Supplementary Information:
Electron pumping in graphene mechanical resonators

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We consider a suspended graphene resonator as described in [1, 2]. The dynamical equation for the out-of-plane deformation from equilibrium $a(t)$ is given by (see the respective Appendixes on the form of the forces),

$$\rho\frac{\partial^2 a}{\partial t^2} = \mathcal{F}_S + \mathcal{F}_D + \mathcal{F}_E \tag{1}$$

where $\mathcal{F}_S, \mathcal{F}_D, \mathcal{F}_E$ are the time dependent restoring elastic, damping and electrostatic forces (or pressures) respectively, in units of m^{-2}. ρ is the mass density. The total restoring elastic force, ignoring second order terms due to bending forces κ and $O(h^2)$, reduces to,

$$\mathcal{F}_S = -\frac{64}{3}\frac{\lambda+2\mu}{L^4}\left(a^3+3a^2h_0+3ah_0^2\right)+\frac{8\Delta L}{L^3}(\lambda+2\mu)a \tag{2}$$

obtained by minimizing elastic energy. Damping is treated phenomenologically via,

$$\mathcal{F}_D = -\frac{\rho}{\tau_d}\frac{\partial a}{\partial t} \tag{3}$$

τ_d can be easily obtained from quality factor Q measured in experiments. Q is defined to be $Q = \omega_0/\Delta\omega = \omega_0\tau_d/2$, where $\Delta\omega$ is the so called bandwidth of the resonance peak. Reported Q is around 100 [1, 2] at room temperature, depending on many factors. For example, temperature dependence of $Q \propto T^{-0.36}$ was found, for $T < 100K$ [2]. And a record $Q \approx 100,000$ at $90mK$ was reported [3]. Mass density of graphene assumed to be $\rho \approx 7.4\times10^{-6}kgm^{-2}$. Lastly, the electrostatic force is modelled as,

$$\mathcal{F}_E = \frac{C_T^2 V_{dc}V_{ac}}{\epsilon_0}\cos(\omega t) \tag{4}$$

neglecting $O(V_{ac}^2)$ and other non-linear terms. $C_T = [\epsilon_0^{-1}(d+h_0)+\epsilon_{SiO2}^{-1}t_{SiO2}]^{-1}$ is the total effective capacitance due to the back-gate oxide and air dielectric, d being the perpendicular distance of the unstrained graphene from the substrate.

We are interested in the steady state solution to the dynamical equation i.e. $a(\omega) = |a|\exp(i\phi)$. We seek an approximate solution through an iterative technique used in the Duffing model [4, 5]. For convenience, we rewrite the dynamical equation as,

$$\rho\ddot{a} = -k_0 a - k_1 a^2 - k_2 a^3 - \frac{\rho}{\tau_d}\dot{a} + f\cos(\omega t) \tag{5}$$

where

$$k_0 = \frac{64(\lambda+2\mu)h_0^2}{L^4} - \frac{8\Delta L}{L^3}(\lambda+2\mu)\ ,\ k_1 = \frac{64(\lambda+2\mu)h_0}{L^4}\ ,\ k_2 = \frac{64(\lambda+2\mu)}{3L^4}\ ,\ f = \frac{C_T^2 V_{dc}V_{ac}}{\epsilon_0} \tag{6}$$

In the spring constant term k_0, we consider only the case for $\Delta L \leq 0$. k_2 contributes to the Duffing force, and renders the spring more stiff (soft) if positive (negative). In the former, the effect will be a shift of resonance with increasing driving force. And at larger driving force would lead to bistability and hysteresis [2].

The frequency response $a(\omega)$ around the resonant frequency $\omega_0 \equiv \sqrt{k_0/\rho}$ has the following approximate solution,

$$\left[(\omega_0^2-\omega^2)|a|+\frac{3k_2}{4\rho}|a|^3\right]^2+\left(\frac{\omega|a|}{\tau_d}\right)^2=\left(\frac{f}{\rho}\right)^2 \tag{7}$$

$$\tan\phi=\frac{\omega|a|}{\tau_d\left[(\omega_0^2-\omega^2)|a|+\frac{3k_2}{4\rho}|a|^3\right]} \tag{8}$$

Note that a^2 terms affects the higher harmonics $2\omega_0$. These converge to the Lorentz model solutions if we set $k_2=0$. In the linear regime, i.e. $k_2=0$, the response at $\omega=\omega_0$ goes as $|a|=2Qf/\rho\omega_0^2$. When $k_2\neq 0$, $|a|$ follows,

$$\frac{9k_2^2}{16}|a|^6+\frac{k_0^2}{4Q^2}|a|^2=f^2 \tag{9}$$

From this, we can define a threshold driving force f_{th} where $|a(\omega=\omega_0)|$ starts to deviate from linearity i.e. $|a|\propto f$,

$$f_{th}\approx\sqrt{\frac{k_0^3}{36Q^3k_2}} \tag{10}$$

As evident, ρ has no effect on f_{th}, and a larger Q yields a smaller f_{th} i.e. increased sensitivity to non-linearity. However, a larger Q is desireable to acheive a larger resonance $|a(\omega=\omega_0)|$.

