Due Thursday 09/29/11

1. Consider the following convection-diffusion equation on $L_2[-1,1]$:

$$\begin{aligned} \phi_t(x,t) &= c\,\phi_{xx}(x,t) \;+\; b\,\phi_x(x,t) \;+\; u(x,t) \\ \phi(x,0) &= f(x) \\ \phi(-1,t) &= 0, \quad \phi_x(1,t) \;=\; 0, \end{aligned} \tag{1}$$

with constant coefficients b and $c > 0$.

(a) Show that the following coordinate transformation

$$\phi(x,t) \;=\; \psi(x,t)\,\mathrm{e}^{-b\,x/(2c)}, \quad u(x,t) \;=\; w(x,t)\,\mathrm{e}^{-b\,x/(2c)},$$

can be used to bring system (1) to the following reaction-diffusion equation:

$$\psi_t(x,t) \;=\; c\,\psi_{xx}(x,t) \;-\; \frac{b^2}{4\,c}\,\psi(x,t) \;+\; w(x,t). \tag{2}$$

Determine the initial and boundary conditions that the field ψ has to satisfy.

(b) Determine the adjoint with respect to the standard L_2 inner product of the operator:

$$\begin{cases} [\mathcal{A}\,f]\,(x) \;=\; \left[\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right](x) \\ \mathcal{D}(\mathcal{A}) \;=\; \left\{ f \,\in\, L_2\,[-1,\,1],\; \dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} \,\in\, L_2\,[-1,\,1],\; f(-1) \,=\, 0,\; f'(1) - f(1) \,=\, 0 \right\} \end{cases}$$

Find the equation that the eigenvalues (λ_n) need to satisfy and determine the expression for the corresponding eigenfunctions $(v_n(x))$. Use Newton's method to determine numerically first ten eigenvalues. Check your results using MATLAB's function fsolve.

(c) For $\{b = 2,\, c = 1\}$ determine the solution to system (1) by expanding the field ψ in (2) in terms of the first ten above determined eigenfunctions v_n (please make sure that they are of unit length). For several different time instants, use MATLAB to plot the dependence on spatial coordinates of the convolution kernel of the operator that maps $\phi(\cdot,0)$ to $\phi(\cdot,t)$.

(d) Represent the solution to system (1) in the (x,t)-plane for

$$\phi(x,0) \;=\; x^2 \;-\; 2x \;-\; 3, \quad u(x,t) \;=\; \delta(x-0.5)\,\delta(t).$$

2. For the wave equation on the infinite spatial extent

$$\begin{aligned} \phi_{tt}(x,t) &= c^2\,\phi_{xx}(x,t) \;+\; u(x,t) \\ \phi(x,0) &= f(x), \quad \phi_t(x,0) \;=\; g(x), \quad x \,\in\, \mathbb{R} \end{aligned}$$

(a) Determine the evolution form representation (i.e., the state-space representation).

(b) Use the spatial Fourier transform to show that the wave-number parameterized state-transition matrix is determined by

$$\hat{T}(\kappa,t) \;=\; \begin{bmatrix} \cos\left(c\,\kappa\,t\right) & \sin\left(c\,\kappa\,t\right)/\left(c\,\kappa\right) \\ -c\,\kappa\,\sin\left(c\,\kappa\,t\right) & \cos\left(c\,\kappa\,t\right) \end{bmatrix}.$$

(c) For the initial conditions that are square integrable, show that the solution of the *unforced* wave equation may not be square-integrable.

(d) Determine the dependence of the solution $\phi(x,t)$ on both the initial conditions and the input. Examine the dependence of the solution to the unforced problem with

$$f(x) \;=\; \mathrm{e}^{-x^2}\cos\left(2 \;-\; 1.5x\right), \quad g(x) \;=\; \begin{cases} 1+x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(e) Show that the energy of the wave

$$E(t) \;=\; \frac{1}{2}\int_{-\infty}^{\infty}\big(c^2\,\phi_x^2(x,t) \;+\; \phi_t^2(x,t)\big)\,\mathrm{d}x \;=\; \frac{1}{2}\int_{-\infty}^{\infty}\big(c^2\,\psi_{1x}^2(x,t) \;+\; \psi_2^2(x,t)\big)\,\mathrm{d}x$$

is constant along the solutions of the *unforced* wave equation.