Distributed Control: Optimality and Architecture

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Large Arrayed Systems of Sensors and Actuators

- **New (and old) technologies**
  - Micro-Electro-Mechanical-Systems (MEMS) → Large Arrays
  - Vehicular Platoons
  - Cross Directional (CD) control in pulp and paper processes

- **Modeling and control issues**
  - Complexity (Control-Oriented Modeling)
  - Overall *System Design* (vs. individual device design)
  - Controller architecture

- **Distributed Systems Theory**
  - Infinite-dimensional systems with special structure
  - Controller architecture
Currently feasible: Very large arrays of MEMS with integrated control circuitry

Issues:

- Tightly coupled dynamics

  Current designs avoid this with large spacing

- Controller architecture
  - Layout of sensors/actuators
  - Communication between actuators/sensors

  *how to decentralize or localize*
2. The Millipede concept

The 2D AFM cantilever array storage technique [8, 9] called "Millipede" is illustrated in Figure 1. It is based on a mechanical parallel $x/y$ scanning of either the entire cantilever array chip or the storage medium. In addition, a feedback-controlled $z$-approaching and -leveling scheme brings the entire cantilever array chip into contact with the storage medium. This tip–medium contact is maintained and controlled while $x/y$ scanning is performed for write/read. It is important to note that the Millipede approach is not based on individual $z$-feedback for each cantilever; rather, it uses a feedback control for the entire chip, which greatly simplifies the system. However, this requires stringent control and uniformity of tip height and cantilever bending. Chip approach and leveling make use of four integrated approaching cantilever sensors in the corners of the array chip to control the approach of the chip to the storage medium. Signals from three sensors (the fourth being a spare) provide feedback signals to adjust three magnetic $z$-actuators until the three approaching sensors are in contact with the medium. The three sensors with the individual feedback loop maintain the chip leveled and in contact with the surface while $x/y$ scanning is performed for write/read operations. The system is thus leveled in a manner similar to an antivibration air table. This basic concept of the entire chip approach/leveling has been tested and demonstrated for the first time by parallel imaging with a $5 \times 5$ array chip [10]. These parallel imaging results have shown that all 25 cantilever tips have approached the substrate within less than 1 micrometer of $z$-activation. This promising result has led us to believe that chips with a tip-apex height control of less than 500 nm are feasible. This stringent requirement for tip-apex uniformity over the entire chip is a consequence of the uniform force needed to minimize or eliminate tip and medium wear due to large force variations resulting from large tip-height nonuniformities [4].

During the storage operation, the chip is raster-scanned over an area called the storage field by a magnetic $x/y$ scanner. The scanning distance is equivalent to the cantilever $x/y$ pitch, which is currently 92 micrometers. Each cantilever/tip of the array writes and reads data only in its own storage field. This eliminates the need for lateral...
Design and Control Issues in MEMS Arrays

- More tightly packed arrays → more dynamical coupling
  - Micro-cantilever arrays
  - Micro-mirror arrays

- Current fixes:
  - Large spacings
  - Complex design to isolate elements

- Experimental effort at UCSB:
  design deliberately coupled arrays

- Demonstrate "electronic" decoupling using feedback
Capacitively actuated micro-cantilevers: *Combined* actuator and sensor

**Important Considerations:**

- Higher throughput, faster “access time” \(\rightarrow\) Tightly packed cantilevers

- For tightly packed cantilevers, significant dynamical coupling due to
  - Mechanical coupling
  - Fringe fields
    (Napoli & Bamieh, ’01)

- Large arrays \(\approx 10,000\) devices
  \(\Rightarrow\) must use localized control
Distributed Systems with Special Structure

- General Infinite-dimensional Systems Theory
  - Well posedness issues (semi-group theory)
  - Constructive (convergent) approximation techniques

  **Theme:** Make infinite-dimensional problems look like finite-dimensional ones

- Special Structure
  - Distributed control and measurement *(now more feasible)*
  - Regular (lattice) arrangement of devices

  Together \(\implies\) **Spatial Invariance**

  - Control of “Vehicular Strings”, (Melzer & Kuo, 71)
  - Discretized PDEs, (Brockett, Willems, Krishnaprasd, El-Sayed, ’74, ’81)
  - “Systems over rings”, (Kamen, Khargonekar, Sontag, Tannenbaum, ...)
  - Systems with “Dynamical Symmetry”, (Fagniani & Willems)