APPENDIX A: ELECTROSTATIC FORCES

The electrostatic force is modelled as,

$$\mathcal{F}_E^{tot}=\frac{1}{2}\frac{C_T^2}{\epsilon_0}V_{bg}^2\approx\frac{1}{2}\frac{C_T^2}{\epsilon_0}\left(V_{dc}^2+2V_{dc}V_{ac}\cos(\omega t)\right)\equiv\mathcal{F}_E^{eq}+\mathcal{F}_E \tag{A1}$$

neglecting $O(V_{ac}^2)$. $C_T=[\epsilon_0^{-1}(d+h_0)+\epsilon_{SiO2}^{-1}t_{SiO2}]^{-1}$ is the total effective capacitance (per unit area) due to the back-gate oxide and air dielectric. d is the perpendicular distance of graphene from the substrate when unstrained.

APPENDIX B: ELASTIC FORCES

The position of a 2D membrane can be described by the in-plane and out-of-plane deformation field given by $\mathbf{u}(x,y)=[u_x(x,y),u_y(x,y)]$ and $h(x,y)$. In the linear approximation, the strain tensor is given by,

$$u_{\alpha\beta}=\frac{1}{2}(\partial_\alpha u_\beta+\partial_\beta u_\alpha+\partial_\alpha\partial_\beta h) \tag{B1}$$

The elastic free energy is given by [6, 7],

$$\begin{aligned}\mathcal{E} &= \int dxdy\left[\frac{1}{2}\kappa(\nabla^2h)^2+\frac{1}{2}\lambda(u_{xx}+u_{yy})^2+\mu(u_{xx}^2+u_{yy}^2+2u_{xy}^2)-\mathcal{F}_E^{eq}h\right]\\ &\approx \int dx\left[\frac{1}{2}\kappa(\nabla^2h)^2+\frac{1}{2}(\lambda+2\mu)u_{xx}^2-\mathcal{F}_E^{eq}h\right]\end{aligned} \tag{B2}$$

where $\kappa\approx 1eV$ is the bending rigidity, $\mu\approx 9eV\AA^{-2}$ and $\lambda\approx 2eV\AA^{-2}$ are the Lame constants of graphene. Experiments [8] measures an elastic constant for graphite $c_{11}=106\times10^{10}Nm^{-2}\approx 1TPa$. For graphene, we have the relation $c_{11}d_{in}=\lambda+2\mu$. Using an interlayer separation distance of $d_{in}=0.335nm$ [9], we $c_{11}d_{in}\approx 355Nm^{-1}$. This is in good agreement with the values of Lame constants we assumed. Recent measurement [10] of the Young's modulus of graphene yields $E_{2d}=342Nm^{-1}$. Note that by definition, $E_{2d}=c_{11}d_{in}$.

$\mathcal{F}_E^{eq}$ is the pressure induced by the bottom electrostatic gate. Assuming homogenuity along transverse direction, we arrive to a one-dimensional problem. Minimizing $\mathcal{E}$ in Eq. B2 yields us the following Euler-Lagrange equations,

$$\begin{aligned}\kappa\partial_x^4h-(\lambda+2\mu)\left[\frac{3}{2}(\partial_xh)^2\partial_x^2h+\partial_xu_x\partial_x^2h+\partial_xh\partial_x^2u_x\right]&=\mathcal{F}_E^{eq}\\ \partial_x^2u_x+\partial_xh\partial_x^2h&=0\end{aligned} \tag{B3}$$

where the latter implies the longitudinal strain is constant i.e.

$$u_{xx} = \text{constant} \tag{B4}$$

Finally, the differential equation governing $h(x)$ is,

$$\kappa \partial_x^4 h - (\lambda + 2\mu) u_{xx} \partial_x^2 h = \mathcal{F}_E^{eq} \tag{B5}$$

