1. In HW2 you have shown that the dynamics of infinitesimal "streamwise-constant" fluctuations around the mean velocity \((U(y), 0, 0)\) in a channel flow are governed by the following evolution model

\[
\begin{bmatrix}
\psi(\tau) \\
u(\tau)
\end{bmatrix} =
\begin{bmatrix}
L \\
ReC_p S
\end{bmatrix}
A
\begin{bmatrix}
\psi(\tau) \\
u(\tau)
\end{bmatrix}.
\]

(LNSE)

Here, \(u\) and \(\psi\) denote the streamwise velocity and stream-function fluctuations, respectively, and the operators in (LNSE) are given by

Orr-Sommerfeld: \(L = \Delta^{-1}\Delta^2\),

Squire: \(S = \Delta\),

Coupling: \(C_p = -jk_z U'(y)\),

with \(k_z\) denoting the spanwise wave number and,

\[
\Delta = \partial_{yy} - k_z^2,
\]

\[
\Delta^2 = \partial_{yyy} - 2k_z^2 \partial_{yy} + k_z^4.
\]

We will consider the generator \(A\) in (LNSE) on the following Hilbert space

\[
\mathbb{H} = \left[ \mathbb{H}_{os} \right]_{L_2[-1, 1]}, \quad \mathbb{H}_{os} = \left\{ g \in L^2[-1, 1]; \ g'' \in L^2[-1, 1], \ g(\pm 1) = 0 \right\},
\]

with the domain of \(A\) determined by

\[
D(A) = \begin{bmatrix}
D(L) \\
D(S)
\end{bmatrix},
\]

\[
\begin{align*}
D(L) &= \left\{ g \in \mathbb{H}_{os}; \ g^{(4)} \in L^2[-1, 1], \ g'(\pm 1) = 0 \right\} \\
D(S) &= \mathbb{H}_{os}.
\end{align*}
\]

We endow the state-space \(\mathbb{H}\) with an inner product

\[
\langle \phi_1, \phi_2 \rangle_e = \langle \phi_1, Q \phi_2 \rangle,
\]

where \(\langle \cdot, \cdot \rangle\) is the standard \(L^2[-1, 1]\) inner product, and \(Q\) is a block diagonal positive self-adjoint operator,

\[
Q = \begin{bmatrix}
-\Delta & 0 \\
0 & I
\end{bmatrix}.
\]

(a) Determine the adjoint of the Squire operator \(S\) (with respect to the standard \(L_2[-1, 1]\) inner product) and the adjoint of the Orr-Sommerfeld operator \(L\) (with respect to \(\langle \psi_1, \psi_2 \rangle_{os} = -\langle \psi_1, \Delta \psi_2 \rangle\)).

(b) Determine the adjoint of the operator \(A\) with respect to \(\langle \cdot, \cdot \rangle_e\).

(c) Show that the Orr-Sommerfeld operator \(L\) has two sets of the eigenvalues \(\{\lambda_k\}_{k \in \mathbb{N}}\) and \(\{\gamma_k\}_{k \in \mathbb{N}}\),

\[
\lambda_k = -(p_k^2 + k_z^2), \quad \gamma_k = -(q_k^2 + k_z^2),
\]

with the corresponding eigenfunctions,

\[
\psi_{os1,k}(y, k_z, \lambda_k) = A_k \left( \cos(p_k y) - \frac{\cos(p_k y)}{\cosh(k_z y)} \cosh(k_z y) \right)
\]

\[
\psi_{os2,k}(y, k_z, \gamma_k) = B_k \left( \sin(q_k y) - \frac{\sin(q_k y)}{\sinh(k_z y)} \sinh(k_z y) \right),
\]

where \(p_k\) and \(q_k\) are obtained as solutions to the following two equations,

\[
p_k \tan(p_k) = -k_z \tanh(k_z),
\]

\[
q_k \cot(q_k) = k_z \coth(k_z).
\]
Convince yourself that the following choices of \( \{ A_k \}_{k \in \mathbb{N}} \) and \( \{ B_k \}_{k \in \mathbb{N}} \),

\[
A_k = \left( p_k^2 + k_2^2 \right) \left( 1 + \frac{\sin(2p_k)}{2p_k} \right)^{-\frac{1}{2}}, \\
B_k = \left( q_k^2 + k_2^2 \right) \left( 1 - \frac{\sin(2q_k)}{2q_k} \right)^{-\frac{1}{2}},
\]
give the orthonormal sets of eigenfunctions \( \{ \psi_{\mathrm{os}, 1, k} \}_{k \in \mathbb{N}} \) and \( \{ \psi_{\mathrm{os}, 2, k} \}_{k \in \mathbb{N}} \) (with respect to \( \langle \cdot, \cdot \rangle_{\mathrm{os}} \)).