More recently:

- Controller architecture and localization, (Bamieh, Paganini, Dahleh)
- LMI techniques, localization, (D’Andrea, Dullerud, Lall)
**Example: Distributed Control of the Heat Equation**

\[ y_i: \text{ input to heating elements.} \quad y_i: \text{ signal from temperature sensor.} \]

**Dynamics are given by:**

\[
\begin{bmatrix}
\vdots \\
y_{-1} \\
y_0 \\
y_1 \\
\vdots 
\end{bmatrix}
\begin{bmatrix}
\vdots \\
H_{0,-1} \\
H_{0,0} \\
H_{0,1} \\
\vdots 
\end{bmatrix}
\begin{bmatrix}
\vdots \\
u_{-1} \\
u_0 \\
u_1 \\
\vdots 
\end{bmatrix}
= 
\begin{bmatrix}
\vdots \\
y_{-1} \\
y_0 \\
y_1 \\
\vdots 
\end{bmatrix}
\]

Each \( H_{i,j} \) is an infinite-dimensional SISO system.

**Fact:** Dynamics are spatially invariant \( \Rightarrow H \) is Toeplitz

The input-output relation can be written as a **convolution over the actuator/sensor index:**

\[
y_i = \sum_{j=-\infty}^{\infty} \bar{H}_{(i-j)} u_j,
\]
The limit of large actuator sensor array:

\[
\frac{\partial \psi}{\partial t}(x,t) = c \frac{\partial^2 \psi}{\partial x^2}(x,t) + u(x,t)
\]

\[
\psi_x = \int_{-\infty}^{\infty} H_{x-\zeta} u_\zeta d\zeta,
\]
Vehicular Platoons

\[ w_{-1} \quad w_o \quad w_1 \quad w_2 \]

\[ x_{-1} \quad L \quad x_o \quad L \quad x_1 \quad L \quad x_2 \]

**Objective:** Design a controller for each vehicle to:

- Maintain constant small slot length $L$.
- Reject the effect of disturbances $\{w_i\}$ (wind gusts, road conditions, etc...)

**Warning:** Designs based on two vehicle models may lack “string stability”, i.e. disturbances get amplified as they propagate through the platoon.

**Problem Structure:**

- **Actuators:** each vehicle’s throttle input.
- **Sensors:** position and velocity of each vehicle.
Vehicular Platoons Set-up

$x_i$: $i$’th vehicle’s position.

\[ \tilde{x}_i := x_i - x_{i-1} - L - C \]
\[ \tilde{x}_{1,i} := \tilde{x}_i \]
\[ \tilde{x}_{2,i} := \dot{\tilde{x}}_i \]

Structure of generalized plant:

\[ H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} \times & \times & 0 \\ \times & \cdots & h_o \\ \times & \cdots & h_1 \end{bmatrix} \]

The generalized plant has a Toeplitz structure!

\[ z = \mathcal{F}(H, C) \]
Example: Centralized $\mathcal{H}^2$ optimal controller gains for a 50 vehicle platoon (From: Shu and Bamieh ’96)

Remarks:

- For large platoons, optimal controller is approximately Toeplitz.
- Optimal centralized controller has some inherent decentralization (“localization”)

Controller gains decay away from the diagonal

Q: Do the above 2 results occur in all “such” problems?
Indexing of actuator and sensor signals:

\[ u_i(t) := u(i_1,\ldots,i_n)(t), \quad y_i(t) := y(i_1,\ldots,i_n)(t). \]

\( i := (i_1, \ldots, i_n) \) a spatial multi-index, \( i \in \mathbb{G} := \mathbb{G}_1 \times \ldots \times \mathbb{G}_n. \)

Linear input-output relations:

A general linear system from \( u \) to \( y \):

\[ y_i = \sum_{j \in \mathbb{G}} H_{i,j} u_j, \quad \Leftrightarrow \quad y(i_1,\ldots,i_n) = \sum_{j_1 \in \mathbb{G}_1} \ldots \sum_{j_n \in \mathbb{G}_n} H_{(i_1,\ldots,i_n), (j_1,\ldots,j_n)} u_{(j_1,\ldots,j_n)}. \]

Spatial Invariance:

**Assumption 1:** Set of spatial indices \( \mathbb{G} \) = commutative group

\[ \mathbb{G} := \mathbb{G}_1 \times \ldots \times \mathbb{G}_n, \quad \text{each } \mathbb{G}_i \text{ a commutative group.} \]

