Evolution of Mixed Strategies for Social Dilemmas on Structured Networks

Ahmet Yasin Yazicioglu
School of Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332–0250
Email: yasin@ece.gatech.edu

Xiaoli Ma
School of Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332–0250
Email: xiaoli@ece.gatech.edu

Yucel Altunbasak
School of Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332–0250
Email: yucel@ece.gatech.edu

Abstract—Cooperation in social dilemma games has been shown to be favored through evolution by various means such as topological heterogeneities or high benefit to cost ratio. As an extension to the previous studies with pure strategies, we incorporate the mixed strategies in the evolutionary dynamics and provide a higher resolution for the quantitative comparison of different scenarios with varying network topologies or game parameters. We show that this extension helps to reveal more information about the cooperation tendencies of different networks which can not be obtained from the pure strategy analysis. Comparison of steady state behaviors for different networks and game parameters are presented through simulation results.

I. INTRODUCTION

Cooperation is one of the fundamental attributes of many social systems. As these systems consist of various individuals, emergence of social dilemmas and conflicts is inevitable. When faced with a dilemma or a conflict, participating individuals usually have two options, they either cooperate or defect and their decisions affect the overall outcome. Cooperators are the people who contribute to the collective behavior at a personal expense whereas defectors are the ones who do not. In game theory, there are widely used metaphors which are used to represent such conditions. Prisoner’s Dilemma, Stag-Hunt game, and Hawk-Dove (also known as Chicken or Snowdrift) game are among the most popular metaphors that are utilized in analyzing social dilemmas [1], [2], [3].

Classical analysis of these games is based on the rational agents assumption. However, rationality assumption is controversial when very large populations are considered [4]. In such cases, the analysis of the population dynamics is rather attained via evolutionary games [5], [6]. Suppose that instead of being a rational maximizer each player comes to the game with a particular strategy attached to it. The strategies that provide higher payoffs to the players programmed to play them spread faster, while the number of players with strategies that fare worse decline. Ultimately, population ends up in an equilibrium state, namely an evolutionary stable configuration [7].

In conventional evolutionary games, society is considered to be infinite and homogeneous, where every agent has equivalent positions on the graph. However, this is not the case for real populations [8], [9]. In real populations, interactions among agents are determined by social and spatial constraints. For such networks, underlying topology affects the outcome significantly. For instance, when Prisoner’s Dilemma game is considered, cooperation can never invade the population in evolutionary sense for infinite homogeneous societies as it is a strictly dominated strategy [1]. However, recent studies in the literature show that based on the topological properties of structured networks and the payoff parameters of the game table [10] - [15], dynamics of the network topology through addition or removal of some nodes and edges [16], [17] or the heterogeneities and asymmetries in the interactions of agents [18], [19], [20] cooperation can be favored through the evolution.

When analyzing the influence of network topology and game parameters on the favorability of cooperation, utilization of pure strategies in earlier studies eliminated the uncertainties induced by randomness of mixed strategies. However, real agents can be better characterized by their ‘likeliness’ to cooperate rather than being absolute cooperators or defectors and mixed strategies incorporate this nature. The expected cooperation probability at steady state throughout many realizations of a particular simulation provides efficient grounds and higher resolution to analyze a given network for a social dilemma problem. Such analysis can be used in various applications. Biological, economical, political studies as well as many others may benefit from estimating the level and strength of cooperative behavior in a population of interest. On the other hand, when it is possible to control the topology or game parameters this analysis can be used for design purposes. In problems such as deciding the organizational structure of a company, transportation planning or design of many other systems where large number of autonomous agents will participate, it is desirable to attain a topology which implies a more cooperative behavior as it increases the overall performance of the system.

In this work, we explore the evolutionary favored mixed strategies on structured networks with constant topology for Prisoner’s Dilemma game. Note that this is an extensive
approach to the previous studies with pure strategies, however we also show that the results of this extension are not simply predictable as they would be for well-mixed populations. Infinite and well-mixed population assumption justifies equivalency of the case where every agent plays mixed strategy of cooperating with probability \( p \) and the pure strategy case where cooperation is played in the population with the frequency \( p \), as probabilities of occurrence for every possible outcome are equal in both cases. On the other hand, when heterogeneous populations are considered, this equivalent treatment can not be justified. Hence the evolution dynamics and results for mixed strategies are quite different and can not be directly obtained from pure strategy results.

