Parameterization of Stabilizing Controllers for Interconnected Systems

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Control, Estimation, and Optimization of Interconnected Systems:
From Theory to Industrial Applications
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Overview

- Motivation: Optimal Constrained Control
- **Linear Time-Invariant**
  - Quadratic invariance
  - Optimal Control over Networks
  - *Summation*
- **Nonlinear Time-Varying**
  - *Iteration*
  - New condition
Block Diagrams

Two-Input Two-Output

Classical
Standard Formulation

\[
\begin{align*}
\text{minimize} & \quad \| P_{11} + P_{12}K(\mathbf{I} - GK)^{-1}P_{21} \| \\
\text{subject to} & \quad K \text{ stabilizes } P
\end{align*}
\]
Communicating Controllers

Control design problem is to find $K$ which is block tri-diagonal.

$$
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4 \\
  u_5 \\
\end{bmatrix} = \begin{bmatrix}
  K_{11} & DK_{12} & 0 & 0 & 0 \\
  DK_{21} & K_{22} & DK_{23} & 0 & 0 \\
  0 & DK_{32} & K_{33} & DK_{34} & 0 \\
  0 & 0 & DK_{43} & K_{44} & DK_{45} \\
  0 & 0 & 0 & DK_{54} & K_{55} \\
\end{bmatrix} \begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  y_4 \\
  y_5 \\
\end{bmatrix}
$$
**General Formulation**

The set of $K$ with a given decentralization constraint is a subspace $S$, called the *information constraint*.

We would like to solve

$$
\begin{align*}
\text{minimize} & \quad \| P_{11} + P_{12}K(I - GK)^{-1}P_{21} \| \\
\text{subject to} & \quad K \text{ stabilizes } P \\
& \quad K \in S
\end{align*}
$$

- For general $P$ and $S$, there is no known tractable solution.
Change of Variables - Stable Plant

minimize \[ \| P_{11} + P_{12}K(I - GK)^{-1}P_{21} \| \]
subject to \[ K \text{ stabilizes } P \]

Using the change of variables

\[ R = K(I - GK)^{-1} \]

we obtain the following equivalent problem

minimize \[ \| P_{11} + P_{12}RP_{21} \| \]
subject to \[ R \text{ stable} \]

This is a convex optimization problem.
Using the change of variables

\[ R = K(I - GK)^{-1} \]

we obtain the following equivalent problem

This is a convex optimization problem.
Breakdown of Convexity - Stable Plant

minimize \[ \|P_{11} + P_{12}K(I - GK)^{-1}P_{21}\| \]
subject to \[ K \text{ stabilizes } P \]
\[ K \in S \]

Using the change of variables

\[ R = K(I - GK)^{-1} \]

we obtain the following equivalent problem

minimize \[ \|P_{11} + P_{12}RP_{21}\| \]
subject to \[ R \text{ stable} \]
\[ R(I + GR)^{-1} \in S \]
Breakdown of Convexity - Stable Plant

Using the change of variables

\[ R = K(I - GK)^{-1} \]

we obtain the following equivalent problem

\[
\begin{align*}
& \text{minimize} & \quad \| P_{11} + P_{12}RP_{21} \| \\
& \text{subject to} & \quad R \text{ stable} \\
& & \quad R(I + GR)^{-1} \in S
\end{align*}
\]
Quadratic Invariance

The set $S$ is called quadratically invariant with respect to $G$ if

$$K GK \in S \quad \text{for all} \quad K \in S$$
Quadratic Invariance

The set $S$ is called quadratically invariant with respect to $G$ if

$$KGK \in S \quad \text{for all } K \in S$$

Main Result

$S$ is quadratically invariant with respect to $G$ if and only if

$$K \in S \quad \iff \quad K(I - GK)^{-1} \in S$$
**Quadratic Invariance**

The set $S$ is called quadratically invariant with respect to $G$ if

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**Main Result**

$S$ is quadratically invariant with respect to $G$ if and only if

$$K \in S \quad \iff \quad K(I - GK)^{-1} \in S$$

**Parameterization**

$$\left\{ K \mid K \text{ stabilizes } P, \ K \in S \right\} = \left\{ R(I + GR)^{-1} \mid R \text{ stable, } R \in S \right\}$$
Optimal Stabilizing Controller

Suppose $G \in \mathcal{R}_{sp}$ and $S \subseteq \mathcal{R}_p$.