**Remark:** “spatial shifting” of signals

\[ (S_\sigma u)_i := u_{i-\sigma} \]

Compare with: *Time shift by \( \tau \) \( \left( S_\tau u \right)(t) := u(t - \tau) \)

**Assumption 2:** Spatial invariance \( \Leftrightarrow \) Commute with spatial shifts

\[ \forall \sigma \in \mathbb{G}, \quad H S_\sigma = S_\sigma H \quad \Leftrightarrow \quad S_\sigma^{-1} H S_\sigma = H \]
Examples of Spatial Invariance

**Generally:** Spatial invariance easily ascertained from basic physical symmetry!

- Vehicular platoons: signals index over \( \mathbb{Z} \).

- Channel flow: Signals indexed over \( \{0, 1\} \times \mathbb{Z} : \)

\[
y(l, i) = \sum_{j=-\infty}^{\infty} H(l - 0, i - j) u(0, j) + \sum_{j=-\infty}^{\infty} H(l - 1, i - j) u(1, j), \quad l = 0, 1.
\]

**Remark:** The input-output mapping of a spatially invariant system can be rewritten:

\[
y_i = \sum_{j \in G} \tilde{G}_{i - j} \ u_j, \quad \Leftrightarrow \quad y(i_1, \ldots, i_n) = \sum_{j_1 \in G_1} \cdots \sum_{j_n \in G_n} \tilde{G}(i_1 - j_1, \ldots, i_n - j_n) \ u(j_1, \ldots, j_n).
\]

A spatial convolution
Many-body systems always have some inherent dynamical symmetries: e.g. equations of motion are invariant to certain coordinate transformations.

**Question:** Given an unstable dynamical system with a certain symmetry, is it possible to stabilize it with a controller that has the same symmetry? (i.e. without “breaking the symmetry”)

**Answer:** Yes! (Fagnani & Willems ’93)

**Remark:** Spatial invariance is a dynamical symmetry. This answer applies to optimal design as well.

i.e. For best achievable performance, need only consider spatially-invariant controllers.
The standard problem:

**Signal norms:**

\[
\|w\|_p^p := \sum_{i \in G} \left( \int_{\mathbb{R}} |w_i(t)|^p dt \right) = \sum_{i \in G} \|w\|_p^p
\]

**Induced system norms:**

\[
\| \mathcal{F}(G, C) \|_{p-i} := \sup_{w \in L^p} \frac{\|z\|_p}{\|w\|_p}.
\]

The \( \mathcal{H}^2 \) norm:

\[
\| \mathcal{F}(G, C) \|^2_{\mathcal{H}^2} = \|z\|^2_2 = \sum_{i \in G} \|z_i\|^2_{L^2},
\]

with impulsive disturbance input \( w_i(t) = \delta(i)\delta(t) \).

**Note:** In the platoon problem: finite system norm \( \Rightarrow \) string stability.
Spatially-Invariant vs. Spatially-Varying Controllers

**Question:** Are spatially-varying controllers better than spatially-invariant ones?

**Answer:** If plant is spatially invariant, no!

$LSI :=$ The class of Linear Spatially-Invariant systems.

$LSV :=$ The class of Linear Spatially-Varying systems.

Compare the two problems:

\[
\gamma_{si} := \inf_{\text{stabilizing } C} \| F(G, C) \|_{p-i} \quad \gamma_{sv} := \inf_{\text{stabilizing } C} \| F(G, C) \|_{p-i}
\]

The best achievable performance with spatially-invariant controllers

The best achievable performance with spatially-varying controllers

**Theorem 1.** If the plant and performance objectives are spatially invariant, i.e. if the generalized plant $G$ is spatially invariant, then the best achievable performance can be approached with a spatially invariant controller. More precisely

\[
\gamma_{si} = \gamma_{sv}.
\]
Related Problem: *Time-Varying vs. Time-Invariant Controllers*

**Fact:** For time-invariant plants, time-varying controllers offer no advantage over time-invariant ones! 

*for norm minimization problems*

Proofs based on use of YJBK parameterization. Convert to

\[
\gamma_{ti} := \inf_{\text{stable } Q} \| T_1 - T_2 Q T_3 \|, \quad \gamma_{tv} := \inf_{\text{stable } Q} \| T_1 - T_2 Q T_3 \|, \quad Q \in \mathcal{LT}_I
\]

\[
Q \in \mathcal{LT}_V
\]

\(T_1, T_2, T_3\) determined by plant, therefore time invariant.
Spatially-Invariant vs. Spatially-Varying Controllers (Cont.)

