quad \dfrac{1}{(\Delta y)^2}\begin{bmatrix} \ddots & & & & \\ \ddots & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$

- In Psuedo-spectral:

$$\mathrm{diag}(U) \cdot D_2 \cdot \underline{\psi}$$

$$\underline{\psi} = \begin{bmatrix} \vdots \\ \psi(\bar{y}-\Delta y) \\ \psi(\bar{y}) \\ \psi(\bar{y}+\Delta y) \\ \vdots \end{bmatrix}$$

$$\begin{cases} f(x) \approx P_N(x) \\ f(x_i) = P_N(x_i) \quad ; \quad i = \{0, \dots, N\} \\ x_i \quad \dots \quad \text{interpolation points} \end{cases}$$

$$f(x) = P_N(x) + \overbrace{R_N(x)}^{\text{Remainder}}$$

$$f(x_i) = P_N(x_i) + \underbrace{R_N(x_i)}_{0}$$

So

$$R_N(x_i) = 0$$

Why is this important?

$$f(x) = \sum_{n=0}^{N} a_n \phi_n(x)$$

$$P_N(x) = \sum_{n=0}^{N} b_n \phi_n(x)$$

$$a_{m,G} = \langle \phi_m, f \rangle_G = \sum_{i=0}^{N} w_i \phi_m(x_i) f(x_i)$$

at x_i we have $f(x_i) = P_N(x_i)$

$$= \sum_{i=0}^{N} w_i \phi_m(x_i) P_N(x_i)$$

G stands for Gaussian Quadrature

$$= \langle \phi_m, P_N \rangle_G = b_n$$

Therefore

if we decide to compute spectral coefficients a_n with Gaussian Quadrature, there is no difference or error between a_n and b_n. So, we can leave $f(x)$ and work with $f(x_i)$ or $P_N(x_i)$ with no error.

Introduction to chebfun

11 - 08 - 11

Example: state - transition operator

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 1-x^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{cases} \dfrac{d\Phi(x,\eta)}{dx} = A(x) \cdot \Phi(x,\eta) \\ \Phi(\eta,\eta) = I \end{cases}$$

Can use chebfun to find $\Phi(x,\eta)$

11-10-11

<u>Evolution equation</u> that we will consider:

$$\mathcal{E}\,\phi_t(y,t) = \mathcal{F}\,\phi(y,t) + \mathcal{G}\,d(y,t)$$

$$\varphi(y,t) = \mathcal{C}\,\phi(y,t)$$

$\mathcal{E}$ can be invertible, it may also be not invertible.

example: Navier-Stokes equations

$$\underbrace{\begin{bmatrix} \Delta & 0 \\ 0 & I \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \phi_{1t} \\ \phi_{2t} \end{bmatrix} = \underbrace{\begin{bmatrix} \Delta^2 & 0 \\ -jk_z U' & \Delta \end{bmatrix}}_{\mathcal{F}} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\Delta^2 = \mathcal{D}^{(4)} - 2k_z^2\,\mathcal{D}^{(2)} + k_z^4$$

$$\Delta = \mathcal{D}^{(2)} - k_z^2$$

$$\mathcal{F} = \begin{bmatrix} k_z^4 & 0 \\ -jk_z U' & -k_z^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\mathcal{D}^{(1)} + \begin{bmatrix} -2k_z^2 & 0 \\ 0 & 1 \end{bmatrix}\mathcal{D}^{(2)}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\mathcal{D}^{(3)} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathcal{D}^{(4)}$$

<u>Input-output map</u>: $\mathcal{T}(\omega): \mathbb{H}_{in} \longrightarrow \mathbb{H}_{out}$

$$\varphi(y,\omega) = \left[\mathcal{T}(\omega)\, d(\cdot,\omega)\right](y)$$

$$= \sum_{n=1}^{\infty} \sigma_n(\omega)\, u_n(y,\omega) \left\langle v_n(y,\omega), d \right\rangle$$

σ_1 ... largest eigenvalue of $\mathcal{T}(\omega)\,\mathcal{T}^*(\omega)$

v_1 ... the input direction that yields largest response

u_1 ... the spatial pattern that is generated by v_1, i.e. the most energetic pattern that one expects to see if the system is forced by a stochastic forcing.

Remark in situation that we don't have a normal operator, eigenvalue decomposition is not a good measure of the system response. (because we cannot obtain eigen-directions that evolve independently.)

For non-normal operators, singular-value decomposition is the right notion. It is a description of the input-output maps.

The input-output gains are obtained from singular values of $\mathcal{T}(\omega)$.

