Due Tuesday 11/01/11

1. In HW2 you have shown that the dynamics of infinitesimal "streamwise-constant" fluctuations around the mean velocity $(U(y), 0, 0)$ in a channel flow are governed by the following evolution model

$$\begin{bmatrix} \psi_\tau(\tau) \\ u_\tau(\tau) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{L} & 0 \\ Re\, \mathcal{C}_p & \mathcal{S} \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \psi(\tau) \\ u(\tau) \end{bmatrix}. \qquad \text{(LNSE)}$$

Here, u and ψ denote the streamwise velocity and stream-function fluctuations, respectively, and the operators in (LNSE) are given by

$$\begin{aligned} \text{Orr-Sommerfeld:} \quad \mathcal{L} &= \Delta^{-1}\Delta^2, \\ \text{Squire:} \quad \mathcal{S} &= \Delta, \\ \text{Coupling:} \quad \mathcal{C}_p &= -\,\mathrm{j} k_z\, U'(y), \end{aligned}$$

with k_z denoting the spanwise wave number and,

$$\begin{aligned} \Delta &= \partial_{yy} - k_z^2, \\ \Delta^2 &= \partial_{yyyy} - 2\, k_z^2\, \partial_{yy} + k_z^4. \end{aligned}$$

We will consider the generator $\mathcal{A}$ in (LNSE) on the following Hilbert space

$$\mathbb{H} = \begin{bmatrix} \mathbb{H}_{os} \\ L_2[-1,\, 1] \end{bmatrix}, \quad \mathbb{H}_{os} = \left\{ g \in L^2[-1,1];\ g'' \in L^2[-1,1],\ g(\pm 1) = 0 \right\},$$

with the domain of $\mathcal{A}$ determined by

$$\mathcal{D}(\mathcal{A}) = \begin{bmatrix} \mathcal{D}(\mathcal{L}) \\ \mathcal{D}(\mathcal{S}) \end{bmatrix}, \quad \left\{ \begin{aligned} \mathcal{D}(\mathcal{L}) &= \left\{ g \in \mathbb{H}_{os};\ g^{(4)} \in L^2[-1,1],\ g'(\pm 1) = 0 \right\} \\ \mathcal{D}(\mathcal{S}) &= \mathbb{H}_{os}. \end{aligned} \right.$$

We endow the *state-space* $\mathbb{H}$ with an *inner product*

$$\langle \phi_1, \phi_2 \rangle_e = \langle \phi_1, \mathcal{Q}\, \phi_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard $L^2[-1,\, 1]$ inner product, and $\mathcal{Q}$ is a block diagonal positive self-adjoint operator,

$$\mathcal{Q} = \begin{bmatrix} -\Delta & 0 \\ 0 & I \end{bmatrix}.$$

   (a) Determine the adjoint of the Squire operator $\mathcal{S}$ (with respect to the standard $L_2[-1,\, 1]$ inner product) and the adjoint of the Orr-Sommerfeld operator $\mathcal{L}$ (with respect to $\langle \psi_1, \psi_2 \rangle_{os} = -\langle \psi_1, \Delta \psi_2 \rangle$).

   (b) Determine the adjoint of the operator $\mathcal{A}$ with respect to $\langle \cdot, \cdot \rangle_e$.

   (c) Show that the Orr-Sommerfeld operator $\mathcal{L}$ has two sets of the eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$ and $\{\gamma_k\}_{k \in \mathbb{N}}$,

$$\begin{aligned} \lambda_k &= -(p_k^2 + k_z^2), \\ \gamma_k &= -(q_k^2 + k_z^2), \end{aligned}$$

   with the corresponding eigenfunctions,

$$\begin{aligned} \psi_{os1,k}(y, k_z, \lambda_k) &= A_k \left(\cos(p_k y) - \frac{\cos(p_k)}{\cosh(k_z)} \cosh(k_z y) \right) \\ \psi_{os2,k}(y, k_z, \gamma_k) &= B_k \left(\sin(q_k y) - \frac{\sin(q_k)}{\sinh(k_z)} \sinh(k_z y) \right), \end{aligned}$$

   where p_k and q_k are obtained as solutions to the following two equations,

$$\begin{aligned} p_k \tan(p_k) &= -k_z \tanh(k_z), \\ q_k \cot(q_k) &= k_z \coth(k_z). \end{aligned}$$

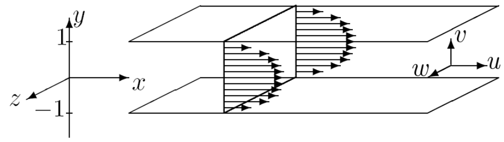


Figure 1: Channel flow geometry.