Related Problem: *Time-Varying vs. Time-Invariant Controllers*

**Fact:** For time-invariant plants, time-varying controllers offer no advantage over time-invariant ones! *for norm minimization problems*

Proofs based on use of YJBK parameterization. Convert to

\[
\gamma_{ti} := \inf_{\text{stable } Q \in LT I} \| T_1 - T_2 QT_3 \| \\
\gamma_{tv} := \inf_{\text{stable } Q \in LT V} \| T_1 - T_2 QT_3 \|
\]

\( T_1, T_2, T_3 \) determined by plant, therefore time invariant.

- **The \( H_\infty \) case:** (Feintuch & Francis, ’85), (Khargonekar, Poolla, & Tannenbaum, ’85). A consequence of Nehari’s theorem
- **The \( \ell^1 \) case:** (Shamma & Dahleh, ’91). Using an averaging technique
- **Any induced \( \ell^p \) norm:** (Chapellat & Dahleh, ’92). Generalization of the averaging technique
Idea of proof: After YJBK parameterization:

$$
\gamma_{si} := \inf_{\text{stable } Q, Q \in LSI} \|T_1 - T_2 Q T_3\| \quad \geq \quad \gamma_{sv} := \inf_{\text{stable } Q, Q \in LSV} \|T_1 - T_2 Q T_3\|,
$$

Let $\bar{Q}$ achieve a performance level $\bar{\gamma} = \|T_1 - T_2 \bar{Q} T_3\|$.

Averaging $\bar{Q}$:

- If $G$ is finite: define

$$
Q_{av} := \frac{1}{|G|} \sum_{\sigma \in G} \sigma^{-1} \bar{Q} \sigma. \quad \rightarrow \quad Q_{av} \text{ is spatially invariant, i.e. } \forall \sigma \in G, \quad \sigma^{-1} Q_{av} \sigma = Q_{av}
$$

Then

$$
\|T_1 - T_2 Q_{av} T_3\| = \|T_1 - T_2 \left( \frac{1}{|G|} \sum_{\sigma \in G} \sigma^{-1} \bar{Q} \sigma \right) T_3\| = \left\| \frac{1}{|G|} \sum_{\sigma \in G} \sigma^{-1} (T_1 - T_2 \bar{Q} T_3) \sigma \right\| \\
\leq \frac{1}{|G|} \sum_{\sigma \in G} \left\| \sigma^{-1} (T_1 - T_2 \bar{Q} T_3) \sigma \right\| = \|T_1 - T_2 \bar{Q} T_3\|
$$
If $\mathcal{G}$ is infinite, take a sequence of finite subsets $M_1 \subset M_2 \subset \cdots$, with $\bigcup_n M_n = \mathcal{G}$.

Then define: $$Q_n := \frac{1}{|M_n|} \sum_{\sigma \in M_n} \sigma^{-1} \bar{Q}_\sigma.$$

$Q_n$ converges weak $*$ to a spatially-invariant $Q_{av}$ with the required norm bound.
Poiseuille flow stabilization:

\[ u_i = \sum_{j} C_{i-j} y_j \]
Implications of the Structure of Spatial Invariance

Poiseuille flow stabilization:

\[ u_i = \sum_j C_{i-j} y_j \]
Uneven distribution of sensors and actuators

Consider the following geometry of sensors and actuators:

- Sensor
- Actuator

What kind of spatial invariance do optimal controllers have?
Uneven distribution of sensors and actuators (Cont.)

Consider the following geometry of sensors and actuators:

- Sensor
- Actuator

Each “cell” is a 1-input, 2-output system. The underlying group is $\mathbb{Z} \times \mathbb{Z}$.
Consider the following PDE with distributed control:
\[
\frac{\partial \psi}{\partial t}(x_1, \ldots, x_n, t) = A \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \psi(x_1, \ldots, x_n, t) + B \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) u(x_1, \ldots, x_n, t)
\]
\[
y(x_1, \ldots, x_n, t) = C \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \psi(x_1, \ldots, x_n, t),
\]
where \( A, B, C \) are matrices of polynomials in \( \frac{\partial}{\partial x_i} \).