H_∞ norm: $\phi(y,t)$

$$\|H\|_\infty^2 = \sup_{\substack{d \neq 0 \\ \int_0^\infty \langle d, d\rangle\, dt \leq 1}} \frac{\int_0^\infty \langle \phi(\cdot,t), \phi(\cdot,t)\rangle\, dt}{\int_0^\infty \langle d(\cdot,t), d(\cdot,t)\rangle\, dt}$$

L_2-induced gain of an LTI system

$$[\text{in } L_2] = \sup \frac{\int_0^\infty \int_{-1}^1 \phi^*(y,t)\,\phi(y,t)\,dy\,dt}{\int_0^\infty \int_{-1}^1 d^*(y,t)\,d(y,t)\,dy\,dt} = \sup_\omega \sigma_1^2(\omega)$$

- H_∞ defined with time-integrals can be interpreted as $\underbrace{\text{Worst-case}}_{\text{Largest}}$ energy that can be obtained by arbitrary deterministic inputs.

- H_∞ defined with $\sup_\omega \sigma_1^2(\omega)$ can be interpreted as the Largest amplification of persistant sinousoidal inputs.

- Robustness interpretation: (Small-gain theorem)

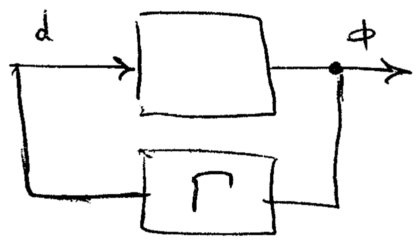
if you have a modeling uncertainty Γ,

$$\|\Gamma\|_\infty \leq \nu \iff \nu < \frac{1}{\|\Gamma\|_\infty}$$

meaning that system remains stable

the amount of uncertainty one can handle in the presence of all unstructured uncertainty Γ.

The Larger $\|\Gamma\|_\infty$, the smaller uncertainty can destabilize the system.



$\boxed{\text{Power spectral density:}}$

$$\| \mathcal{T}(\omega) \|^2_{HS} = \text{trace}\left(\mathcal{T}(\omega) \mathcal{T}^{+}(\omega) \right) = \sum_{n=1}^{\infty} \sigma_n^2 (\omega)$$

What is trace of an operator P?

if P is a matrix $[P_{ij}]$, then

$$\text{trace}(P) = \sum_{i=1}^{n} P_{ii}$$

if P is an operator: $\quad g(x) = [Pf](x)$

$$= \int_a^b P_{ker}(x, \xi) f(\xi) \, d\xi$$

$$= \int_a^b \left[P_{ker}(x,\xi) \right] \left[f \right] d\xi$$

$$\text{trace}(P) = \int_a^b P_{ker}(x, x) \, dx$$

$\rightarrow$ if $f(x) \in \mathbb{C}$, $g(x) \in \mathbb{C}$

Now, if $f(x) \in \mathbb{C}^m$; $g(x) \in \mathbb{C}^m$

$$\boxed{\underbrace{\text{trace}(P)}_{\text{operator trace}} = \int_a^b \underbrace{\text{tr}\left(P_{ker}(x,x) \right)}_{\text{matrix trace}} dx}$$

here, $P(\omega) = \mathcal{T}(\omega) \mathcal{T}^{+}(\omega)$

H_2 norm

$$\|\mathcal{T}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\mathcal{T}(\omega)\|_{HS}^2 \, d\omega \quad \leftarrow \quad \text{definition}$$

Computation : $\|\mathcal{T}\|_2^2 = \operatorname{trace}(\mathcal{C}\mathcal{X}\mathcal{C}^\dagger)$

$$\underbrace{\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^\dagger = -\mathcal{B}\mathcal{B}^\dagger}$$

operator Lyapunov equation: in general it is difficult to solve explicitely.

if $\mathcal{A}\mathcal{A}^\dagger = \mathcal{A}^\dagger\mathcal{A}$; $\mathcal{B}\mathcal{B}^\dagger = I$

Then $\mathcal{X} = -(\mathcal{A} + \mathcal{A}^\dagger)^{-1}$

Example Heat equation :

$$\mathcal{A} = \Delta = \begin{cases} \dfrac{d^2}{dy^2}, & 1D \\ \dfrac{d^2}{dy^2} - k_z^2, & 2D \end{cases}$$

$\oplus$

Dirichlet BCs

$$\mathcal{X} = \frac{-1}{2}\Delta^{-1}.$$

Example $\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^\dagger = -Q$

$\mathcal{A}\mathcal{A}^\dagger = \mathcal{A}^\dagger\mathcal{A}$

$$\|\mathcal{T}\|_2^2 = \operatorname{trace}(\mathcal{X}) = -\operatorname{trace}((\mathcal{A}+\mathcal{A}^\dagger)^{-1}Q).$$

Exponential stability:

$$(*) \qquad \|\mathcal{T}\| < M e^{-\alpha t} \quad , \qquad M, \alpha > 0$$

induced operator norm

$\mathcal{T}$: $\mathcal{C}_0$-semigroup generated by $\mathcal{A}$.