We would like to solve

$$\begin{align*}
\text{minimize} & \quad \|P_{11} + P_{12}K(I - GK)^{-1}P_{21}\| \\
\text{subject to} & \quad K \text{ stabilizes } P \\
& \quad K \in S
\end{align*}$$
Optimal Stabilizing Controller

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We would like to solve

\[
\begin{align*}
\text{minimize} & \quad \| P_{11} + P_{12}K(I - GK)^{-1}P_{21} \| \\
\text{subject to} & \quad K \text{ stabilizes } P \\
& \quad K \in S
\end{align*}
\]

If $S$ is quadratically invariant with respect to $G$, we may solve

\[
\begin{align*}
\text{minimize} & \quad \| P_{11} + P_{12}RP_{21} \| \\
\text{subject to} & \quad R \text{ stable} \\
& \quad R \in S
\end{align*}
\]

which is convex.
Assuming transmission delays satisfy the triangle inequality (i.e. transmissions take the quickest path)

$S$ is quadratically invariant with respect to $G$ if

$$t_{ij} \leq p_{ij} \quad \text{for all } i, j$$
Distributed Control with Delays

- $K_3$ sees information from $G_3$ immediately
  - $G_2, G_4$ after a delay of $t$
  - $G_1, G_5$ after a delay of $2t$

- $G_3$ is affected by inputs from $K_3$ immediately
  - $K_2, K_4$ after a delay of $p$
  - $K_1, K_5$ after a delay of $2p$
Distributed Control with Delays

\[ S \text{ is quadratically invariant with respect to } G \text{ if } t \leq p \]
Distributed Control with Delays

- $K_3$ sees information from $G_3$ after a delay of $c$
  
  $G_2$, $G_4$ after a delay of $c + t$
  
  $G_1$, $G_5$ after a delay of $c + 2t$

- $G_3$ is affected by inputs from $K_3$ immediately
  
  $K_2$, $K_4$ after a delay of $p$
  
  $K_1$, $K_5$ after a delay of $2p$
Without computational delay, $S$ is quadratically invariant with respect to $G$ if

$$\quad t \leq p$$
Without computational delay, \( S \) is quadratically invariant with respect to \( G \) if

\[
t \leq p
\]

If computational delay is also present, then \( S \) is quadratically invariant with respect to \( G \) if

\[
t \leq p + \frac{c}{n-1}
\]
Two-Dimensional Lattice

Assuming controllers communicate along edges
Two-Dimensional Lattice

Assuming controllers communicate along edges

Assuming dynamics propagate along edges
Two-Dimensional Lattice

$S$ is quadratically invariant with respect to $G$ if

$$t \leq p$$
Assuming controllers communicate along edges
Two-Dimensional Lattice

Assuming controllers communicate along edges

Assuming dynamics propagate outward so that delay is proportional to geometric distance
Two-Dimensional Lattice

$S$ is quadratically invariant with respect to $G$ if

$$t \leq \frac{p}{\sqrt{2}}$$
Sparsity Example

Suppose

\[ G \sim \begin{bmatrix}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet
\end{bmatrix} \]

\[ S = \left\{ K \mid K \sim \begin{bmatrix}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet
\end{bmatrix} \right\} \]

For arbitrary \( K \in S \)

\[ KGK \sim \begin{bmatrix}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet
\end{bmatrix} \]

Hence \( S \) is quadratically invariant with respect to \( G \).
Suppose

\[ G \sim \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \quad S = \left\{ K \mid K \sim \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \right\} \]

Then

\[ KGK \sim \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \]

so \( S \) is not quadratically invariant with respect to \( G \).

In fact,

\[ K(I - GK)^{-1} \sim \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \]
Small Gain

If $|a| < 1$ then

$$(1 - a)^{-1} = 1 + a + a^2 + a^3 + \ldots$$

for example

$$\left(1 - \frac{1}{2}\right)^{-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

more generally, if $\|A\| < 1$ then

$$(1 - A)^{-1} = I + A + A^2 + A^3 + \ldots$$
Not So Small Gain

Let

\[ Q = I + A + A^2 + A^3 + \ldots \]

then

\[ AQ = A + A^2 + A^3 + \ldots \]

subtracting we get

\[ (I - A)Q = I \]

and so

\[ Q = (I - A)^{-1} \]
Convergence to Unstable Operator

- Let $W(s) = \frac{2}{s+1}$
- Plot shows impulse response of $\sum_{k=0}^{N} W^k$ for $N = 1, \ldots, 7$
- Converges to that of $\frac{I}{I-W} = \frac{s+1}{s-1}$
- In what topology do the associated operators converge?
Conditions for Convergence: Inert

- Very broad, reasonable class of plants and controllers
- Basically, impulse response must be finite at any time $T$
- Arbitrarily large
- Includes the case $G \in \mathcal{R}_{sp}, \ K \in \mathcal{R}_p$
Proof Sketch

\[ KGK \in S \quad \text{for all } K \in S \]