Consider also combined PDE difference equations such as:
\[
\frac{\partial \psi}{\partial t}(x_1, \ldots, x_m, k_1, \ldots, k_n, t) = A \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, z_1^{-1}, \ldots, z_n^{-1} \right) \psi(x_1, \ldots, x_n, k_1, \ldots, k_n, t)
\]
\[
+ B \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, z_1^{-1}, \ldots, z_n^{-1} \right) u(x_1, \ldots, x_n, k_1, \ldots, k_n, t)
\]

We only require that the spatial variables \( x, k \), belong to a commutative group.

Taking the Fourier transform:
\[
\hat{\psi}(\lambda, t) := \int_G e^{-j<\lambda, x>} \psi(x, t) \, dx,
\]
The above system equations become:

\[ \frac{d\hat{\psi}(\lambda, t)}{dt} = A(\lambda) \hat{\psi}(\lambda, t) + B(\lambda) \hat{u}(\lambda, t) \]

\[ \hat{y}(\lambda, t) = C(\lambda) \hat{\psi}(\lambda, t), \]

where \( \lambda \in \hat{G} \), the dual group to \( G \).

Remark: This can be thought of as a parameterized family of finite-dimensional systems.
The Fourier transform converts:

spatially-invariant operators on $L_2(G) \rightarrow$ multiplication operators on $L_2(\hat{G})$

In general:

<table>
<thead>
<tr>
<th>group: $G$</th>
<th>dual group: $\hat{G}$</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>$\mathbb{R}$</td>
<td>Fourier Transform</td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>$\partial \mathbb{D}$</td>
<td>Z-Transform</td>
</tr>
<tr>
<td>$\partial \mathbb{D}$</td>
<td>$\mathbb{Z}$</td>
<td>Fourier Series</td>
</tr>
<tr>
<td>$\mathbb{Z}_n$</td>
<td>$\mathbb{Z}_n$</td>
<td>Discrete Fourier Transform</td>
</tr>
</tbody>
</table>

and the transforms preserve $L_2$ norms:

$$\|f\|_2^2 = \int_G |f(x)|^2 dx = \int_{\hat{G}} |\hat{f}(\lambda)|^2 d\lambda = \|\hat{f}\|_2^2$$

The system operation is then spatially decoupled or “block diagonalized”:

$$\frac{\partial}{\partial t} \psi(x, t) = A \psi(x, t) + B u(x, t) \quad \frac{\partial}{\partial t} \hat{\psi}(\lambda, t) = \hat{A}(\lambda)\hat{\psi}(\lambda, t) + \hat{B}(\lambda)\hat{u}(\lambda, t)$$

$$y(x, t) = C \psi(x, t) + D u(x, t) \quad \hat{y}(\lambda, t) = \hat{C}(\lambda)\hat{\psi}(\lambda, t) + \hat{D}(\lambda)\hat{u}(\lambda, t)$$

A distributed, spatially-invariant system \hspace{1cm} A parameterized family of finite-dimensional systems
**TRANSFORM METHODS**

In physical space

\[ \frac{d}{dt} \psi_n = A_n \ast \psi_n + B_n \ast u_n \]

\[ y_n = C_n \ast \psi_n \]

After spatial Fourier trans. (FT)

\[ \frac{d}{dt} \hat{\psi}(\theta) = \hat{A}(\theta) \hat{\psi}(\theta) + \hat{B}(\theta) \hat{u}(\theta) \]

\[ \hat{y}(\theta) = \hat{C}(\theta) \hat{\psi}(\theta) \]

**IMPLICATIONS**

- Dynamics are decoupled by FT \((The \ A, \ B, \ C \ operators \ are \ “diagonalized”)\)
- Quadratic forms preserved by FT \(\implies\) Quadratically optimal control problems are equivalent for FT
- Yields a parametrized family of mutually independent problems

**TRANSFER FUNCTIONS**

operator-valued transfer function

\[ \mathcal{H}(s) = C (sI - A)^{-1} B \]

spatio-temporal transfer function

\[ H(s, \theta) = \hat{C}(\theta) \left( sI - \hat{A}(\theta) \right)^{-1} \hat{B}(\theta) \]