if $(*)$ is satisfied, solutions of the system decay exponentially with time, because:

$$\|\mathcal{T}\| = \sup_{0 < \|\psi_0\| \leq 1} \frac{\|\mathcal{T}(t)\psi_0\|}{\|\psi_0\|} < M e^{-\alpha t}$$

$$\Rightarrow \quad \underbrace{\|\mathcal{T}(t)\psi_0\|} < M \|\psi_0\| e^{-\alpha t}$$

norm of the solution, $\psi(t)$,
starting at $\psi(0) = \psi_0$

Lyapunov-based Characterization

$\mathcal{T}(t)$ on $\mathcal{H}$ is exponentially stable

$$\Updownarrow$$

$\exists$ bounded positive operator $\mathcal{P}$ s.t.

$$\langle \mathcal{A}\psi, \mathcal{P}\psi \rangle + \langle \mathcal{P}\psi, \mathcal{A}\psi \rangle = -\langle \psi, \psi \rangle$$

for all $\psi \in \mathcal{D}(\mathcal{A})$

equivalently:

$$\mathcal{A}^\dagger \mathcal{P} + \mathcal{P}\mathcal{A} = -\mathcal{I} \quad \text{on} \quad \mathcal{D}(\mathcal{A})$$

$\downarrow$

observability Gramian on an infinite time horizon.

$$\mathcal{P}\psi_0 = \int_0^\infty \mathcal{T}^\dagger(t)\cdot\mathcal{I}\cdot\mathcal{T}(t)\psi_0 \, dt \quad : \text{observability Gramian}$$

$\boxed{\Rightarrow}$ proof

"Boundedness"

$$\langle \psi_0, \mathcal{P}\psi_0 \rangle = \int_0^\infty \psi_0^* \mathcal{T}^*(t)\mathcal{T}(t)\psi_0 \, dt$$

$$= \int_0^\infty \| \mathcal{T}(t)\psi_0 \|^2 \, dt \leq \gamma_\psi < \infty$$

$\downarrow$ Datko's Lemma

Also, $\langle \psi_0, \mathcal{P}\psi_0 \rangle \geq 0$: "Positivity"

Note

$$\langle \psi_0, \mathcal{P}\psi_0 \rangle = 0 \iff \| \mathcal{T}(t)\psi_0 \| = 0 \quad \underbrace{\text{a.e.}}_{\text{almost everywhere}}$$

From strong continuity of $\mathcal{T}(t) \Rightarrow \| \mathcal{T}(t)\psi_0 \| = 0$ (a.e)

$\boxed{\Leftarrow}$ proof $\quad \psi_0 = 0 \Rightarrow \mathcal{P} > 0$.

Lyapunov functional candidate:

$$V(\psi(t)) = \langle \psi(t), \mathcal{P}\psi(t) \rangle$$

Note: $\boxed{\mathcal{P} \neq \mathcal{P}(t)}$

$$\boxed{\frac{dV(\psi(t))}{dt}} = \langle \psi_t(t), \mathcal{P}\psi(t) \rangle + \langle \psi(t), \mathcal{P}\psi_t(t) \rangle$$

$$= \langle \mathcal{A}\psi(t), \mathcal{P}\psi(t) \rangle + \langle \psi(t), \mathcal{P}\mathcal{A}\psi(t) \rangle$$

$$= \langle \mathcal{A}\psi, \mathcal{P}\psi \rangle + \langle \mathcal{P}\psi, \mathcal{A}\psi \rangle$$

$$= -\langle \psi, \psi \rangle = -\|\psi(t)\|^2$$

$$= \boxed{-\|\mathcal{T}(t)\psi_0\|^2}$$

$$V(\psi(t)) = V(\psi(0)) - \int_0^t \|\mathcal{T}(t)\psi_0\|^2 \, dt$$

$$0 \leq V(\psi(t)) = V(\psi(0)) - \int_0^t \|\mathcal{T}(t)\psi_0\|^2 \, dt$$

$$\int_0^t \|\mathcal{T}(t)\psi_0\|^2 \, dt \leq V(\psi_0) \quad \text{on } \mathcal{D}(\mathcal{A})$$

Example: Diffusion on $L_2[-1,1]$

$$\phi'' = \frac{-1}{2}\psi, \quad \phi(\pm 1) = 0$$

$$\begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \psi$$

$$\phi = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(-1) \\ \phi_2(-1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(1) \\ \phi_2(1) \end{bmatrix}$$

$$\phi(x) = \int_{-1}^{1} P_{ker}(x, \xi)\, \psi(\xi)\, d\xi$$

Optimal Control of Distributed Systems

$\boxed{11-29-11}$

LQR : $\begin{cases} \min & J = \int_0^\infty \left(\langle \psi(t), Q\psi(t)\rangle + \langle u(t), Ru(t)\rangle \right) dt \\ \text{s.t.} & \psi_t = A\psi + Bu \; ; \; \psi(0) \in \mathbb{R} \end{cases}$