**Note:** Even though your analysis will imply that the spectral decompositions of \( L \) in terms of \( \{ \psi_{\mathrm{os}, 1, k} \}_{k \in \mathbb{N}} \) and \( \{ \psi_{\mathrm{os}, 2, k} \}_{k \in \mathbb{N}} \) can be considered separately, you will need to use both sets of eigenfunctions to determine the solution to the Orr-Sommerfeld equation.

(d) Show that the Squire operator \( S \) has the eigenvalues \( \{ \theta_k \}_{k \in \mathbb{N}} \),

\[
\theta_k = -\left( \frac{k\pi}{2} \right)^2 - k_2^2,
\]
with the corresponding eigenfunctions,

\[
u_{\mathrm{sq}, k}(y) = \sin \left( \frac{k\pi}{2} (y + 1) \right).
\]

(e) Using results obtained in Parts (c) and (d) determine the eigenvalue decomposition of the operators \( A \) and \( A^\dagger \). Make sure that bi-orthogonality between the eigenfunctions of \( A \) and \( A^\dagger \) holds.

2. For the convection-diffusion equation that you studied in HW2,

\[
\phi_t(x,t) = \phi_{xx}(x,t) - \phi_x(x,t) \\
\phi(0,0) = f(x) \\
\phi(\pm 1, t) = 0
\]

(a) Show that the generator of the dynamics,

\[
\left\{ [A f](x) = \left[ \frac{d^2 f}{dx^2} - \frac{df}{dx} \right](x) \\
D(A) = \left\{ f \in L_2[-1,1], \frac{d^2 f}{dx^2} \in L_2[-1,1], f(\pm 1) = 0 \right\}
\]

is self-adjoint with respect to the following inner product:

\[
\langle f, g \rangle_w = \int_{-1}^{1} f(x) e^{-x} g(x) \, dx.
\]

(b) Determine the kernel representation of the inverse of the operator \( A \).

3. The cantilever beam, shown in Figure 3, of length \( l \) deflects due to atomic forces between the sample and the cantilever tip. The model of this system is given by

\[
\left\{ \begin{array}{l}
\mu \psi_{tt} + \alpha EI \psi_{xxxx} + EI \psi_{xxxx} = 0, \\
\psi(0,t) = u_1(t), \ \psi_x(0,t) = 0, \\
\psi(\ell, t) = u_2(t), \ \psi_x(\ell, t) = 0,
\end{array} \right.
\]

where \( \mu \) is the mass per unit length of the beam, \( E \) is Young’s modulus, \( I \) is the area moment of inertia of the beam (\( EI \) is also known as the flexural stiffness), and \( \alpha \) is the structural damping factor.
(a) Determine the transfer functions $H_1(s)$ and $H_2(s)$ from the inputs $u_1$ and $u_2$ to the output $y(t) = \psi(l, t)$.

(b) Plot the magnitudes and the phases of the frequency responses $H_1(j\omega)$ and $H_2(j\omega)$ for

$$\mu = 1.88 \times 10^{-7} \text{ kg/m, } EI = 7.55 \times 10^{-12} \text{ Nm}^2, \alpha = 5 \times 10^{-8} \text{ s}.$$ 

(c) Comment on whether the resonant peaks that you uncovered in Part (b) are captured well by the eigenvalues of the dynamical generator in (1) with the following homogeneous boundary conditions:

$$\begin{align*}
\psi(0, t) &= 0, \quad \psi_x(0, t) = 0, \\
\psi_{xx}(l, t) &= 0, \quad \psi_{xxx}(l, t) = 0.
\end{align*}$$