A multi-dimensional system with temporal, but not spatial causality
\[
\frac{\partial}{\partial t} \psi(x, t) = c \frac{\partial^2}{\partial x^2} \psi(x, t) + u(x, t) \quad \rightarrow \quad \frac{d}{dt} \hat{\psi}(\lambda, t) = -c\lambda^2 \hat{\psi}(\lambda, t) + \hat{u}(\lambda, t)
\]

Solve the LQR problem with \( Q = qI \), \( R = I \). The corresponding ARE family:

\[-2c\lambda^2 \hat{p}(\lambda) - \hat{p}(\lambda)^2 + q = 0,\]

and the positive solution is:

\[\hat{p}(\lambda) = -c\lambda^2 + \sqrt{c^2 \lambda^4 + q}.\]

**Remark:** In general \( \hat{P}(\lambda) \) an irrational function of \( \lambda \), even if \( \hat{A}(\lambda), \hat{B}(\lambda) \) are rational. **i.e.** PDE systems have optimal feedbacks which are *not* PDE operators.

Let \( \{k(x)\} \) be the inverse Fourier transform of the function \( \{-\hat{p}(\lambda)\} \).
Then *optimal (temporally static) feedback*

\[
   u(x, t) = \int_{\mathbb{R}} k(x - \xi) \psi(\xi, t) \, d\xi
\]

**Remark:** The “spread” of \( \{k(x)\} \) indicates information required from distant sensors.
Important Observation: \( \{ k(x) \} \) is "localized". It decays exponentially!!

\[
\hat{k}(\lambda) = c\lambda^2 - \sqrt{c^2\lambda^4 + q}.
\]

This can be analytically extended by:

\[
\hat{k}_e(s) = cs^2 - \sqrt{c^2s^4 + q},
\]

which is analytic in the strip

\[
\left\{ s \in \mathbb{C} ; \text{Im}\{s\} < \frac{\sqrt{2}}{2} \left( \frac{q}{c^2} \right)^{\frac{1}{4}} \right\}.
\]

Therefore: \( \exists M \) such that

\[
|k(x)| \leq Me^{-\alpha|x|}, \quad \text{for any } \alpha < \frac{\sqrt{2}}{2} \left( \frac{q}{c^2} \right)^{\frac{1}{4}}.
\]

This results is true in general: under mild conditions

Solutions of AREs always inverse transform to exponentially decaying convolution kernels
Parameterized ARE solutions yield “localized” operators!

Consider unbounded domains, i.e. $\mathbb{G} = \mathbb{R}$ (or $\mathbb{Z}$).

**Theorem 2.** Consider the parameterized family of Riccati equations:

$$A^*(\lambda)P(\lambda) + P(\lambda)A(\lambda) - P(\lambda)B(\lambda)R(\lambda)B^*(\lambda)P(\lambda) + Q(\lambda) = 0, \quad \lambda \in \mathbb{G}.$$  

*Under mild conditions:* 
there exists an analytic continuation $P(s)$ of $P(\lambda)$ in a region 

$$\{|Im(s)| < \alpha\}, \quad \alpha > 0.$$ 

Convolution kernel resulting from Parameterized ARE has exponential decay. That is, they have some degree of inherent decentralization ("localization")!

**Comparison:**

- **Modal truncation:** In the transform domain, ARE solutions decay algebraically.

- **Spatial truncation:** In the spatial domain, convolution kernel of ARE solution decays exponentially.

**Therefore:** Use transform domain to design $\forall \lambda$. Approximate in the spatial domain!
DISTRIBUTED ARCHITECTURE OF QUADRATICALLY OPTIMAL CONTROLLERS

Observer based controller has the following structure:

\[
\frac{d}{dt} \psi_n = A_n \psi_n + B_n u_n \\
y_n = C_n \psi_n
\]

\[
u_i = K_i \hat{\psi}_i
\]

\[
\frac{d}{dt} \hat{\psi}_n = A_n \hat{\psi}_n + B_n u_n \\
+ L_n (y_n - \hat{y}_n)
\]

REMARKS:

• Optimal Controller is “locally” finite dimensional.

• The gains \( \{K_i\}, \{L_i\} \) are localized (exponentially decaying) → “spatial truncation”

• After truncation, local controller need only receive information from neighboring subsystems.