Convince yourself that the following choices of $\{A_k\}_{k\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$,

$$A_k \;=\; \left((p_k^2 \,+\, k_z^2) \left(1 \,+\, \frac{\sin(2p_k)}{2p_k}\right)\right)^{-\frac{1}{2}},$$
$$B_k \;=\; \left((q_k^2 \,+\, k_z^2) \left(1 \,-\, \frac{\sin(2q_k)}{2q_k}\right)\right)^{-\frac{1}{2}},$$

give the orthonormal sets of eigenfunctions $\{\psi_{os1,k}\}_{k\in\mathbb{N}}$ and $\{\psi_{os2,k}\}_{k\in\mathbb{N}}$ (with respect to $\langle\cdot,\cdot\rangle_{os}$).

Note: Even though your analysis will imply that the spectral decompositions of $\mathcal{L}$ in terms of $\{\psi_{os1,k}\}_{k\in\mathbb{N}}$ and $\{\psi_{os2,k}\}_{k\in\mathbb{N}}$ can be considered separately, you will need to use both sets of eigenfunctions to determine the solution to the Orr-Sommerfeld equation.

(d) Show that the Squire operator $\mathcal{S}$ has the eigenvalues $\{\theta_k\}_{k\in\mathbb{N}}$,

$$\theta_k \;=\; -\left(\left(\frac{k\pi}{2}\right)^2 \,+\, k_z^2\right),$$

with the corresponding eigenfunctions,

$$u_{sq,k}(y) \;=\; \sin\left(\frac{k\pi}{2}\,(y \,+\, 1)\right).$$

(e) Using results obtained in Parts (c) and (d) determine the eigenvalue decomposition of the operators $\mathcal{A}$ and $\mathcal{A}^\dagger$. Make sure that bi-orthogonality between the eigenfunctions of $\mathcal{A}$ and $\mathcal{A}^\dagger$ holds.

2. For the convection-diffusion equation that you studied in HW2,

$$\begin{aligned} \phi_t(x,t) \;&=\; \phi_{xx}(x,t) \;-\; \phi_x(x,t) \\ \phi(x,0) \;&=\; f(x) \\ \phi(\pm 1,t) \;&=\; 0 \end{aligned}$$

(a) Show that the generator of the dynamics,

$$\left\{ \begin{aligned} [\mathcal{A}\,f]\,(x) \;&=\; \left[\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \,-\, \frac{\mathrm{d}f}{\mathrm{d}x}\right](x) \\ \mathcal{D}(\mathcal{A}) \;&=\; \left\{f \,\in\, L_2\,[-1,\,1],\; \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \,\in\, L_2\,[-1,\,1],\; f(\pm 1) \,=\, 0\right\} \end{aligned} \right.$$

is self-adjoint with respect to the following inner product:

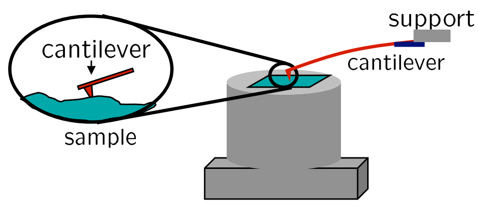
$$\langle f, g\rangle_w \;=\; \int_{-1}^{1} f(x)\,\mathrm{e}^{-x} g(x)\,\mathrm{d}x.$$

(b) Determine the kernel representation of the inverse of the operator $\mathcal{A}$.

3. The cantilever beam, shown in Figure 3, of length l deflects due to atomic forces between the sample and the cantilever tip. The model of this system is given by

$$\left\{ \begin{aligned} \mu\,\psi_{tt} \,+\, \alpha EI\,\psi_{txxxx} \,+\, EI\,\psi_{xxxx} \;&=\; 0, \\ \psi(0,t) \;&=\; u_1(t), \;\; \psi_x(0,t) \;=\; 0, \\ \alpha EI\,\psi_{txxx}(l,t) \,+\, EI\psi_{xxx}(l,t) \;&=\; u_2(t), \;\; \psi_{xx}(l,t) \;=\; 0, \end{aligned} \right. \qquad (1)$$

where μ is the mass per unit length of the beam, E is Young's modulus, I is the area moment of inertia of the beam (EI is also known as the flexural stiffness), and α is the structural damping factor.



(a) Determine the transfer functions $H_1(s)$ and $H_2(s)$ from the inputs u_1 and u_2 to the output

$$y(t) \;=\; \psi(l,t).$$

(b) Plot the magnitudes and the phases of the frequency responses $H_1(\mathrm{j}\omega)$ and $H_2(\mathrm{j}\omega)$ for

$$\mu \;=\; 1.88\times 10^{-7}\,\mathrm{kg/m}, \quad EI \;=\; 7.55\times 10^{-12}\,\mathrm{Nm}^2, \quad \alpha \;=\; 5\times 10^{-8}\,\mathrm{s}.$$

(c) Comment of whether the resonant peaks that you uncovered in Part (b) are captured well by the eigenvalues of the dynamical generator in (1) with the following homogeneous boundary conditions:

$$\left\{ \begin{array}{rcl} \psi(0,t) & = & 0, \;\; \psi_x(0,t) \;=\; 0, \\ \psi_{xx}(l,t) & = & 0, \;\; \psi_{xxx}(l,t) \;=\; 0. \end{array} \right.$$