$\boxed{\text{finite dimensions:}}$

$$u(t) = -K\psi(t)$$

$$K = R^{-1}B^*P$$

$$P = P^* \quad ; \quad A^*P + PA + Q - PBR^{-1}B^*P = 0$$

ARE : $A^*P + PA + Q - PB\underbrace{R^{-1}B^*P}_{K} = 0$

Add & subtract $PBR^{-1}B^*P$ to ARE

$$(A - BK)^*P + P(A - BK) = -(Q + K^*RK)$$

$$A_{cl}^*P + PA_{cl} = -(Q + K^*RK)$$

So P is observability Gramian with respect to an ~~[illegible]~~ appropriate output

$$A_{cl}^*P + PA_{cl} = -C^*C \qquad \begin{cases} \dot{x} = Ax \\ z = Cx \end{cases}$$

$$z = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u$$

$$J = \int_0^\infty z^*(t)\, z(t)\, dt$$

$$\begin{cases} \dot{\psi} = (A - BK)\psi \\ z = \begin{bmatrix} Q^{1/2} \\ -R^{1/2}K \end{bmatrix} \psi \\ K = R^{-1}B^*P \end{cases}$$

Infinite dimensions

$$u(t) = -\mathcal{K}\,\psi(t)$$
$$\mathcal{K} = \mathcal{R}^{-1}\mathcal{B}^{\dagger}P$$

$$\langle \mathcal{A}\psi_1, \mathcal{P}\psi_2\rangle + \langle \mathcal{P}\psi_1, \mathcal{A}\psi_2\rangle + \underbrace{\langle\quad\rangle}_{\langle Q^{+1/2}\psi_1,\, Q^{1/2}\psi_2\rangle} + \langle \mathcal{B}^{\dagger}P\psi_1, \mathcal{R}^{-1}\mathcal{B}^{\dagger}P\psi_2\rangle = 0$$

- optimal Controller that is obtained for spatially-invariant systems is centralized.

$$\psi_t(x,t) = [\mathcal{A}\psi(\cdot,t)](x) + [\mathcal{B}u(\cdot,t)](x)$$

translation-invariant operators $\mathcal{A}$, $\mathcal{B}$.

Spatial Fourier Transform

$$\dot{\hat{\psi}}(k,t) = \hat{\mathcal{A}}(k)\,\hat{\psi}(k,t) + \hat{\mathcal{B}}(k)\,\hat{u}(k,t)$$

k ↓ spatial frequency

multiplication operators: $\hat{\mathcal{A}}(k)$, $\hat{\mathcal{B}}(k)$

- The appropriate Fourier Transforms in space, effectively block-diagonalize the system.
- Similarly, the appropriate Fourier Transform can decouple the ARE associated with an infinite-dimensional system. The requirement is that $\mathcal{A}$, $\mathcal{B}$, Q, $\mathcal{R}$ are jointly, unitarily block diagonalizable.

Penalty term in physical & frequency domains.

Example:

$$Q_p = I + \overbrace{C}^{\text{circulant}}$$

$$C_{4\times 4} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$\hat{q}_p(k) = 1 + 2\left(1 - \cos\frac{2\pi k}{N}\right)$$

here, $N = 4$

Position penalty

$$Q = \begin{bmatrix} Q_p & 0 \\ 0 & Q_v \end{bmatrix}$$

velocity penalty

example: $\hat{K}(k) = \hat{R}^{-1}(k)\,\hat{B}^*(k)\,\hat{P}(k)$

$$= \frac{1}{\hat{r}(k)}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{P}_1(k) & \hat{P}_0^*(k) \\ \hat{P}_0(k) & \hat{P}_2(k) \end{bmatrix}$$

$$= \Bigg[\underbrace{\frac{1}{\hat{r}(k)}\hat{P}_0(k)}_{\text{Position feedback gain}} \quad \underbrace{\frac{1}{\hat{r}(k)}\hat{P}_2(k)}_{\text{velocity feedback gain}}\Bigg]$$

PDEs on $L_2(-\infty,\infty)$:

ex: heat equation

$$\psi_t(x,t) = \psi_{xx}(x,t) + u(x,t)$$

$$u(x,t) = -\left[K\psi(\cdot,t)\right](x) = -\int_{-\infty}^{\infty} K_{ker}(x,\xi)\,\psi(\xi,t)\,d\xi$$

For spatially-invariant systems:

$$u(x,t) = -\int_{-\infty}^{\infty} K_{ker}(x-\xi)\psi(\xi,t)\,d\xi$$

$$\Big\downarrow \text{F.T.}$$

$$\hat{u}(k,t) = -\hat{K}(k)\,\hat{\psi}(k,t)$$

$$K_{ker}(x) = \mathcal{F}^{-1}\{\hat{K}(k)\}$$

Heat equation:

$$\dot{\hat{\psi}}(k,t) = -k^2\hat{\psi}(k,t) + \hat{u}(k,t)$$

$$\begin{cases} \hat{A}(k) = -k^2 \\ \hat{B}(k) = 1 \end{cases} \qquad \begin{cases} \hat{Q}(k) = \hat{q}(k) \\ \hat{R}(k) = \hat{r}(k) \end{cases}$$

$$\underbrace{-2k^2\hat{p}}_{\hat{A}^*\hat{P}+\hat{P}\hat{A}} + \underbrace{\hat{q}}_{\hat{Q}} - \frac{1}{\hat{r}}\hat{p}^2 = 0 \qquad \leftarrow \text{ARE}$$

$$\hat{P} = \hat{r}\left(-k^2 \pm \sqrt{k^4 + \frac{\hat{q}}{\hat{r}}}\right)$$

Choose $(+)$ to get $\hat{p} > 0$.