Organization of this paper is as follows: Section 2 presents the game model and the evolution of strategies along with the conditions for favorability of cooperation. Section 3 presents the simulation results and some discussions. Finally Section 4 concludes the paper with some remarks and indicates possible future directions.

II. GAME MODEL, EVOLUTION DYNAMICS AND FAVORABILITY OF COOPERATION

Among the metaphors of public goods games, Prisoner’s Dilemma is perhaps the most widely used model as it can represent the underlying phenomenon for various public goods problems and social dilemmas. In Prisoner’s Dilemma, a defector pays a cost, \( c \), for another person to receive a benefit, \( b \), where \( b > c \). On the other hand, a defector does not pay any cost and does not distribute any benefit. This scheme is depicted by the game table shown in Table I. When this game is played by a social group it induces dynamics on the underlying network which can be represented via a graph. Players in the network occupy the vertices and links exist between the nodes that play the game against each other. At each round, nodes play their current strategies against all of their neighbors and get an accumulated payoff.

If mixed strategies are played, payoffs are random variables rather than being deterministic and a node’s strategy can be defined by the probability that it chooses to cooperate. Let us consider a node which plays a mixed strategy with probability of cooperation \( p \), meets another node playing a mixed strategy with probability of cooperation \( q \). Expected payoff for this node is:

\[
pq(b - c) + (1 - p)qb - p(1 - q)c = qb - pc \quad (1)
\]

Based on Eq. (1), the expected value of accumulated payoff for such a node with \( l \) neighbors is:

\[
b \sum_{i=1}^{l} q_i - lpc \quad (2)
\]

Analysis of this game on networks can be obtained through the evolutionary game theory. Classical evolutionary game dynamics is defined for infinite well-mixed populations, and governed by the following differential equation:

\[
x_i' = x_i(\pi_i - \bar{\pi}) \quad (3)
\]

where \( x_i \) is the fraction of phenotype \( i \) in population, \( \pi_i \) is the fitness of this phenotype which is defined as the average accumulated payoff for the members of this phenotype and \( \bar{\pi} \) is the average accumulated payoff in the whole population. This dynamics is also known as the replicator dynamics and defines how the different phenotypes multiply or decline in the well-mixed society based on their fitness and average fitness of the population.

When the evolution of cooperation in Prisoner’s Dilemma game is studied on finite populations, the replicator dynamics given in Eq. (3) does not apply directly but rather there is an analog for finite populations which converges to the replicator dynamics in the limit of infinite complete graphs [12]. This evolutionary dynamics involves the following events: At the beginning of each time step (generation), nodes play a single round of PD game against each of their neighbors and they accumulate the resulting payoffs. After that, each node, \( x_i \), randomly picks one of its neighbors, \( y \), and compares their accumulated payoffs \( U_x \) and \( U_y \). Node \( x \) adopts the strategy of node \( y \) only if \( U_y > U_x \) with a transition probability that increases monotonically to 1 as \( U_y - U_x \) increases. Different functions can be used to define the transition probabilities [21]. One possible option is \( 1/(1 + \exp[-(U_y - U_x)/K]) \) where \( K \) characterizes the possible noise effects [15]. Alternatively \((U_y - U_x)/(\max(k_x, k_y)(b + c))\), where \( k_x \) and \( k_y \) are the node degrees, can be used [12].

Based on this evolution dynamics, we analyze the conditions that increase the favorability of cooperation. There are the conditions which increase the chances that a more cooperative node can accumulate greater payoff than its less cooperative neighbors and possibly spread its strategy. In analyzing these conditions, we focus on the difference of accumulated payoffs for neighbors. For arbitrary neighbors \( x \) and \( y \), transition probabilities depend on the accumulated payoff difference of nodes, \( U_y - U_x \). The strategy that fares better to its player will be evolutionary favored through the connection between \( x \) and \( y \). Based on the expected value of accumulated payoff difference, \( E[U_y - U_x] \), it is possible to see which strategy, on the average, will be favored. If \( E[U_y - U_x] < 0 \) we have, on the average, probability of transition for \( x \) adopting strategy of \( y \) is less than the
probability of transition for $y$ adopting strategy of $x$. When $E[U_y - U_x] > 0$, relation is flipped for transition probabilities. In light of Eq. (2), $E[U_y]$ and $E[U_x]$ are obtained as:

$$E[U_x] = b(p_y + \sum_{i=1}^{k_x-1} q_{xi}) - c k_x p_x = b(p_y + \bar{q}_x) - c k_x p_x$$

$$E[U_y] = b(p_x + \sum_{i=1}^{k_y-1} q_{yi}) - c k_y p_y = b(p_x + \bar{q}_y) - c k_y p_y \quad (4)$$

where $q_{xi}$, $q_{yi}$ are the probability of cooperation for neighbors (except each other) of nodes $x$, $y$, and $\bar{q}_x = \sum_{i=1}^{k_x-1} q_{xi}$, $\bar{q}_y = \sum_{i=1}^{k_y-1} q_{yi}$. $E[U_y - U_x] > 0$ is attained when the following is satisfied:

$$c(k_x p_x - k_y p_y) + b(\bar{q}_y - \bar{q}_x + p_x - p_y) > 0, \quad (5)$$

as $b$, $c$ and $k_y$ are positive, we can divide the inequality by $-(b + c k_y)$ and rearrange to obtain the condition as

$$p_y = \frac{b + c k_x}{b + c k_y} p_x + \frac{b}{b + c k_y} (\bar{q}_x - \bar{q}_y) < 0 \quad (6)$$

By definition both $p_x$ and $p_y$ are bounded within interval $[0, 1]$, and the inequality (6) gives us subregions separated by a line which is obtained by setting the left side equal to zero. For the points $(p_x, p_y)$ which are located below this line we have $p_y$ being favored as, on the average, it fares better than $p_x$. For the points $(p_x, p_y)$ which are located above this line we have $p_x$ favored. Figure 1 depicts some possible scenarios. In Fig. 1 (a), for every possible value of $p_x$ and $p_y$ in $[0, 1]$ node $y$ accumulates better expected payoff than $x$ and its strategy is likely to be adopted by $x$. Fig. 1 (d) presents the opposite scenario where node $x$ gets a significant advantage.

Fig. 1 (b) and (c) depict some possible cases which occur if $E[U_x] = E[U_y]$ line passes through the feasible region. When this line passes through the feasible region, at most 4 regions can emerge as in Fig. 1 (c). Regions I and III are the regions where defective strategy has more evolutionary advantage. Regions II and IV, on the other hand, are the regions where cooperative strategy has more evolutionary advantage.

To interpret Eq. (6) and the effect of different parameters on the cooperation, let us first consider the well-mixed homogeneous case, where the cooperative behavior is known to be eliminated through evolution. For such a population, every node has the same degree ($k_x=k_y$) and every node is connected to all other nodes ($\bar{q}_x=\bar{q}_y$), which makes the $E[U_y] = E[U_x]$ line coincident with $p_y = p_x$ line. As the strategy of node $y$ is favored below this line and strategy of $x$ is favored above this line, it is possible to see that the strategy with smaller probability of cooperation is favored for this setting. Hence, cooperation is strongly opposed by evolution.

Parameters of the $E[U_y] = E[U_x]$ line depends on the topology and game parameters. As $(b + c k_x)/(b + c k_y)$ is always positive, the slope of this line is always positive. Hence, if this line passes through the feasible region (where $p_x, p_y \in [0, 1]$) the portion where cooperative strategy is favored is smaller than the portion where defective strategy is favored. In Fig. 1 (c), it can be seen that the total area of the regions II and IV is smaller than total area of the regions I and III. Similarly, in Fig. 1 (b), area of the region II is smaller than the total area of the regions I and III. However, the portion where more cooperative strategy is favored can still be altered via the slope of the $E[U_y] = E[U_x]$ line. In order to see the effect of slope, let us consider cases where the $E[U_x] = E[U_y]$ line passes through the origin. In this setting both $x$ and $y$ intercepts of the line are zero and none of the nodes can attain a possible advantage due to the intercepts. If the slope is equal to 1, we have $k_x = k_y$ and the $E[U_x] = E[U_y]$ line is coincident with $p_x = p_y$ line. For that case we know that more cooperative strategy is strongly opposed by evolution. However, if the slope is smaller or larger than 1, which can be obtained through degree differences between the nodes, a region where the cooperative strategy is favored emerges. For a slope smaller than 1 a region where the more cooperative strategy of node $x$ accumulates a higher expected payoff is formed, whereas for a slope greater than 1 a region where node $y$ accumulates a higher expected payoff with a higher probability of cooperation is formed.