• Quadratically optimal controllers are inherently distributed and semi-decentralized \((\text{localized})\)
The many remaining issues

- Various heterogeneities
  - Spatial variance
  - Irregular arrangements of sensors and actuators

- How to specify “localization” apriori

- The complexities of “high order”
  - The phenomenology of linear infinite dimensional systems can be arbitrarily complex
Outline

- **Background**
  - Distributed control and sensing
  - Useful idealizations, e.g. spatial invariance

- **Structured problems**
  - Constrained information passing structures
    - Decentralized, Localized, etc..
  - Information passing structures which lead to convex problems

- **Issues of large scale**
  - Performance as a function of system size
  - Ex: Fundamental limitations in controlling *Vehicular Platoons*
Centralized vs. Decentralized control: An old and difficult problem
CENTRALIZED:

BEST PERFORMANCE
EXCESSIVE COMMUNICATION

FULLY DECENTRALIZED:

WORST PERFORMANCE
NO COMMUNICATION

LOCALIZED:

MANY POSSIBLE ARCHITECTURES
System Representations

All signals are spatio-temporal, e.g. $u(x, t)$, $\psi(x, t)$, $y(x, t)$, etc.
Spatially distributed inputs, states, and outputs

- State space description

$$\frac{d}{dt} \psi(x, t) = A \psi(x, t) + B u(x, t)$$
$$y(x, t) = C \psi(x, t) + D u(x, t)$$

$A, B, C, D$ translation invariant operators

$\rightarrow$ spatially invariant system

- Spatio-temporal impulse response $h(x, t)$

$$y(x, t) = \int \int h(x - \xi, t - \tau) u(\xi, \tau) \, d\tau \, d\xi,$$

- Transfer function description

$$Y(\kappa, \omega) = H(\kappa, \omega) U(\kappa, \omega)$$
Spatio-temporal Impulse Response

Spatio-temporal impulse response $h(x, t)$

$$y(x, t) = \int \int h(x - \xi, t - \tau) u(\xi, \tau) \, d\tau \, d\xi,$$

**Interpretation**

$h(x, t)$: effect of input on output a distance $x$ away and time $t$ later

**Example:** Constant maximum speed of effects
Def: A system is \textit{funnel-causal} if impulse response $h(.,.)$ satisfies

$$h(x, t) = 0 \quad \text{for} \quad t < f(x),$$

where $f(.)$ is

1. non-negative
2. $f(0) = 0$
3. $\{f(x), x \geq 0\}$ and $\{f(x), x \leq 0\}$ are concave

i.e. $\text{supp}(h)$ is a “funnel shaped” region
Properties of funnel causal systems

Let $S_f$ be a funnel shaped set

- $\text{supp} (h_1) \subset S_f \& \text{supp} (h_2) \subset S_f \Rightarrow \text{supp} (h_1 + h_2) \subset S_f$

- $\text{supp} (h_1) \subset S_f \& \text{supp} (h_2) \subset S_f \Rightarrow \text{supp} (h_1 \ast h_2) \subset S_f$

- $(I + h_1)^{-1}$ exists & $\text{supp} (h_1) \subset S_f \Rightarrow \text{supp} ((I + h_1)^{-1}) \subset S_f$

i.e.

The class of funnel-causal systems is closed under

*Parallel, Serial, & Feedback interconnections*
• Given a plant $G$ with $\text{supp } (G_{22}) \subset S_{fg}$

• Let $S_{fk}$ be a set such that $S_{fg} \subset S_{fk}$
  
  *i.e. controller signals travel at least as fast as the plant’s*

Solve

$$\inf_{K \text{ stabilizing}} \| \mathcal{F}(G; K) \|,$$

$$\text{supp } (K) \subset S_{fk}$$
\( L_f := \text{class of linear systems w/ impulse response supported in } S_f \)

- Let \( G_{22} \in L_{fg} \)
  \[ G_{22} = NM^{-1} \text{ and } XM -YN = I \] with \( N, M, X, Y \in L_{fg} \) and stable
- Let \( S_{fg} \subset S_{fk} \)
- Then all stabilizing controllers \( K \) such that \( K \in L_{fk} \) are given by
  \[ K = (Y + MQ)(X + NQ)^{-1}, \]
  where \( Q \) is a stable system in \( L_{fk} \).
- The problem becomes
  \[ \inf_{\substack{Q \text{ stable} \\ Q \in L_{fk}}} \| H - UQV \|, \]
  \[ A \text{ convex problem!} \]
Bezout identity: Find $K$ and $L$ such that $A + LC$ and $A + BK$ stable.

\[
\begin{bmatrix}
X & -Y
\end{bmatrix}
:=
\begin{bmatrix}
A + LC & -B & L \\
K & I & 0
\end{bmatrix},
\begin{bmatrix}
M \\
N
\end{bmatrix}
:=
\begin{bmatrix}
A + BK & B \\
K & I \\
C & 0 \\
I & 0
\end{bmatrix},
\]

then $G = NM^{-1}$ and $XM - YN = I$.