① $$\boxed{\hat{K} = \frac{1}{\hat{r}}\hat{P} = -k^2 + \sqrt{k^4 + \frac{\hat{q}}{\hat{r}}}}$$ feedback gain

Check boundedness of $\hat{K}$

② $$\hat{K} = \frac{-\hat{q}/\hat{r}}{k^2 + \sqrt{k^4 + \frac{\hat{q}}{\hat{r}}}}$$

even though $\hat{K}$ written in form ① looks like a 2nd derivative operator and may indicate unboundedness of $\hat{K}$, when written in form ②, it is clear that it can be written in terms of integral operators.

Consensus and vehicular formulation

12-05-11

- use spatially-invarient theory to answer some of the questions that arise in these problems.

- These systems can be thought as spatio-temporal systems, where signals depend on time and discrete spatial variable 'n'.

- Coupling between subsystems can come either from mathematical modeling or from distributed control at the level of control objective.

- optimal control of finite platoons

$$J = \int_0^\infty \left(p^T(t) Q_p p(t) + q_v v^T(t) v(t) + r u^T(t) u(t) \right) dt$$

$\to 1$ (if lead fict. veh. is removed)

$$Q_p = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$\to 1$ (if follow fict. veh. is removed)

$\to$ when lead & follow fictitious vehicles are added to formation.

If both lead & follow fictitious vehicles are removed Q_p becomes singular.

• infinite platoons

$$\begin{bmatrix} \dot{p}_n \\ \dot{v}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_n \\ v_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n \quad ; \quad n \in \mathbb{Z}$$

$$J = \int_0^\infty \sum_{n \in \mathbb{Z}} \left((p_n(t) - p_{n-1}(t))^2 + v_n^2(t) + u_n^2(t) \right) dt$$

$\downarrow$ Spatial Z_θ- transform

$$A_\theta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q_\theta = \begin{bmatrix} 2(1-\cos\theta) & 0 \\ 0 & 1 \end{bmatrix}$$

$$0 \leq \theta < 2\pi$$

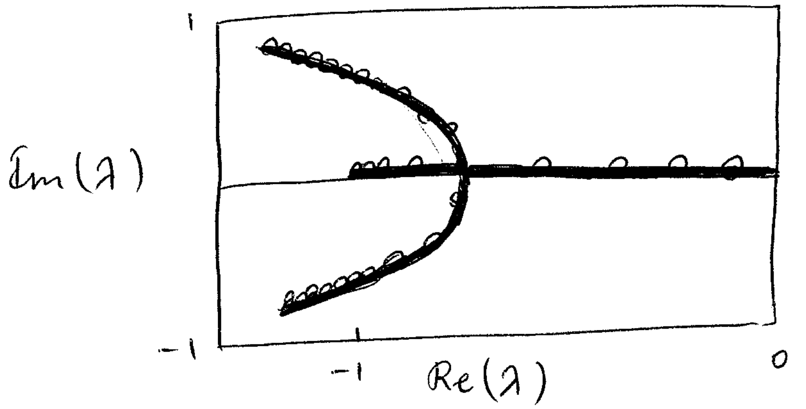
• The coupling comes from the performance index.

☒ if $\theta = 0$, $Q_p = 2(1-\cos\theta) = 0$
meaning that the pair (A_θ, Q_θ) is not detectable.

☒ fix: penalize global position errors:

$$Q_p = q + 2(1-\cos\theta)$$

$q = 0 \Rightarrow$ many modes have slow rate of convergence.



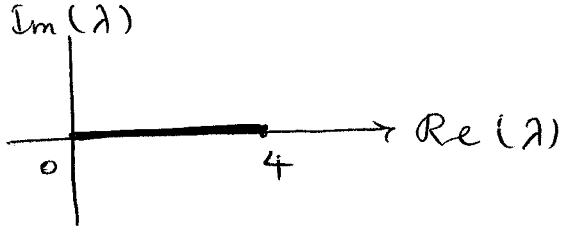
thick solid line → infinite platoons

symbols → finite platoons

Q_p is symmetric - Toeplitz matrix

$$Q_p = V \Lambda V^* \quad ; \quad V V^* = I$$

$$q_p(\theta) = 2(1 - \cos\theta)$$



Finite platoons :

$$Q_p = V \Lambda V^*$$

$$Q_p \sim \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{M \times M}$$

$$\boxed{\lambda_n(Q_p) = 2\left(1 - \cos \frac{n\pi}{M+1}\right)}$$

Can show : solution to ARE

$$P = \begin{bmatrix} P_1 & P_0^* \\ P_0 & P_2 \end{bmatrix} \; ; \quad P_j = V \Lambda_j V^* \quad j = 1, 2, 0$$

$$P = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \Lambda_1 & \Lambda_0 \\ \Lambda_0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & V^* \end{bmatrix}$$

where Λ_j are diagonal matrices with elements determined by $\lambda_n(Q_p)$

$$\boxed{\begin{aligned} \hat{A} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ; \quad \hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hat{Q}_p &= \begin{bmatrix} \lambda_n(Q_p) & 0 \\ 0 & q_v \end{bmatrix} ; \quad \hat{R} = r \end{aligned}} \qquad (*)$$