The $x$ and $y$ intercepts of the $E[U_y] = E[U_x]$ line also depict an advantage of a node over the other. As the slope of the line is always positive, we have either both intercepts equal to zero (the line passes through the origin) or one intercept is negative while the other is positive. Value
of the positive intercept depicts a certain advantage the corresponding node acquires. That is to say, if \( x \) intercept is positive, node \( x \) becomes more influential as the value of this intercept gets larger. For positive \( y \) intercept, node \( y \) becomes more influential as the value of this intercept gets larger. In Fig. 1 (b) \( y \) intercept is positive and one can easily see that the region where \( p_y \) is favored, the total area of the regions II and III, is greater than the region where \( p_x \) is favored, the area of the region I. Ultimately if the positive intercept is greater than 1, the line does not pass through the feasible region and we have a significantly influential node, \( x \) or \( y \). Fig. 1 (a) depicts a case where node \( y \) is significantly influential whereas Fig. 1 (d) shows the opposite. In such cases the influential node has the possibility to make the other node adopt even the pure cooperation strategy regardless of its current strategy. In light of Eq. (6), \( y \) intercept is equal to \( \frac{b}{b + c k_y} (\bar{q}_y - \bar{q}_x) \) whereas \( x \) intercept is \( \frac{b}{b + c k_x} (\bar{q}_x - \bar{q}_y) \). This implies that, for two arbitrary neighbors, one of them can obtain an advantage over the other if it accumulates greater expected benefit from its neighbors than the other node (\( \bar{q}_x \) and \( \bar{q}_y \)). The amount of difference is scaled by a factor \( \frac{b}{b + c k_y} \) (for \( y \) intercept) or \( \frac{b}{b + c k_x} \) (for \( x \) intercept). As the scaling factor gets larger, the necessary benefit difference to obtain a certain level of advantage via the intercept decreases.

Deviating the \( E[U_{iy}] = E[U_{ix}] \) line from the \( p_x = p_y \) line by altering the slope and/or the intercept is significant in promoting the cooperative behavior. The slope and the intercept of the line depends on the node degrees and game parameters. Depending on the advantages obtained by the nodes due to the slope and/or the intercept, for certain \( (p_x, p_y) \) combinations more cooperative strategy provides higher expected payoff. For such cases cooperative strategy has a higher chance to be adopted by the other node and possibly spread throughout the network. This increases the evolutionary favorability of cooperation.

### III. Simulation Results

Simulations were performed for various networks and different game parameters. In the simulations we consider both cases where only pure strategies are allowed and cases with mixed strategies to compare the results. First let us compare the steady state strategy distribution of both cases for a particular scenario. The difference between pure strategy and mixed strategy evolutions on a Barabási-Albert scale-free topology with 1000 nodes is depicted in Fig. 2. When only pure strategies are allowed, this particular network converges to a uniform cooperation (although in 4 of the simulations a minority around 30 nodes survive as defectors). This result is quite consistent with the previous results presented in [3] as the particular game parameters used in the simulations \( (b = 1 \) and \( c = 0.12) \) are in the region where they report a uniform cooperation on Barabási-Albert scale-free networks. Also the average degree of the particular network is \( k = 7.8 \) which satisfies the rule, \( b/c > k \), proposed in [13]. Thus, the strong favorability of cooperation observed from the results with pure strategy evolution is expected. However, when the mixed strategies are also incorporated, results are quite different. It is still obvious that cooperation is evolutionary favored in the network through the heterogeneity, however the tendency to cooperate displays a certain variability where the average value for 20 simulations is 76\%.

Next we consider the pure and mixed strategy evolutions for various cases on Erdős-Rényi random, Watts-Strogatz small world and Barabási-Albert scale-free topologies with 1000 nodes. We check for the expected probability of cooperation (mixed strategy) and frequency of cooperators (pure strategy) at steady state for varying average degree and game parameters. Networks with average degrees \( (k) \) 4,6,8,10 and 12 are generated for each topology. Game parameters are normalized by setting \( b = 1 \) and \( c \) is varied in \( 0.03 - 0.15 \) interval with 0.03 increments. For each combination of \( k \) and \( c \), 10 simulations are run and the average result is reported. Steady state values are obtained through averaging of 1000 time steps after a warm up period of 10,000 time steps, starting from a uniform distribution of strategies among the
nodes.