If \[
\begin{array}{c}
• \ e^{tA}B, \ Ce^{tA} \text{ and } Ce^{tA}B \text{ are funnel causal} \\
• \ K \text{ and } L \text{ are funnel causal} \quad \text{(Easy!)}
\end{array}
\]

then all elements of Bezout identity are funnel-causal.
Example: Wave Equations with Input

1-d wave equation, \( x \in \mathbb{R} \):

\[
\partial^2_t \psi(x, t) = c^2 \partial^2_x \psi(x, t) + u(x, t)
\]

State space representation:

\[
\begin{align*}
\partial_t \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} &= \begin{bmatrix} 0 & I \\ c^2 \partial^2_x & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \\
\psi &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.
\end{align*}
\]

The semigroup:

\[
e^{tA} = \frac{1}{2} \begin{bmatrix} T_{ct} + T_{-ct} & \frac{1}{c} R_{ct} \\ c \partial^2_x R_{ct} & T_{ct} + T_{-ct} \end{bmatrix}.
\]

\[
R_{ct} := \text{spatial convolution with } \text{rec}\left(\frac{1}{ct}x\right)
\]

\[
T_{ct} := \text{translation by } ct
\]

_all components are funnel causal_

e.g. the impulse response \( h(x, t) = \frac{1}{2c} \text{rec}\left(\frac{1}{ct}x\right) \).
\( \kappa := \) spatial Fourier transform variable ("wave number")

\[
A + BK = \begin{bmatrix} 0 & 1 \\ -c^2 \kappa^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \\
= \begin{bmatrix} 0 & 1 \\ -c^2 \kappa^2 + k_1 & k_2 \end{bmatrix}.
\]

Set \( k_1 = 0 \), then

\[
\sigma(A + BK) = \bigcup_{\kappa \in \mathbb{R}} \left( k_2 \pm \frac{1}{2} \sqrt{k_2^2 - 4c^2 \kappa^2} \right) = \begin{bmatrix} 3k_2, \frac{1}{2}k_2 \end{bmatrix} \bigcup (k_2 + j\mathbb{R})
\]

Similarly for \( A + LC \). Therefore, choose e.g.

\[
K = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.
\]
Elements of the Bezout Identity are thus:

\[
\begin{bmatrix}
  X & -Y
\end{bmatrix} = \begin{bmatrix}
  -1 & 1 & 0 & -1 \\
  -c^2 \kappa^2 & 0 & -1 & 0 \\
  0 & -1 & 1 & 0 \\
\end{bmatrix},
\]

\[
\begin{bmatrix}
  M \\
  N
\end{bmatrix} = \begin{bmatrix}
  0 & 1 & 0 \\
  -c^2 \kappa^2 & -1 & 1 \\
  0 & -1 & 1 \\
  1 & 0 & 0
\end{bmatrix}.
\]

Equivalently

\[
M = \frac{s^2 + c^2 \kappa^2}{s^2 + s + c^2 \kappa^2}, \quad X = \frac{s^2 + 2s + c^2 \kappa^2 + 1}{s^2 + s + c^2 \kappa^2},
\]

\[
N = \frac{1}{s^2 + s + c^2 \kappa^2}, \quad -Y = \frac{-c^2 \kappa^2}{s^2 + s + c^2 \kappa^2}.
\]
How easily solvable are the resulting convex problems?

- In general, these convex problems are infinite dimensional
  *i.e. worse than standard half-plane causality*

- In certain cases, problem similar in complexity to half-plane causality
  *e.g. $H^2$ with the causality structure below*

  \[(Voulgaris, Bianchini, Bamieh, SCL ’03)\]
Generalizations

• Quick generalizations:
  – Several spatial dimensions
  – Spatially-varying systems
    \textit{funnel causality} \leftrightarrow \textit{non-decreasing speed with distance}
  – Use relative degree in place of time delay

• Arbitrary graphs

• How to solve the resulting convex problems

Related recent work:

• Rotkowitz \& Lall

• Anders Rantzer