Suppose \( K \in S \). Then

\[ K(I - GK)^{-1} = K + KGK + K(GK)^2 + \ldots \in S \]
Consider Nonlinear

Let

\[ Q = I + A + A^2 + A^3 + \ldots \]

then

\[ AQ = A + A^2 + A^3 + \ldots \]

subtracting we get

\[ (I - A)Q = I \]

and so

\[ Q = (I - A)^{-1} \]
For a given $r$, we seek $y, u$ such that

$$y = r + Gu$$
$$u = Ky$$

and then define $Y, R$ such that

$$y = Yr = (I - GK)^{-1}r$$
$$u = Rr = K(I - GK)^{-1}r$$
Iteration of Signals

We can define the following iteration, commensurate with the diagram

\[
\begin{align*}
  y^{(0)} &= r \\
  u^{(n)} &= Ky^{(n)} \\
  y^{(n+1)} &= r + Gu^{(n)}
\end{align*}
\]
Iteration of Signals and Operators

We can define the following iterations, commensurate with the diagram:

\[
\begin{align*}
y^{(0)} &= r \\
u^{(n)} &= Ky^{(n)} \\
y^{(n+1)} &= r + Gu^{(n)} \\
Y^{(0)} &= I \\
R^{(n)} &= KY^{(n)} \\
Y^{(n+1)} &= I + GR^{(n)}
\end{align*}
\]
Iteration of Signals

We then get the following recursions

\[
\begin{align*}
y^{(0)} & = r \\
y^{(n+1)} & = r + GK y^{(n)} \\
u^{(0)} & = Kr \\
u^{(n+1)} & = K(r + Gu^{(n)})
\end{align*}
\]
Iteration of Signals

We then get the following recursions

\[
\begin{align*}
y^{(0)} &= r \\
y^{(n+1)} &= r + GKy^{(n)} \\
\vdots \\
y &= \lim_{n \to \infty} y^{(n)}
\end{align*}
\]

\[
\begin{align*}
u^{(0)} &= Kr \\
u^{(n+1)} &= K(r + Gu^{(n)}) \\
\vdots \\
u &= \lim_{n \to \infty} u^{(n)}
\end{align*}
\]
Iteration of Operators

We then get the following recursions

\[
Y^{(0)} = I \\
Y^{(n+1)} = I + GKY^{(n)} \\
\vdots \\
Y = \lim_{n \to \infty} Y^{(n)}
\]

\[
R^{(0)} = K \\
R^{(n+1)} = K(I + GR^{(n)}) \\
\vdots \\
R = \lim_{n \to \infty} R^{(n)}
\]
Conditions for Convergence

- Very broad, reasonable class of plants and controllers
- Arbitrarily large
- Includes the case $G$ strictly causal, $K$ causal
- Includes the case $G \in \mathcal{R}_{sp}, K \in \mathcal{R}_p$
Parameterization - Nonlinear

\[ \{ K \mid K \text{ stabilizes } G \} = \left\{ \frac{1}{R(I + GR)} \mid R \text{ stable} \right\} \]

New Invariance Condition

\[ K_1(I \pm GK_2) \in S \quad \text{for all } K_1, K_2 \in S \]
New Invariance Condition

\[ K_1(I \pm GK_2) \in S \quad \text{for all } K_1, K_2 \in S \]

Invariance under Feedback

If this condition is satisfied, then

\[ K \in S \iff K(I - GK)^{-1} \in S \]
New Invariance Condition

\[ K_1(I \pm GK_2) \in S \quad \text{for all } K_1, K_2 \in S \]

Invariance under Feedback

If this condition is satisfied, then

\[ K \in S \quad \iff \quad K(I - GK)^{-1} \in S \]

Parameterization

\[ \{ K \mid K \text{ stabilizes } G, \ K \in S \} = \left\{ R(I + GR)^{-1} \mid R \text{ stable, } R \in S \right\} \]
Proof Sketch

\[ K_1(I \pm GK_2) \in S \quad \forall \ K_1, K_2 \in S \]

Suppose that \( K \in S \).

Then \( R^{(0)} = K \in S \).

If we assume that \( R^{(n)} \in S \), then

\[ R^{(n+1)} = K(I + GR^{(n)}) \in S \]

Thus \( R^{(n)} \in S \) for all \( n \in \mathbb{Z}_+ \).

\[ R = \lim_{n \to \infty} R^{(n)} \in S \]
Conclusions

- Quadratic invariance allows parameterization of all (LTI) stabilizing decentralized controllers.
- Similar condition allows parameterization of all (NLTV) stabilizing decentralized controllers.
- These conditions are satisfied when communications are faster than the propagation of dynamics.
Quadratic Invariance

The set $S$ is called quadratically invariant with respect to $G$ if

$$K G K \in S \quad \text{for all } K \in S$$

$$K_1 (I \pm G K_2) \in S \quad \text{for all } K_1, K_2 \in S$$
Open Questions / Future Work

- Unstable plant
- Is there a weaker condition which achieves the same results? Perhaps
  \[ K(I \pm GK) \in S \quad \forall K \in S \] ?
- When is the optimal (possibly nonlinear) controller linear?
- When (else) does it hold?
- What should we call it?