Fig. 3. Expected probability of cooperation at steady state on Erdős-Rényi random networks for various average degrees and costs of cooperation.

Fig. 4. Frequency of cooperators at steady state on Erdős-Rényi random networks for various average degrees and costs of cooperation.

Fig. 5. Expected probability of cooperation on Watts-Strogatz small world networks for various average degrees and costs of cooperation.

Fig. 6. Frequency of cooperators at steady state on Watts-Strogatz small world networks for various average degrees and costs of cooperation.

Fig. 7. Expected probability of cooperation on Barabási-Albert scale-free networks for various average degrees and costs of cooperation.

Fig. 8. Frequency of cooperators at steady state on Barabási-Albert scale-free networks for various average degrees and costs of cooperation.

When we compare the pure strategy results with mixed strategy results, we can see that in both cases it is possible to observe the degradation in the favorability of cooperation as the average degree and cost of cooperation increase. However, in many of the cases pure strategy results are rather optimistic about the cooperative behavior of the particular network than the mixed strategy simulations. When the cooperative behavior is able to survive, pure strategy simulations give boosted results and it is harder to distinguish the cooperation tendencies of different cases. For example when we compare different topologies for $c < 0.09$, pure strategy results (Figures 4, 6, 8) do not provide a clear comparison as for all the networks frequency of cooperators at steady state are close and in the region $0.8 - 1$. On the other
when we consider the mixed strategy simulations (Figures 3, 5, 7), it is easier to see how significantly cooperation can be degraded for Watts-Strogatz small world networks compared to the other two topologies. Furthermore, with pure strategy results, scale free topology seems to behave more cooperative than the other two topologies for almost every scenario, whereas mixed strategy results rather highlight the robustness of this topology against changes in average degree and cost of cooperation. While this robustness resists the attenuation of cooperation in a quite wider range of parameters, it also resists the higher promotion of cooperation for low values of $c$ and $k$. In the simulations with $c = 0.03$, more cooperative behaviors were obtained with random networks, or even with small world networks for certain values of average degree.

As it can be seen from the results, expected probability of cooperation varies significantly among the different topologies for the same average degree and cost of cooperation. As Barabási-Albert scale-free being the most heterogeneous topology, it shows a more cooperative behavior in a wider range of $k$ and $c$. In this topology, direct links among the hubs also help to promote cooperation. When two hubs are connected, the more cooperative one gets an advantage over the other. This is due to the reluctance of hubs (when $b/c$ is also large enough) when meeting less connected neighbors. Both hubs are likely to be imitated by most of their low degree neighbors through evolution, however as this happens the more cooperactive hub creates itself a more cooperative neighborhood resulting in higher payoffs and ultimately the less cooperactive hub adopts the strategy of more cooperactive hub and its neighbors are also likely to adopt this strategy in the following generations. Note that with their reluctance against less connected neighbors, hubs can also convert their neighbors with higher probability of cooperation which causes the resistance at very low values of $c$ and $k$, where the other two topologies can display more cooperactive behaviors.

In a sense, a more cooperactive strategy played by a hub which is also able to keep the hub resistive (depending on the parameters $b$ and $c$) to imitate its low degree neighbors has high chance of spreading throughout the population for Barabási-Albert scale-free topology.

Results also show that Erdős-Rényi random networks have more cooperactive tendencies than the Watts-Strogatz small world networks. This is an expected result as the small world networks are obtained from rewiring of regular networks, hence their heterogeneity lies somewhere between the two. Results depict that Watts-Strogatz small world topology, as being the most regular topology among the three topologies, shows the least cooperactive behavior and cooperation easily dies out with increasing average degree or cost of cooperation.

**IV. Conclusion and Future Directions**

In this paper, mixed strategy evolution in PD game on structured networks was considered. Steady state value for expected probability of cooperation was proposed as a quantitative comparison measure for different scenarios.

The evolution dynamics and its dependency on the network topology and game parameters were presented along with the simulation results for various cases. Random, small world and scale-free topologies were simulated for different average degrees and costs of cooperation. Scale-free topology presented a higher robustness against the game and network parameters and more cooperactive behavior for a wide range, whereas tendency to cooperate is highly attenuated for small world topology as a consequence of increased regularity.

As a future work, mixed strategy evolution for other widely used social dilemma games can be explored. Moreover, similar analysis can be applied on dynamic topologies where some nodes/edges are added or removed throughout the evolutionary dynamics.

**References**