ARE for the complete system :

$$A^* P + P A^* + Q - P B R^{-1} B^* P = 0$$

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix} ; \quad Q = \begin{bmatrix} Q_p & 0 \\ 0 & q_v I \end{bmatrix} ; \quad R = r I$$

Can write A as :

$$A = \begin{bmatrix} \mathcal{V} & 0 \\ 0 & \mathcal{V} \end{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{V}^* & 0 \\ 0 & \mathcal{V}^* \end{bmatrix}$$

(Since $\mathcal{V}\mathcal{V}^* = I$) .

"key"

> Choose '$\mathcal{V}$' that diagonalizes Q .
> This brings the Large-size ARE into a set of AREs with size 2×2 .

Now, Compare $(*)$ with the following :

$$\hat{A}_\theta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ; \quad \hat{B}_\theta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\hat{Q}_\theta = \begin{bmatrix} 2(1-\cos\theta) & 0 \\ 0 & q_v \end{bmatrix} ; \quad \hat{R}_\theta = r$$

$\boxed{\text{For infinite problem}} \downarrow$

$$J = \int_0^\infty \left(\langle P, Q_p P \rangle + q_v \langle v, v \rangle + r \langle u, u \rangle \right) dt$$

$$P(t) = \begin{bmatrix} \vdots \\ p_{n-1}(t) \\ p_n(t) \\ p_{n+1}(t) \\ \vdots \end{bmatrix} \in l_2$$

$$\langle P, Q_p P \rangle_{\ell_2} = \langle \hat{P}(\theta), \widehat{(Q_p P)}(\theta) \rangle_{L^2[0, 2\pi]}$$

$$[Q_p P](n) = \sum_{k=-\infty}^{\infty} q_p(n-k) P(k), \quad \begin{cases} q_p(0) = 2 \\ q_p(\pm 1) = -1 \end{cases}$$

$$\widehat{[Q_p P]}(\theta) = q_p(\theta) P(\theta)$$

<u>*Note that:*</u>

$$q_p(\theta) = \left[q_p(0) z^0 + q_p(1) z^1 + q_p(-1) z^{-1} \right] \Big|_{z = e^{j\theta}}$$

$$= 2(1 - \cos\theta)$$

Then,

$$J = \int_0^{\infty} \int_0^{2\pi} \Big(\hat{p}^*(\theta,t)\, q_p(\theta)\, \hat{p}(\theta,t) + q_v \hat{v}^*(\theta,t)\, \hat{v}(\theta,t) + r\, \hat{u}^*(\theta,t)\, u(\theta,t) \Big) \, d\theta \, dt$$

<u>Control</u> :

$$\hat{u}(\theta,t) = - \hat{K}(\theta) \hat{\psi}(\theta,t)$$

$$= - \hat{R}^{-1}(\theta)\, \hat{B}^*(\theta)\, \hat{P}(\theta)\, \hat{\psi}(\theta,t)$$

where

$$\hat{K}(\theta) = \left[\hat{k}_p(\theta) \quad \hat{k}_v(\theta) \right]$$

$$= \left[\frac{1}{r} \hat{P}_0(\theta) \quad \frac{1}{r} \hat{P}_2(\theta) \right]$$

In the physical space

$$u_n(t) = -\sum_{k=-\infty}^{\infty} k_p(n-k)\, p_k(t) - \sum_{k=-\infty}^{\infty} k_v(n-k)\, v_k(t)$$

Levine & Athans '66 introduced the variable e_n:

$$e_n = p_n - p_{n-1} \quad ; \quad n = 2, \dots, M$$

$$\dot{e}_n = \dot{p}_n - \dot{p}_{n-1} = v_n - v_{n-1}$$

$$\dot{v}_n = u_n$$

$$\begin{bmatrix} \dot{e} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

with $e = \begin{bmatrix} e_1 \\ \vdots \\ e_M \end{bmatrix} \in \mathbb{R}^{M+1}$

$$\begin{cases} \dot{e}_n = v_n - v_{n-1} \\ \dot{v}_n = u_n \end{cases}$$

$$\begin{cases} \dot{e}_\theta = (1 - e^{-j\theta}) v_\theta \\ \dot{v}_\theta = u_\theta \end{cases}$$

$$\begin{bmatrix} \dot{e}_\theta \\ \dot{v}_\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 - e^{-j\theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_\theta \\ v_\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_\theta$$

$$Q_\theta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} ; \quad R_\theta = r$$