The differential equation for h then reduces to,

$$-(\lambda + 2\mu) u_{xx} \partial_x^2 h = \mathcal{F}_E^{eq} \tag{B6}$$

whose explicit solution with the boundary condition $h(\pm L/2) = 0$ is,

$$h(x) \approx \frac{\mathcal{F}_E^{eq}(L^2 - 4x^2)}{8(\lambda + 2\mu)u_{xx}} \equiv h_0 - \frac{4h_0}{L^2}x^2 \quad \text{where} \quad h_0 = \frac{\mathcal{F}_E^{eq} L^2}{8(\lambda + 2\mu)u_{xx}} \tag{B7}$$

where h_0 is the maximum deflection i.e. at $x = 0$. Including the second order terms, the expression can be rather complicated [11]. With the profile $h(x)$, simple geometry allows us to relate u_{xx} with h_0 as per Eq. C9. Then the elastic force equation reduces from Eq. B6 to,

$$\mathcal{F}_S^{eq} = -\frac{64}{3}(\lambda + 2\mu)\frac{h_0^3}{L^4} + \frac{8\Delta L}{L^3}(\lambda + 2\mu)h_0 = \mathcal{F}_E^{eq} \tag{B8}$$

where we included the possibility of an initial tension, $\Delta L < 0$. Elastic forces due to deformation a away from the equilibrium can then be described by,

$$\mathcal{F}_S = -\frac{64}{3}\frac{\lambda + 2\mu}{L^4}\left(a^3 + 3a^2 h_0 + 3a h_0^2\right) + \frac{8\Delta L}{L^3}(\lambda + 2\mu)a \tag{B9}$$

APPENDIX C: DEFORMATION AND GATING

Next we shall determine u_{xx}. Following [12], it is defined as,

$$u_{xx} = \frac{L' - (L + \Delta L)}{L + \Delta L} \approx \frac{L' - L}{L} - \frac{\Delta L}{L} \tag{C1}$$

where L' is the length of the strained graphene,

$$L' = 2\int_0^{L/2} dx \sqrt{1 + |\nabla h|^2} \approx L + \int_0^{L/2} dx \left(\partial_x h\right)^2 = \frac{L^3 (\mathcal{F}_E^{eq})^2}{24(\lambda + 2\mu)^2 u_{xx}^2} + L \tag{C2}$$

L is trench length and $L + \Delta L$ is length in absense of strain. Hence, solving u_{xx} then reduces to finding the root of,

$$u_{xx}^3 + \frac{\Delta L}{L}u_{xx}^2 - \frac{L^2(\mathcal{F}_E^{eq})^2}{24(\lambda + 2\mu)^2} = 0 \tag{C3}$$

No initial tension or slack: If $\Delta L = 0$, we will obtain,

$$u_{xx} = \left[\frac{L^2(\mathcal{F}_E^{eq})^2}{24(\lambda + 2\mu)^2}\right]^{1/3} = \frac{8h^2}{3L^2} \tag{C4}$$

Electrostatically, one can approximate $\mathcal{F}_E^{eq}$ as,

$$\mathcal{F}_E^{eq} \approx \frac{e^2 n^2}{2\epsilon_{eff}} \tag{C5}$$

where n is the carrier density in graphene and ϵ_{eff} is the effective dielectric due to air gap and back gate oxide. We ignores the curvature of graphene. Assuming that $\Delta L = 0$, we get an expression for maximum deflection h_0 at equilibrium,

$$h_0 = \left[\frac{3L^4e^2n^2}{128\epsilon_{eff}(\lambda+\mu)}\right]^{\frac{1}{3}} \tag{C6}$$

With initial tension or slack: For general case of $\Delta L \neq 0$, To obtain h_0 rigorously requires solving the following the electrostatic and elasticity equations self-consistently.

$$F_E^{eq} = \frac{1}{2\epsilon_0}\left[\left(\overbrace{\frac{\epsilon_0}{d-h_0}}^{C_a}\right)^{-1} + \left(\overbrace{\frac{\epsilon_{ox}}{t_{ox}}}^{C_{ox}}\right)^{-1}\right]^{-2} V_{dc}^2 \tag{C7}$$

$$-\frac{64}{3L^3}h_0^3 + \frac{8\Delta L}{L^2}h_0 + \frac{F_E^{eq}L}{\lambda+2\mu} = 0 \tag{C8}$$

When $F_E^{eq} = 0$, the latter requires $h_0 = 0$ for $\Delta L \leq 0$ (tension), which is expected. Another remark. A series capacitance C_{ox} to C_a is essential as it provides stability to the electrically actuated mechanical system. A relation between u_{xx} and h_0 can be obtained from Eq. B7 and C3, yielding,

$$u_{xx}^3 + \left(\frac{\Delta L}{L} - \frac{8h_0^2}{3L^2}\right)u_{xx}^2 = 0$$

$$\Rightarrow u_{xx} = 0 \text{ or } u_{xx} = \frac{8h_0^2}{3L^2} - \frac{\Delta L}{L} \tag{C9}$$

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