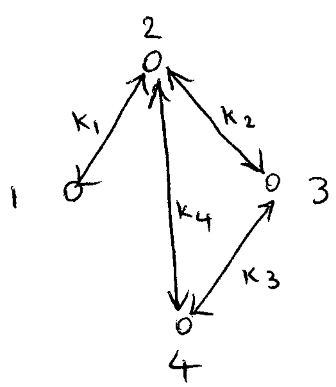
- At the limit of infinite vehicles, Controllability is Lost.

$$Le^{-j\theta} \xrightarrow[\theta \to 0]{} 0$$

This is a consequence of increase in the relative degree of the dynamics of the vehicle located far away from the Leader. In other words, the number of integrators between the vehicles and the Leader increases to infinity which results in Large (close to infinity) delay in the response of the vehicles located infinitely far away from the Leader.

Convergence or deviation from average

Example



$$\left.\begin{aligned}
\dot{x}_1(t) &= -k_1\,(x_1(t) - x_2(t)) \\
\dot{x}_2(t) &= -k_1\,(x_2(t) - x_1(t)) \\
&\quad - k_2\,(x_2(t) - x_3(t)) \\
&\quad - k_4\,(x_2(t) - x_4(t)) \\
\dot{x}_3(t) &= -k_2\,(x_3(t) - x_2(t)) \\
&\quad - k_3\,(x_3(t) - x_4(t)) \\
\dot{x}_4(t) &= -k_3\,(x_4(t) - x_3(t)) \\
&\quad - k_4\,(x_4(t) - x_2(t))
\end{aligned}\right\}$$

Iterations that each node take to update their value.

We can show that

$$\dot{x}(t) = -E K E^T x(t) + d(t)$$

incidence matrix

$K = \mathrm{diag}\{k_1, k_2, k_3, k_4\}$

Can check that $A = -EKE^T$

has rows and columns whose sum are 0.

$$A\mathbb{1} = 0 \cdot \mathbb{1}$$
$$\mathbb{1}^T A = 0 \cdot \mathbb{1}^T$$

example: Let $k_i = 1$; $i = 1, 2, 3, 4$

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

Question:

$$\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t) = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Can all nodes converge to $\bar{x}(t)$?
  Answer: Yes
- How quickly ?
- What would be the effect of disturbances on convergence ?

$$\begin{cases} \dot{\psi}(t) = U^* A U \psi(t) + U^* d \\ \dot{\bar{x}}(t) = 0 \cdot \bar{x}(t) + \frac{1}{N} \mathbb{1}^T \cdot d \end{cases}$$

$$\dot{\psi}(t) = \bar{A} \psi(t) + \bar{B} \cdot d$$

$$\mathrm{Re}(\lambda_i(\bar{A})) < 0, \quad i = 1, \ldots, N-1$$

$$z = (I - \tfrac{1}{N} \mathbb{1}\mathbb{1}^T)\, x(t) =$$

$$(I - \tfrac{1}{N}\mathbb{1}\mathbb{1}^T)[U \ \ \mathbb{1}]\begin{bmatrix} \psi \\ \bar{x} \end{bmatrix}$$

$$= \Big[(U - \tfrac{1}{N}\underbrace{\mathbb{1}\mathbb{1}^T U}_{0}) \ \vdots\ (\mathbb{1} - \tfrac{1}{\cancel{N}} \mathbb{1}\underbrace{\mathbb{1}^T\mathbb{1}}_{\cancel{N}})\Big]\begin{bmatrix} \psi \\ \bar{x} \end{bmatrix}$$

$$= U\psi$$

Thus, if we use DFT on systems with Circulant matrices, we obtain

$$\begin{cases} \dot{\hat{\psi}}_k(t) = \hat{a}_k \hat{\psi}_k(t) + \hat{d}_k \\ \hat{z}_k(t) = \hat{\psi}_k(t) \end{cases}$$

Lyapunov equation:

$$\bar{A}P + P\bar{A}^* = -\bar{B}\bar{B}^*$$

$$\|H\|_2^2 = \mathrm{trace}(\bar{C}^* P \bar{C})$$

Note. For spatially-invariant systems :

$$\hat{a}_k \hat{P}_k + \hat{P}_k \hat{a}_k^* = -1 \ ; \qquad \|H\|_2^2 = \sum_{k=1}^{n} \frac{-1}{\hat{a}_k + \hat{a}_k^*}$$

An example:

Nearest neighbour information exchange.

Q. What would happen if instead we had paid attention to

$$y_n(t) = x_n(t) - x_{n-1}(t)$$

$$\hat{y}_k(t) = \underbrace{\left(1 - e^{-j\frac{2\pi}{N}k}\right)}_{\hat{C}_k} \hat{x}_k(t)$$

$$\hat{C}_k^* \hat{C}_k = 2\left(1 - \cos\frac{2\pi}{N}k\right)$$

Then, $\|H\|_2^2 = \sum_{k=1}^{N-1} \hat{C}_k^* \hat{C}_k \hat{P}_k = \sum_{k=1}^{N-1} \frac{1}{2} = \frac{N-1}{2}$

In this case

So, the total variance amplification of the system, is increasing linearly with N.

But, if we pay attention to the deviation from average or the slot length, it scales badly with N. (refer to lecture slides)

Role of dimensionality

Features:

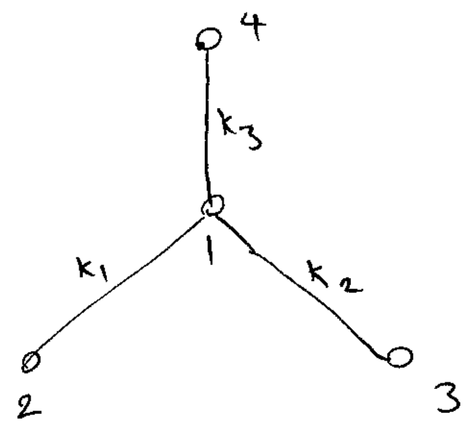
- spatial invariance
- locality $\Rightarrow$ fix the number of neighbours you are communicating with, then increase the total number of nodes.
- miror symmetry $\Rightarrow$ pay same attention to neighbours in all directions.

Incidence matrix:

$E \in \mathbb{R}^{N \times M}$

N ... # of nodes

M ... # of edges



$$E = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$E^T x = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \\ x_1 - x_4 \end{bmatrix} ; \quad E^T \mathbb{1} = 0$$

Laplacian: $\quad L(K) = E K E^T$

where K is structured feedback gain:

$$K = \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_M \end{bmatrix}$$

Reference: Zelazo & Mesbahi CDC'09
TAC'11

E ... incidence matrix

L ... Laplacian

$$L(K) = EKE^T = \sum_{\ell=1}^{m} k_\ell e_\ell e_\ell^T$$

Structured feedback gain $K = \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_m \end{bmatrix}$

Arrive at a structured optimal control problem.

Let's first consider graphs that do not have loops. (trees)

Coordinate transformation:

$$\begin{bmatrix} \Psi(t) \\ \bar{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} E^T \\ \frac{1}{N} \mathbb{1}^T \end{bmatrix}}_{T} x(t)$$

$\Psi(t)$ .... relative difference between adjacent nodes.

$\bar{x}(t)$ .... average mode.

Then,

$$\begin{bmatrix} \dot{\Psi}(t) \\ \dot{\bar{x}}(t) \end{bmatrix} = \begin{bmatrix} -E_t^T E_t K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Psi(t) \\ \bar{x}(t) \end{bmatrix} + \begin{bmatrix} E_t^T \\ \frac{1}{N} \mathbb{1}^T \end{bmatrix} d(t)$$

- $\bar{x}(t)$ is preserved when $d(t) = 0$, otherwise it drifts with random walk.

$$\dot{\Psi}(t) = -E_t^T E_t K \Psi(t) + E_t^T d(t)$$

$$z(t) = \begin{bmatrix} E_t (E_t^T E_t)^{-1} \\ -E_t K \end{bmatrix} \Psi(t)$$

$\mathcal{H}_2$ -norm from d to z:

$$J(K) = \frac{1}{2} \operatorname{trace}\left(G^{-1} K^{-1} + G K \right)$$

where $G = E_t^T E_t$

$$J(K) = \frac{1}{2} \sum_{n=1}^{N-1} \left(\frac{1}{K_n g_n} + k_n g_n \right)$$

$$= \frac{1}{2} \sum_{n=1}^{N-1} \frac{1 + (k_n g_n)^2}{k_n g_n}$$

Can minimize $J(k)$ by minimizing each term $\frac{1+(k_n g_n)^2}{k_n g_n}$, because we have seperability between the index 'n' or between nodes.

So, if we use incidence matrix of a tree graph, we can separate the effect of nodes on the objective function, if J is the difference between the values of each node, and then we can solve the optimal Control problem.

General undirected graphs.

incidence matrix $\quad E = [E_t \quad E_c]$

$E_c \downarrow$ part of the incidence matrix where there is a loop (cycle).

Columns of E_c are linear combination of columns of E_t.

Equality-Constrained Convex optimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{s.t.} \quad & Ax - b = 0 \end{aligned}$$

$$\mathcal{L}(x,y) = f(x) + y^T(Ax-b)$$

if f is differentiable,

$$\nabla_x \mathcal{L}(x,y) = \nabla f(x) + A^T y = 0$$

Ex $\quad f(x) = \frac{1}{2} x^T Q x \; ; \; Q = Q^T \succ 0$

$$Qx + A^T y = 0$$

$$\Leftrightarrow \boxed{x = -Q^{-1} A^T y}$$

dual ascent method

$$\begin{cases} x^{k+1} = -Q^{-1} A^T y^k \\ y^{k+1} = y^k + s^k (A x^{k+1} - b) \end{cases}$$

Advantage it may lead to distributed implementation.